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ASYMPTOTIC PROPERTIES OF STOCHASTIC PREY–PREDATOR MODELS

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Abstract. In this paper, we study three stochastic two-species predator-prey models. We construct stochastic models from deterministic models by introducing three different stochastic perturbations to the growth equations of the prey and predator populations. For the first model, we obtain sufficient conditions for the stochastic model to be asymptotically stable in probability at three different equilibrium points. In addition, using a suitable stochastic Lyapunov method, we study the existence and uniqueness of the solution, the existence of positive recurrence and the ultimate boundedness of the three stochastic systems under certain conditions. we also discuss the global asymptotic stability of the equilibrium point and extinction of the last two stochastic systems. Finally, we provide some numerical simulations to illustrate our mathematical results. We show that stochastic models still retain the desirable stability property of their deterministic counterparts if stochastic perturbations are relatively small.

1. Introduction

Predator-prey systems have been an important topic in mathematical biology due to their universal existence and importance. As a result, interest in mathematical models of interacting populations among species has grown rapidly [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. One of the well-known models is the Lotka-Volterra model, which describes the time evolution of two interacting species: a prey population that grows with a constant birth rate in the absence of a predator species, while the predator population decays with a constant death rate without the presence of a prey species. And the population model has the form

$$\begin{cases} \mathrm{d}x(t) = x\big(a_1 - b_1 x - c_1 y\big)\mathrm{d}t, \\ \mathrm{d}y(t) = y\big(a_2 - b_2 y + c_2 x\big)\mathrm{d}t \end{cases} \tag{1}$$

where $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, $a_2 > 0$, $b_2 > 0$, $c_2 > 0$. The biological meaning of each parameter can be referred to [11]. And $a_2 > 0$ indicates that the predator y has other food source besides x. The functions $x(t)$ and $y(t)$ represent, respectively, the number of prey and predator. Obviously, the model (1) has four equilibrium points: $E^0 = (0,0)$, $E^1 = (\hat{x},0)$, $E^2 = (0,\hat{y})$, $E^* = (x^*,y^*)$, where $\hat{x} = \frac{a_1}{b_1}$, $\hat{y} = \frac{a_2}{b_2}$, $x^* = \frac{a_1b_2 - a_2c_1}{b_1b_2 + c_1c_2}$, $y^* =$

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$\frac{a_2b_1+a_1c_2}{b_1b_2+c_1c_2}$. And the parameters are hypothesized to be deterministic, which ignore the environmental stochasticity.

However, population dynamics in the real world are often affected by some uncertain factors, namely environmental noise [12,13]. Therefore, a large number of scholars have introduced stochasticity into deterministic systems to analyze the role of stochasticity in population dynamics [14, 15, 16, 17, 18, 19, 20]. In this paper, we consider the following stochastic systems:

$$\mathrm{d}X(t) = F\big(X(t)\big)dt + G_i(t)dB(t), \quad i = 1,2,3,$$

where $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $B(t) = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$, $F(X(t)) = \begin{pmatrix} x\big(a_1 - b_1x - c_1y\big) \\ y\big(a_2 - b_2y + c_2x\big) \end{pmatrix}$.

A stochastic perturbation can be proportional to the distance from its equilibrium value [21,22]. Therefore, we assume that stochastic perturbation is directly proportional to distance $(x(t), y(t))$ from value of (x^*, y^*), we get

$$G_1(X(t)) = \begin{pmatrix} \sigma_1\big(x(t) - x^*\big) & 0 \\ 0 & \sigma_2\big(y(t) - y^*\big) \end{pmatrix}.$$

At the same time, motivated by the references [23, 24], we also assume that the stochastic perturbation is proportional to the distance between $(x(t), y(t))$ and (x^*, y^*) and $(0,0)$. We have

$$G_2\big(X(t)\big) = \begin{pmatrix} \sigma_1 x(t)\big(x(t) - x^*\big) & 0 \\ 0 & \sigma_2 y(t)\big(y(t) - y^*\big) \end{pmatrix}$$

and

$$G_3\big(X(t)\big) = \begin{pmatrix} \sigma_1 x(t)\big(y(t) - y^*\big) & 0 \\ 0 & \sigma_2 y(t)\big(x(t) - x^*\big) \end{pmatrix}.$$

