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ON THE CONVEXITY OF SOME TRACE FUNCTIONS

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Abstract. In this paper, we study the convexity of the trace function $A \to \mathrm{Tr}\, B^* f(A) B$. And we get an extension of the Peierls-Bogolyubov inequality and the joint convexity of the trace geometric mean.

1. Introduction

Throughout this paper, we write $\mathbf{M}_n$ to denote the $n \times n$ complex matrices, $\mathbf{H}_n$ the subset consisting of self-adjoint (Hermitain) matrices, $\mathbf{P}_n$ and $\mathbf{P}_n^+$ the subset consisting of positive semidefinite matrices and positive definite matrices, respectively. We write $A \geqslant 0$ if A is positive semidefinite and $A > 0$ if A is positive definite.

It is well known that, when f is convex, the trace function $A \to \mathrm{Tr}\, f(A)$ is convex for self-adjoint matrices. Some related results can be found in [6, Introduction]. See also [7]. Several important concepts in operator theory and quantum information theory are closely related to the convexity of this type of trace functions. For example, the Schatten p-norms $(\mathrm{Tr}\,|A|^p)^{\frac{1}{p}}$, $(p > 1)$, the von Neumann entropy $-\mathrm{Tr}\, A \log A$, and the Peierls-Bogolyubov inequality [5, 1, 4].

The Peierls-Bogolyubov inequality plays a very important role in quantum information theory, especially in the calculation of the partition function. It states that

$$\log \frac{\mathrm{Tr}\, \exp(A+B)}{\mathrm{Tr}\, \exp A} \geqslant \frac{\mathrm{Tr}\, \exp(A) B}{\mathrm{Tr}\, \exp A}.$$

It can also state the log-convexity of the trace function $\mathrm{Tr}\, \exp(A)$, i.e., the map

$$A \to \log \mathrm{Tr}\, \exp(A)$$

is convex on $\mathbf{H}_n$.

In [8], Hansen et al. studied the Peierls-Bogolyubov inequality for deformed exponentials. The deformed logarithm denoted $\log_q$ is defined by setting

$$\log_q a = \begin{cases} \dfrac{a^{q-1} - 1}{q-1} & q \neq 1, \\ \log a & q = 1, \end{cases} \qquad a > 0.$$

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The deformed logarithm is also denoted by the q-logarithm. The deformed exponential function or the q-exponential is defined as the inverse function to the q-logarithm. It is denoted by $\exp_q$ and is given by the formula

$$\exp_q a = \begin{cases} (a(q-1)+1)^{1/(q-1)}, & a > -1/(q-1), \quad q > 1, \\ (a(q-1)+1)^{1/(q-1)}, & a < -1/(q-1), \quad q < 1, \\ \exp a, & a \in \mathbb{R}, \quad q = 1. \end{cases}$$

In [8], the authors studied the convexity of the function

$$f(A) = \log_r \operatorname{Tr} B^* \exp_q(A)B.$$

It has been proved that

(i) If $q \leqslant 0$ and $r \geqslant q$, then f is convex.

(ii) If $\frac{3}{2} \leqslant q \leqslant 2$ and $r \geqslant q$, then f is convex.

(iii) If $q \geqslant 2$ and $r \leqslant q$, then f is concave.

We find there is a gap for $0 \leqslant q \leqslant \frac{3}{2}$ that has not been studied yet. In this paper, we tackle with this problem and complement the corresponding results. We will also study the convexity of the trace function $\exp_q \operatorname{Tr} H^* \log_q AH$.

Moreover, we study the joint convexity of the trace function

$$(A,B) \to \operatorname{Tr} B^{\frac{1}{2}} f(B^{\frac{-1}{2}} AB^{\frac{-1}{2}})B^{\frac{1}{2}}.$$

It enables us to consider the joint convexity of the trace geometric mean.

2. Main results

2.1. The convexity of trace functions

THEOREM 2.1. *Let $B \in \mathbf{M}_n$. If $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then the trace function*

$$F(A) = \operatorname{Tr} B^* f(A)B \tag{2.1}$$

is convex for $A \in \mathbf{H}_n$.

Proof. Let $\{P_i\}$ be a pairwise orthogonal family of minimal projections with $\sum_{i=1}^n P_i = I_n$. Let $\sum_j s_j Q_j$ be the spectral decomposition of a self-adjoint matrix C.