The main aims of this paper are to investigate how different types of stochasticities have different effects when applied to the same deterministic model. The arrangements of this paper are organized as follows: in Section 2, we establish the existence of the unique positive global solution for the model with $G_1\big(X(t)\big)$. In addition, we analyse the stability of different equilibrium points and sufficient conditions for positive recurrence. Finally, we show that the system with $G_1\big(X(t)\big)$ is stochastically ultimately bounded. Different results of the stochastic model with $G_2\big(X(t)\big)$, like the existence and uniqueness of the positive solutions and their boundedness, a set of sufficient conditions for asymptotic stability of equilibrium point, extinction, existence of positive recurrence, are presented in Section 3. In Section 4, we present the existence and uniqueness of the solution and prove the global stability of the equilibrium point of the system with $G_3\big(X(t)\big)$. Then a Lyapunov function is performed to obtain the sufficient conditions for extinction and the positive recurrence. Our analysis result reveals that the system with $G_3\big(X(t)\big)$ is stochastically ultimately bounded. The last section, a number of numerical simulation results of the systems are also given to illustrate the analytical results obtained in Section 2, Section 3 and Section 4.

In conclusion, when the noise intensity is relatively weak, the positive equilibrium point of global asymptotic stability is still globally asymptotically stable. In other words, small stochastic perturbations do not change the stability of the positive equilibrium.

2. Analysis of model with $G_1\big(X(t)\big)$

THEOREM 1. *Suppose that* $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ *and* $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$. *Then for the model with* $G_1\big(X(t)\big)$, *there exists a unique positive solution* $\big(x(t),y(t)\big)^T \in \mathrm{R}^2_+ = \{(x,y)\in \mathrm{R}^2 \mid x>0, y>0\}$ *for any initial value* $\big(x(0),y(0)\big)\in R^2_+$.

Proof. Since the coefficients of the model with $G_1\big(X(t)\big)$ are locally Lipschitz continuous, for any given initial value $\big(x(0),y(0)\big)\in R^2_+$, there is a unique maximal local solution $\big(x(t),y(t)\big)$ on $t\in[0,\tau_e)$, where τ_e is the explosion time. To show the solution is global, we need to show $\tau_e=\infty$. We choose a sufficiently large non-negative number r_0 such that both of $x(0)$ and $y(0)$ lie within the interval $[\frac{1}{r_0}, r_0]$. For each integer $r\geqslant r_0$, we define the stopping time

$$\tau_r = \inf\left\{t\in[0,\tau_e) \mid x\notin\left(\frac{1}{r},r\right) \text{ or } y\notin\left(\frac{1}{r},r\right)\right\}$$

where $\inf\varnothing=\infty$. Clearly, τ_r is increasing as $r\to\infty$. Set $\lim_{r\to+\infty}\tau_r=\tau_\infty$, whence $\tau_\infty\leqslant\tau_e$. If we can show that $\tau_\infty=\infty$, then $\tau_e=\infty$. To complete the proof, we need to show that $\tau_\infty=\infty$. Motivated by [21, Theorem 2.1], let's define a C^2-function $V: R^2_+\to R_+$ by $V(x,y)=(x-x^*-x^*\ln\frac{x}{x^*})\frac{1}{c_1}+(y-y^*-y^*\ln\frac{y}{y^*})\frac{1}{c_2}$. Let $r\geqslant r_0$ and $T>0$ be arbitrary. For $0\leqslant t\leqslant \tau_r\wedge T$, it follows from Itô formula that

$$\begin{aligned}
\mathrm{d}V &= \frac{1}{2}\big(x-x^*\ y-y^*\big)\left[\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\begin{pmatrix}-b_1 & -c_1\\ c_2 & -b_2\end{pmatrix}+\begin{pmatrix}-b_1 & c_2\\ -c_1 & -b_2\end{pmatrix}\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\right.\\
&\quad\left.+\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\begin{pmatrix}x^* & 0\\ 0 & y^*\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\right]\begin{pmatrix}x-x^*\\ y-y^*\end{pmatrix}\mathrm{d}t\\
&\quad+\big(x-x^*\ y-y^*\big)\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}x-x^*\\ y-y^*\end{pmatrix}\begin{pmatrix}\mathrm{d}B_1(t)\\ \mathrm{d}B_2(t)\end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{L}V &= \frac{1}{2}\big(x-x^*\ y-y^*\big)\left[\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\begin{pmatrix}-b_1 & -c_1\\ c_2 & -b_2\end{pmatrix}+\begin{pmatrix}-b_1 & c_2\\ -c_1 & -b_2\end{pmatrix}\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\right.\\
&\quad\left.+\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\begin{pmatrix}x^* & 0\\ 0 & y^*\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\right]\begin{pmatrix}x-x^*\\ y-y^*\end{pmatrix}\\
&= \frac{1}{2}\big(x-x^*\ y-y^*\big)\begin{pmatrix}\frac{1}{c_1}(\sigma_1^2x^*-2b_1) & 0\\ 0 & \frac{1}{c_2}(\sigma_2^2y^*-2b_2)\end{pmatrix}\begin{pmatrix}x-x^*\\ y-y^*\end{pmatrix}\leqslant 0.
\end{aligned}$$

Therefore, we can get that

$$\mathrm{d}V \leqslant \left(x-x^* \ y-y^*\right)\begin{pmatrix}\frac{1}{c_1} & 0\\ 0 & \frac{1}{c_2}\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}x-x^*\\ y-y^*\end{pmatrix}\begin{pmatrix}\mathrm{d}B_1(t)\\ \mathrm{d}B_2(t)\end{pmatrix}.$$

We can now integrate from 0 to $\tau_r \wedge T$ and then take the expressions to get

$$EV\big(x(\tau_r\wedge T), y(\tau_r\wedge T)\big) \leqslant V\big(x(0),y(0)\big) \triangleq K.$$

Note that for every $\omega \in \tau_r \leqslant T$, there is at least one of the $x(\tau_r,\omega)$, $y(\tau_r,\omega)$ which is equal either r or $\frac{1}{r}$ and hence $V\big(x(\tau_r),y(\tau_r)\big)$ is no less than the smallest of $\Big(r-x^*-x^*\ln\frac{r}{x^*}\Big)\frac{1}{c_1}$, $\Big(r-y^*-y^*\ln\frac{r}{y^*}\Big)\frac{1}{c_2}$, $\Big(\frac{1}{r}-x^*-x^*\ln\frac{1}{rx^*}\Big)\frac{1}{c_1}$ and $\Big(\frac{1}{r}-y^*-y^*\ln\frac{1}{ry^*}\Big)\frac{1}{c_2}$.

Consequently,

$$\begin{aligned} K &\geqslant E\big[1_{\{\tau_r\leqslant T\}}(\omega)V\big(x(\tau_r,\omega),y(\tau_r,\omega)\big)\big]\\ &\geqslant P\{\tau_r\leqslant T\}\left[\left(r-x^*-x^*\ln\frac{r}{x^*}\right)\frac{1}{c_1}\wedge\left(r-y^*-y^*\ln\frac{r}{y^*}\right)\frac{1}{c_2}\right.\\ &\qquad\left.\wedge\left(\frac{1}{r}-x^*-x^*\ln\frac{1}{rx^*}\right)\frac{1}{c_1}\wedge\left(\frac{1}{r}-y^*-y^*\ln\frac{1}{ry^*}\right)\frac{1}{c_2}\right]\end{aligned}$$

where 1_{τ_r} is the indicator function of τ_r. Letting $r\to\infty$ gives $\lim_{r\to+\infty}P\{\tau_r\leqslant T\}=0$.