Then for positive definite matrix $K(=BB^*)$, we have $\operatorname{Tr}KP_i > 0$ for $i = 1,\dots,n$. Thus

$$\begin{aligned}\operatorname{Tr}Kf(C) &= \sum_j f(s_j)\operatorname{Tr}KQ_j = \sum_i\left(\sum_j f(s_j)\operatorname{Tr}KQ_jP_i\right)\\ &= \sum_i\left[\frac{\sum_j f(s_j)\operatorname{Tr}KQ_jP_i}{\operatorname{Tr}KP_i}\operatorname{Tr}KP_i\right]\\ &\geqslant \sum_i(\operatorname{Tr}KP_i)f\left(\sum_j s_j\frac{\operatorname{Tr}KQ_jP_i}{\operatorname{Tr}KP_i}\right)\\ &= \sum_i(\operatorname{Tr}KP_i)f\left(\frac{\operatorname{Tr}KCP_i}{\operatorname{Tr}KP_i}\right). \end{aligned} \tag{2.2}$$

Now, assume $\sum_{i=1}^n \mu_iP_i$ is the spectral decomposition of the convex combination

$$A = \lambda C_1 + (1-\lambda)C_2.$$

It follows that

$$\begin{aligned}\operatorname{Tr}Kf(A) &= \sum_i f(\mu_i)\operatorname{Tr}KP_i\\ &= \sum_i(\operatorname{Tr}KP_i)f\left(\frac{\operatorname{Tr}KAP_i}{\operatorname{Tr}KP_i}\right)\\ &= \sum_i(\operatorname{Tr}KP_i)f\left(\frac{\operatorname{Tr}K(\lambda C_1+(1-\lambda)C_2)P_i}{\operatorname{Tr}KP_i}\right)\\ &\leqslant \sum_i(\operatorname{Tr}KP_i)\left[\lambda f\left(\frac{\operatorname{Tr}KC_1P_i}{\operatorname{Tr}KP_i}\right)+(1-\lambda)f\left(\frac{\operatorname{Tr}KC_2P_i}{\operatorname{Tr}KP_i}\right)\right]\\ &\leqslant \lambda\operatorname{Tr}Kf(C_1)+(1-\lambda)\operatorname{Tr}Kf(C_2),\end{aligned}$$

where the first inequality follows from the convexity of f, and the second inequality follows from applying the inequality (2.2) twice. Hence we have that

$$\operatorname{Tr}Kf(\lambda C_1+(1-\lambda)C_2) \leqslant \lambda\operatorname{Tr}Kf(C_1)+(1-\lambda)\operatorname{Tr}Kf(C_2), \tag{2.3}$$

with K positive definite.

For arbitrary matrix $B \in \mathbf{M}_n$, $K = BB^*$ is positive semidefinite. We set $K_\varepsilon = K + \varepsilon I$ for an arbitrary $\varepsilon \in (0,1)$. Then K_ε is positive definite and satisfies

$$\operatorname{Tr}K_\varepsilon f(\lambda C_1+(1-\lambda)C_2) \leqslant \lambda\operatorname{Tr}K_\varepsilon f(C_1)+(1-\lambda)\operatorname{Tr}K_\varepsilon f(C_2).$$

Letting $\varepsilon \to 0$, we have the inequality (2.3) holds for positive semidefinite matrix K. Hence, we have $A \to \operatorname{Tr}B^*f(A)B$ is convex. $\square$

Now we consider the Jensen inequalities related to the trace function $\operatorname{Tr}B^*f(A)B$. In this part, we denote the set of $n\times m$ matrices by $\mathbf{M}_{nm}$.

THEOREM 2.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be convex. Then the following statements hold and they are equivalent:*

(i) For $B\in\mathbf{M}_n,\ F(A)=\operatorname{Tr}B^*f(A)B$ is convex on $\mathbf{H}_n$.