Hence

$$P\{\tau_\infty\leqslant T\}=0.$$

Since $T>0$ is arbitrary, we must have $P\{\tau_\infty<\infty\}=0$. So $P\{\tau_\infty=\infty\}=1$ as required. □

The stochastic system can be centered at its positive equilibrium point by the following change in variables:

$$u_1 = x-x^*,\ \ u_2 = y-y^*.$$

The linearized counterpart of the nonlinear SDE system with $G_1\big(X(t)\big)$ about (x^*,y^*) reads:

$$\begin{cases}\mathrm{d}u_1 = \big(\hat{A}u_1+\hat{B}u_2\big)\mathrm{d}t+\sigma_1u_1\mathrm{d}B_1(t),\\ \mathrm{d}u_2 = \big(\hat{C}u_1+\hat{D}u_2\big)\mathrm{d}t+\sigma_1u_2\mathrm{d}B_2(t)\end{cases} \tag{2}$$

where $\hat{A}=a_1-2b_1x^*-c_1y^*$, $\hat{B}=-c_1x^*$, $\hat{C}=c_2y^*$, $\hat{D}=a_2-2b_2y^*+c_2x^*$. Note that the stability of the zero solution of (2) is equivalent to the stability property of the equilibrium solution (x^*,y^*) of system with $G_1\big(X(t)\big)$. One can state the following theorem for the stability of different equilibrium points of the model with $G_1\big(X(t)\big)$.

THEOREM 2. *(I) Equilibrium* $E^1=(\hat{x},0)$ *is asymptotically mean square stable if (i)* $a_2+\frac{a_1c_2}{b_1}-\frac{3\sigma_2^2}{4}>0$, *(ii)* $\sigma_1\leqslant\sqrt{\frac{2b_1}{x^*}}$ *and* $\sigma_2\leqslant\sqrt{\frac{2b_2}{y^*}}$.

(II) Equilibrium $E^2 = (0, \hat{y})$ is asymptotically mean square stable if
(i) $\sigma_1 < \sqrt{-2\hat{A}}$, $\hat{A} < 0$, (ii) $\sigma_2 < \sqrt{-2\hat{D}}$, $\hat{D} < 0$,
(iii) $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ and $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$.
(III) Equilibrium $E^* = (x^*, y^*)$ is asymptotically mean square stable if
(i) $\sigma_1 < \sqrt{-2\hat{A}}$, $\hat{A} < 0$, (ii) $\sigma_2 < \sqrt{-2\hat{D}}$, $\hat{D} < 0$,
(iii) $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ and $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$.

Proof. For the axial equilibrium $E^1 = (\hat{x}, 0)$, we have $\hat{A} = -b_1\hat{x}$, $\hat{B} = -c_1\hat{x}$, $\hat{C} = 0$, $\hat{D} = a_2 + c_2\hat{x}$. Define the function $V(u_1, u_2) = u_1^{\frac{1}{2}} + u_2^{\frac{-1}{2}}$, then

$$\begin{aligned}\mathscr{L}V &= \frac{1}{2}u_1^{\frac{-1}{2}}(-b_1\hat{x}u_1 - c_1\hat{x}u_2) - \frac{1}{8}\sigma_1^2 u_1^{\frac{1}{2}} - \frac{1}{2}u_2^{\frac{-3}{2}}(a_2 + c_2\hat{x})u_2 + \frac{3}{8}\sigma_2^2 u_2^{\frac{-1}{2}} \\ &= -\left(\frac{1}{2}b_1\hat{x} + \frac{1}{8}\sigma_1^2\right)u_1^{\frac{1}{2}} - \frac{1}{2}c_1 u_1^{\frac{-1}{2}}u_2 - \frac{1}{2}\left(a_2 + c_2\hat{x} - \frac{3}{4}\sigma_2^2\right)u_2^{\frac{-1}{2}}.\end{aligned}$$

Therefore $\mathscr{L}V$ is negative definite. The equilibrium solution $(0,0)$ of the model (2) is globally asymptotically stable. Hence the equilibrium solution $E^1 = (\hat{x}, 0)$ of the model with $G_1(X(t))$ is globally asymptotically stable.

We define a Lyapunov function $V = \frac{1}{2}(\omega_1 u_1^2 + \omega_2 u_2^2)$, where ω_1 and ω_2 are positive real constants to determined. One can express $V = \frac{1}{2}\begin{pmatrix}u_1 & u_2\end{pmatrix} Q \begin{pmatrix}u_1 \\ u_2\end{pmatrix}$, where $Q = \begin{pmatrix}\omega_1 & 0 \\ 0 & \omega_2\end{pmatrix}$.

Let $\lambda_1(Q)$ and $\lambda_2(Q)$ be the smallest and largest eigenvalues of Q, respectively, and then we have

$$\frac{1}{2}\lambda_1(Q) \leqslant V \leqslant \frac{1}{2}\lambda_2(Q).$$

Define $K_1 = \frac{1}{2}\lambda_1(Q)$ and $K_2 = \frac{1}{2}\lambda_2(Q)$, so that the inequality $K_1|u|^2 \leqslant V \leqslant K_2|u|^2$ holds, where $|u|^2 = u_1^2 + u_2^2$. Meanwhile

$$\begin{aligned}\mathscr{L}V &= \begin{pmatrix}\hat{A}u_1 + \hat{B}u_2 & \hat{C}u_1 + \hat{D}u_2\end{pmatrix}\begin{pmatrix}\omega_1 u_1 \\ \omega_2 u_2\end{pmatrix} + \frac{1}{2}Tr\begin{pmatrix}\sigma_1^2 u_1^2 \omega_1 & 0 \\ 0 & \sigma_2^2 u_2^2 \omega_2\end{pmatrix} \\ &= \left(\hat{A} + \frac{\sigma_1^2}{2}\right)\omega_1 u_1^2 + \left(\hat{D} + \frac{\sigma_2^2}{2}\right)\omega_2 u_2^2 + (\hat{B}\omega_1 + \hat{D}\omega_2)u_1 u_2 \\ &= -u^T M u\end{aligned}$$

where the symmetric matrix M is defined as

$$M = \begin{pmatrix} -\left(\hat{A} + \frac{\sigma_1^2}{2}\right)\omega_1 & -\frac{1}{2}(\hat{B}\omega_1 + \hat{C}\omega_2) \\ -\frac{1}{2}(\hat{B}\omega_1 + \hat{C}\omega_2) & -\left(\hat{D} + \frac{\sigma_2^2}{2}\right)\omega_2 \end{pmatrix}.$$

Now we choose ω_1 and ω_2 in such a manner that M becomes positive definite. Meanwhile, we will find some conditions so that all the principal minors of M become positive.

For the axial equilibrium $E^2 = (0,\hat{y})$, we obtain $\hat{A} = a_1 - c_1\hat{y}$, $\hat{B} = 0$, $\hat{C} = c_2\hat{y} > 0$, $\hat{D} = a_2 - 2b_2\hat{y}$. Principal minors of M are

$$M_{11} = -\left(\hat{A} + \frac{\sigma_1^2}{2}\right)\omega_1 \text{ and } M_{22} = \left(\hat{A} + \frac{\sigma_1^2}{2}\right)\left(\hat{D} + \frac{\sigma_2^2}{2}\right)\omega_1\omega_2 - \frac{\hat{C}^2\omega_2^2}{4}.$$

Positivity of the first principal minor M_{11} forces $\hat{A} + \frac{\sigma_1^2}{2} < 0$ and $\hat{A} < 0$. Since $\hat{A} + \frac{\sigma_1^2}{2}$ is negative and ω_1, ω_2 are chosen to be positive, we select $\omega_1 = \frac{1}{(\hat{A}+\frac{\sigma_1^2}{2})(\hat{D}+\frac{\sigma_2^2}{2})}$ and $\omega_2 = \frac{1}{\hat{C}^2}$. Note that ω_2 is positive, and ω_1 will be positive under the condition $\sigma_2 < \sqrt{-2\hat{D}}$ with $\hat{D} < 0$. Hence M becomes positive definite if $\sigma_1 < \sqrt{-2\hat{A}}$ and $\sigma_2 < \sqrt{-2\hat{D}}$ with $\hat{A} < 0$ and $\hat{D} < 0$.