(ii) For $A_i\in\mathbf{H}_n,\ B\in\mathbf{M}_m$, and $H_i\in\mathbf{M}_{nm}$, with $\sum_i H_i^*H_i=I_m$,

$$\operatorname{Tr}B^*f(\sum_i H_i^*A_iH_i)B\leqslant\sum_i\operatorname{Tr}B^*H_i^*f(A_i)H_iB.$$

(iii) For $A\in\mathbf{H}_n$, and $V\in\mathbf{M}_{nm},B\in\mathbf{M}_m$ with $V^*V=I_m$,

$$\operatorname{Tr}B^*f(V^*AV)B\leqslant\operatorname{Tr}B^*V^*f(A)VB.$$

Proof. According to Theorem 2.1, we should only prove the equivalence.

$(i)\to(iii)$. Set $Z=\begin{pmatrix}A&0\\0&M\end{pmatrix}$, $C=\begin{pmatrix}B&0\\0&0\end{pmatrix}$, $U=\begin{pmatrix}V&P\\0&-V^*\end{pmatrix}$, $W=\begin{pmatrix}V&-P\\0&V^*\end{pmatrix}$, where $M\in\mathbf{H}_m$ and $P=I-VV^*$. Note that V is isometry, and $V^*P=0,\ PV=0$, then U and W are unitary. By calculation,

$$U^*ZU=\begin{pmatrix}V^*AV&V^*AP\\PAV&PAP+VMV^*\end{pmatrix},\quad W^*ZW=\begin{pmatrix}V^*AV&-V^*AP\\-PAV&PAP+VMV^*\end{pmatrix},$$

$$\frac{1}{2}(U^*ZU+W^*ZW)=\begin{pmatrix}V^*AV&0\\0&PAP+VMV^*\end{pmatrix}.$$

Thus

$$\begin{aligned}
&\operatorname{Tr}B^*f(V^*AV)B\\
&=\operatorname{Tr}\begin{pmatrix}B^*&0\\0&0\end{pmatrix}f\begin{pmatrix}V^*AV&0\\0&PAP+VMV^*\end{pmatrix}\begin{pmatrix}B&0\\0&0\end{pmatrix}\\
&=\operatorname{Tr}C^*f\left(\frac{1}{2}(U^*ZU+W^*ZW)\right)C\\
&\leqslant\frac{1}{2}\operatorname{Tr}C^*f(U^*ZU)C+\frac{1}{2}\operatorname{Tr}C^*f(W^*ZW)C\\
&=\operatorname{Tr}C^*\left(\frac{1}{2}U^*f(Z)U+\frac{1}{2}W^*f(Z)W\right)C\\
&=\operatorname{Tr}\begin{pmatrix}B^*&0\\0&0\end{pmatrix}\begin{pmatrix}V^*f(A)V&0\\0&Pf(A)P+Vf(M)V^*\end{pmatrix}\begin{pmatrix}B&0\\0&0\end{pmatrix}\\
&=\operatorname{Tr}B^*V^*f(A)VB.
\end{aligned}$$

$(iii)\to(ii)$. Set

$$A=\begin{bmatrix}A_1&&&\\&A_2&&\\&&\ddots&\\&&&A_n\end{bmatrix},\quad V=\begin{bmatrix}H_1&0\cdots0\\H_2&0\cdots0\\\vdots&0\cdots0\\H_n&0\cdots0\end{bmatrix},\quad D=\begin{bmatrix}B&0\cdots0\\0&0\cdots0\\\vdots&0\cdots0\\0&0\cdots0\end{bmatrix}.$$

Then

$$V^*AV = \begin{bmatrix} \sum_i H_i^* A_i H_i & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & 0 \cdots 0 \\ 0 & 0 \cdots 0 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \operatorname{Tr} B^* f(\sum_i H_i^* A_i H_i) B &= \operatorname{Tr} D^* f(V^*AV) D \\ &\leqslant \operatorname{Tr} D^* V^* f(A) V D \\ &= \operatorname{Tr} B^* \left(\sum_i H_i^* f(A_i) H_i \right) B \\ &= \sum_i \operatorname{Tr} B^* H_i^* f(A_i) H_i B. \end{aligned}$$

$(ii) \to (i)$ is easy to prove. We omit it. □

2.2. An extension of Peierls-Bogolyubov's inequality

By Theorem 2.1, we know that the trace function $A \to \operatorname{Tr} B^* A^p B$ is concave for $0 < p \leqslant 1$, and convex for $p < 0$, and $p > 1$. In [8], the authors obtained the following proposition.

PROPOSITION 2.3. ([8, Proposition 1]) *Let f be a real positive function defined on $\mathbf{P}_n^+$ and assume f is homogeneous of degree $p \neq 0$.*

(i) If f is convex and $p > 0$, then $f^{1/p}$ is convex.

(ii) If f is convex and $p < 0$, then $f^{1/p}$ is concave.

(iii) If f is convex and $p < 0$ and $r > 0$, then f^r is convex.

(iv) If f is concave and $p > 0$, then $f^{1/p}$ is concave.

(v) If f is concave and $p < 0$, then $f^{1/p}$ is convex.

(vi) If f is concave and $p > 0$ and $r < 0$, then f^r is convex.

According to Proposition 2.3, we have

PROPOSITION 2.4. *Let $B \in \mathbf{M}_n$ be an arbitrary operator and consider the function*

$$F(A) = \left(\operatorname{Tr} B^* A^p B\right)^{1/r}$$

defined on positive definite matrices $\mathbf{P}_n^+$. Then

(i) F is concave for $r \leqslant p < 0$;

(ii) F *is convex for* $p<0$ *and* $r>0$;

(iii) F *is concave for* $0<p\leqslant 1$ *and* $r\geqslant p$;

(iv) F *is convex for* $0<p\leqslant 1$ *and* $r<0$;

(v) F *is convex for* $p\geqslant 1$ *and* $0<r\leqslant p$.

Note that (iii) and (iv) and the cases of $-1\leqslant p<0$ and $1\leqslant p\leqslant 2$ in $(i),(ii),(v)$ have already been discussed in [8]. And the cases of $p<-1$ in $(i),(ii)$, and the case of $p>2$ in (v) are new.

Consider the function

$$\begin{aligned} F(A) &= \log_r \operatorname{Tr} B^* \exp_q(A)B \\ &= \frac{1}{r-1}\left(\left(\operatorname{Tr}\left(B^*(A(q-1)+1)^{\frac{1}{q-1}}B\right)\right)^{r-1}-1\right). \end{aligned}$$

Let

$$\frac{1}{q-1}=p, \quad \frac{1}{r-1}=r.$$

Then by Proposition 2.4, we have

THEOREM 2.5. *Let* B *be an arbitrary matrix and consider the function*

$$F(A)=\log_r \operatorname{Tr} B^* \exp_q(A)B$$

defined on self-adjoint $A>-(q-1)^{-1}$.

(i) If $q<1$ *and* $r\geqslant q$, *then* F *is convex.*

(ii) If $1<q\leqslant 2$ *and* $r\geqslant q$, *then* F *is convex.*

(iii) If $q\geqslant 2$ *and* $r\leqslant q$, *then* F *is concave.*

When $q=r$, and letting $q\to 1$, we have

THEOREM 2.6. *Let* B *be an arbitrary matrix, then the function*

$$F(A)=\log \operatorname{Tr} B^* \exp(A)B$$

is convex.

That is, we have an extension of the Peierls-Bogoliubov inequality as

COROLLARY 2.7. *Let* $A,B\in \mathbf{H}_n$ *be self-adjoint operators, and* $C\in \mathbf{M}_n$. *Then we have the following inequality*

$$\log \frac{\operatorname{Tr} C^* \exp(A+B)C}{\operatorname{Tr} C^* \exp(A)C} \geqslant \frac{\operatorname{Tr} C^*(d\exp(A)B)C}{\operatorname{Tr} C^* \exp(A)C}, \tag{2.4}$$

where $d\exp(A)$ *is the Fréchet derivative of the function* $\exp(A)$.

Proof. Take self-adjoint matrices $A,B \in \mathbf{H}_n$ and define the function

$$g(t) = \log \operatorname{Tr} C^* \exp(A+tB)C.$$

Since $g(t)$ is convex, we obtain the inequality

$$g(1) - g(0) \geqslant g'(0).$$

Hence the inequality (2.4) follows. □

Now we consider the concavity of another trace function related to exponential and logarithmic.