For the interior equilibrium point $E^* = (x^*,y^*)$, we have $\hat{A} = -b_1x^* < 0$, $\hat{B} = -c_1x^* < 0$, $\hat{C} = c_2x^* > 0$, $\hat{D} = -b_2y^* < 0$. Since $\hat{A} < 0$ and $\hat{D} < 0$, it is possible to choose suitable values of ω_1 and ω_2 so that M becomes positive definite. We choose $\omega_1 = -\frac{1}{\hat{B}}$ and $\omega_2 = \frac{1}{\hat{C}}$. Then the corresponding principle minors of M become

$$M_{11} = -\left(\hat{A} + \frac{\sigma_1^2}{2}\right)\omega_1 \text{ and } M_{22} = \left(\hat{A} + \frac{\sigma_1^2}{2}\right)\left(\hat{D} + \frac{\sigma_2^2}{2}\right)\omega_1\omega_2.$$

One can easily observe that the first principle minor is positive if $\sigma_1 < \sqrt{-2\hat{A}}$, and the second principle minor will be positive when $\sigma_2 < \sqrt{-2\hat{D}}$. Hence the positivity of both principle minors imply the positive definiteness of M subject to the conditions $\sigma_1 < \sqrt{-2\hat{A}}$ and $\sigma_2 < \sqrt{-2\hat{D}}$ with $\hat{A} < 0$ and $\hat{D} < 0$. Therefore, all eigenvalues of M become positive. Thus, we have

$$\mathscr{L}V < -\lambda(M)|u|^2$$

where $\lambda(M)$ is the smallest eigenvalues of M and $\lambda(M) > 0$. Therefore the equilibrium solution $(0,0)$ of the (2) is globally asymptotically stable. Hence the equilibrium solutions $E^2 = (0,\hat{y})$ and $E^* = (x^*,y^*)$ of the model with $G_1\big(X(t)\big)$ are globally asymptotically stable.

For the equilibrium point $E^0 = (0,0)$, the stochastic system reads:

$$\begin{cases} \mathrm{d}x(t) = x\big(a_1 - b_1x - c_1y\big)\mathrm{d}t + \sigma_1 x\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y\big(a_2 - b_2y + c_2x\big)\mathrm{d}t + \sigma_2 y\mathrm{d}B_2(t). \end{cases}$$

And many scholars have studied it well (e.g. [5, 11, 25, 26, 27, 28, 29, 30]). □

THEOREM 3. *Suppose that* $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ *and* $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$ *hold. Then the solution* $\big(x(t),y(t)\big)$ *of the system with* $G_1\big(X(t)\big)$ *is positively recurrent.*

Proof. Define $V(x,y) = (x - x^* - x^* \ln \frac{x}{x^*})\frac{1}{c_1} + (y - y^* - y^* \ln \frac{y}{y^*})\frac{1}{c_2}$. $D = \{(x,y) \in R_+^2 \mid \frac{1}{N} \leqslant x \leqslant N, \frac{1}{N} \leqslant y \leqslant N\}$. Applying Itô formula, we have

$$\begin{aligned}
\mathscr{L}V &= \frac{1}{2}\begin{pmatrix} x - x^* & y - y^* \end{pmatrix} \left[\begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \end{pmatrix} \begin{pmatrix} -b_1 & -c_1 \\ c_2 & -b_2 \end{pmatrix} + \begin{pmatrix} -b_1 & c_2 \\ -c_1 & -b_2 \end{pmatrix} \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \end{pmatrix} \begin{pmatrix} x^* & 0 \\ 0 & y^* \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right] \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} x - x^* & y - y^* \end{pmatrix} \begin{pmatrix} \frac{1}{c_1}(\sigma_1^2 x^* - 2b_1) & 0 \\ 0 & \frac{1}{c_2}(\sigma_2^2 y^* - 2b_2) \end{pmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&= -\frac{1}{2c_1}(2b_1 - \sigma_1^2 x^*)(x - x^*)^2 - \frac{1}{2c_2}(2b_2 - \sigma_2^2 y^*)(y - y^*)^2.
\end{aligned}$$

When $(x,y) \to (0,0)$, we have

$$\mathscr{L}V = -\frac{1}{2c_1}(2b_1 - \sigma_1^2 x^*)(x^*)^2 - \frac{1}{2c_2}(2b_2 - \sigma_2^2 y^*)(y^*)^2 < 0.$$

When $x \to 0$ and $y \to +\infty$, we get

$$\mathscr{L}V = -\frac{1}{2c_1}(2b_1 - \sigma_1^2 x^*)(x^*)^2 - \frac{1}{2c_2}(2b_2 - \sigma_2^2 y^*)(y - y^*)^2 \to -\infty.$$

When $x \to +\infty$ and $y \to 0$, we obtain

$$\mathscr{L}V = -\frac{1}{2c_1}(2b_1 - \sigma_1^2 x^*)(x - x^*)^2 - \frac{1}{2c_2}(2b_2 - \sigma_2^2 y^*)(y^*)^2 \to -\infty.$$

When $x \to +\infty$ and $y \to +\infty$, we deduce

$$\mathscr{L}V = -\frac{1}{2c_1}(2b_1 - \sigma_1^2 x^*)(x - x^*)^2 - \frac{1}{2c_2}(2b_2 - \sigma_2^2 y^*)(y - y^*)^2 \to -\infty.$$

Therefore, for $(x,y) \in D^c$, $\mathscr{L}V < 0$. Using the similar proof of [Theorem 3.26] of [31], it is a sufficient and necessary condition for positive recurrence. The proof is complete. □

THEOREM 4. *Suppose that* $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ *and* $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$ *hold. The system with* $G_1(X(t))$ *is stochastically ultimately bounded for any initial value* $(x(0), y(0)) \in R_+^2$.

Proof. We first claim that there is a positive constant $K = K(\theta)$, which is independent of the initial value $(x(0), y(0))$, such that the solution $X = (x,y)$ of the system with $G_1(X(t))$ has the property that

$$\lim_{t \to +\infty} \sup E\,|x|^\theta \leqslant K.$$

Define $V = x^\theta + \frac{c}{c_2}y^\theta$, where $c < \min\{b_1, c_1\}$. It follows from Itô formula that

$$\begin{aligned} \mathrm{d}V = \Big[&\theta x^\theta(a_1 - b_1x - c_1y) - \frac{\theta(1-\theta)}{2}\sigma_1^2(x-x^*)^2x^{\theta-2} \\ &+ \theta y^\theta(a_2 - b_2y + c_2x)\frac{c}{c_2} - \frac{\theta(1-\theta)}{2}\sigma_2^2(y-y^*)^2y^{\theta-2}\frac{c}{c_2}\Big]\mathrm{d}t \\ &+ \theta\sigma_1(x-x^*)x^{\theta-1}\mathrm{d}B_1(t) + \frac{c}{c_2}\theta\sigma_2(y-y^*)y^{\theta-1}\mathrm{d}B_2(t). \end{aligned} \tag{3}$$

Denote

$$\begin{aligned} \mathscr{L}V &= \theta x^\theta(a_1 - b_1x - c_1y) - \frac{\theta(1-\theta)}{2}\sigma_1^2(x-x^*)^2x^{\theta-2} \\ &\quad + \theta y^\theta(a_2 - b_2y + c_2x)\frac{c}{c_2} - \frac{\theta(1-\theta)}{2}\sigma_2^2(y-y^*)^2y^{\theta-2}\frac{c}{c_2} \\ &= \theta a_1x^\theta - \theta b_1x^{\theta+1} - \theta c_1x^\theta y - \frac{\theta(1-\theta)}{2}\sigma_1^2(x-x^*)^2x^{\theta-2} \\ &\quad + \theta a_2y^\theta\frac{c}{c_2} - \theta b_2y^{\theta+1}\frac{c}{c_2} + \theta cy^\theta x - \frac{\theta(1-\theta)}{2}\sigma_2^2(y-y^*)^2y^{\theta-2}\frac{c}{c_2} \\ &\triangleq F - V \end{aligned}$$

where

$$\begin{aligned} F = {}&\theta a_1x^\theta - \theta b_1x^{\theta+1} - \theta c_1x^\theta y - \frac{\theta(1-\theta)}{2}\sigma_1^2(x-x^*)^2x^{\theta-2} + \theta a_2y^\theta\frac{c}{c_2} \\ &- \theta b_2y^{\theta+1}\frac{c}{c_2} + \theta cy^\theta x - \frac{\theta(1-\theta)}{2}\sigma_2^2(y-y^*)^2y^{\theta-2}\frac{c}{c_2} + x^\theta + \frac{c}{c_2}y^\theta. \end{aligned}$$