THEOREM 2.8. *Let* $\operatorname{Tr} H^*H = 1$. *Consider the trace function*

$$G(A) = \exp_q \operatorname{Tr} H^* \log_q AH.$$

We have

(i) If $q < 1$, *then* G *is concave.*

(ii) If $1 < q \leqslant 2$, *then* G *is concave.*

(iii) If $q \geqslant 2$, *then* G *is convex.*

Proof. Since

$$\begin{aligned}
\exp_q \operatorname{Tr} H^* \log_q AH &= \left[\left(\operatorname{Tr} H^* \frac{A^{q-1}-1}{q-1} H\right)(q-1)+1\right]^{\frac{1}{q-1}} \\
&= \left[\frac{\operatorname{Tr} H^*A^{q-1}H - \operatorname{Tr} H^*H}{q-1}(q-1)+1\right]^{\frac{1}{q-1}} \\
&= \left(\operatorname{Tr} H^*A^{q-1}H\right)^{\frac{1}{q-1}},
\end{aligned}$$

by Proposition 2.4, we obtain the conclusions. □

Letting $q \to 1$, we have

COROLLARY 2.9. *Let* $\operatorname{Tr} H^*H = 1$. *The trace function*

$$G(A) = \exp \operatorname{Tr} H^* \log AH$$

is concave on self-adjoint matrices.

2.3. Trace geometric mean

Now we study the joint convexity of some trace functions.

THEOREM 2.10. *Let f be a convex function. Then the trace function*

$$(A,B) \to \operatorname{Tr} B^{\frac{1}{2}} f(B^{\frac{-1}{2}} A B^{\frac{-1}{2}}) B^{\frac{1}{2}}$$

is jointly convex.

Proof. Let $\lambda \in (0,1)$, $A = \lambda A_1 + (1-\lambda)A_2$, $B = \lambda B_1 + (1-\lambda)B_2$. And set $(\lambda B_1)^{\frac{1}{2}} B^{-\frac{1}{2}} = K_1$, $[(1-\lambda)B_2]^{\frac{1}{2}} B^{-\frac{1}{2}} = K_2$. Then K_1, K_2 satisfy $K_1^* K_1 + K_2^* K_2 = I$. Since f is convex, it follows from Theorem 2.2,

$$\begin{aligned}
&\operatorname{Tr} B^{\frac{1}{2}} f(B^{\frac{-1}{2}} A B^{\frac{-1}{2}}) B^{\frac{1}{2}} \\
&= \operatorname{Tr} B^{\frac{1}{2}} f(K_1^* B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}} K_1 + K_2^* B_2^{-\frac{1}{2}} A_2 B_2^{-\frac{1}{2}} K_2) B^{\frac{1}{2}} \\
&\leqslant \operatorname{Tr} B^{\frac{1}{2}} K_1^* f(B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}}) K_1 B^{\frac{1}{2}} + \operatorname{Tr} B^{\frac{1}{2}} K_2^* f(B_2^{-\frac{1}{2}} A_2 B_2^{-\frac{1}{2}}) K_2 B^{\frac{1}{2}} \\
&= \lambda \operatorname{Tr} B_1^{\frac{1}{2}} f(B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}}) B_1^{\frac{1}{2}} + (1-\lambda) \operatorname{Tr} B_2^{\frac{1}{2}} f(B_2^{-\frac{1}{2}} A_2 B_2^{-\frac{1}{2}}) B_2^{\frac{1}{2}}. \quad \square
\end{aligned}$$

Now we consider the jointly convexity of the trace geometric mean

$$\hat{Q}_\alpha(A,B) = \operatorname{Tr} B^{\frac{1}{2}} (B^{\frac{-1}{2}} A B^{\frac{-1}{2}})^\alpha B^{\frac{1}{2}}. \tag{2.5}$$

From Theorem 2.10, we have the following corollary.

COROLLARY 2.11. *The trace geometric mean $\hat{Q}_\alpha(A,B)$ is jointly convex for $\alpha \geqslant 1$ and $\alpha < 0$, and is jointly concave for $\alpha \in (0,1)$.*

Note that the operator geometric mean $G(A,B) = B^{\frac{1}{2}} (B^{\frac{-1}{2}} A B^{\frac{-1}{2}})^\alpha B^{\frac{1}{2}}$ is jointly concave for $\alpha \in (0,1)$, and jointly convex for $1 \leqslant \alpha \leqslant 2$ and $-1 \leqslant \alpha < 0$. See [3].

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