When $x \leqslant y$, we can get

$$\begin{aligned} F \leqslant {}&-\theta c_1x^\theta y^\theta(y^{1-\theta} - x^{1-\theta}) - \theta b_1x^{1+\theta} - \theta b_2y^{1+\theta}\frac{c}{c_2} + \theta a_1x^\theta + \theta a_2y^\theta\frac{c}{c_2} \\ &+ x^\theta + \frac{c}{c_2}y^\theta - \frac{\theta(1-\theta)}{2}\sigma_1^2(x-x^*)^2x^{\theta-2} - \frac{\theta(1-\theta)}{2}\sigma_2^2(y-y^*)^2y^{\theta-2}\frac{c}{c_2}. \end{aligned}$$

Since the coefficient of the highest of x is $-\theta b_1 < 0$ and the coefficient of the highest of y is $-\theta b_2\frac{c}{c_2} < 0$, then there exists a constant K_1 such $F \leqslant K_1$ in R_+^2. When $x > y$, we can get

$$\begin{aligned} F \leqslant {}&-\theta(b_1 - c)x^{1+\theta} - \theta b_2y^{1+\theta}\frac{c}{c_2} - \theta c_1x^\theta y + \theta a_1x^\theta + \theta a_2y^\theta\frac{c}{c_2} \\ &- \frac{\theta(1-\theta)}{2}\sigma_1^2(x-x^*)^2x^{\theta-2} - \frac{\theta(1-\theta)}{2}\sigma_2^2(y-y^*)^2y^{\theta-2}\frac{c}{c_2} + x^\theta + \frac{c}{c_2}y^\theta. \end{aligned}$$

Since the coefficient of the highest of x is $-\theta(b_1 - c) < 0$ and the coefficient of the highest of y is $-\theta b_2\frac{c}{c_2} < 0$, then there exists a constant K_2 such $F \leqslant K_2$ in R_+^2.

Therefore $F \leqslant K'$, $\forall (x,y) \in R_+^2$, where $F' = \min\{K_1, K_2\}$. Hence we have $\mathscr{L}V \leqslant K' - V$. Substituting this into (3) yields

$$dV \leqslant (K' - V)dt + \theta\sigma_1(x - x^*)x^{\theta-1}dB_1(t) + \frac{c}{c_2}\theta\sigma_2(y - y^*)y^{\theta-1}dB_2(t). \tag{4}$$

From (4) and once again by Itô formula, we obtain

$$\begin{aligned} d[e^t V] = e^t(V dt + dV) \leqslant K' e^t dt + e^t \theta\sigma_1(x - x^*)x^{\theta-1}dB_1(t) \\ + e^t \frac{c}{c_2}\theta\sigma_2(y - y^*)y^{\theta-1}dB_2(t). \end{aligned}$$

Taking expectation of both side of the above inequality, we get $e^t EV \leqslant V\big(x(0), y(0)\big) + K' e^t$. This implies that

$$\lim_{t\to+\infty} \sup EV \leqslant K'.$$

On the other hand, we deduce $|X|^2 \leqslant 2\max\{x,y\}$. Thus $|X|^\theta \leqslant 2^{\frac{\theta}{2}} \max\{x^\theta, y^\theta\} \leqslant 2^{\frac{\theta}{2}} V$. We have

$$\lim_{t\to+\infty} \sup E|X|^\theta \leqslant 2^{\frac{\theta}{2}} K' \triangleq K.$$

Then, for any $\varepsilon > 0$, let $H = \frac{K^2}{\varepsilon^2}$. By Chebyshev's inequality, we attain $P\{|X| > H\} \leqslant \frac{E(\sqrt{|X|})}{\sqrt{H}}$. Hence

$$\lim_{t\to+\infty} \sup P\{|X| > H\} \leqslant \frac{K}{\sqrt{H}} = \varepsilon.$$

This means

$$\lim_{t\to+\infty} \sup P\{|X| \leqslant H\} \geqslant 1 - \varepsilon. \quad \square$$

3. Analysis of model with $G_2\big(X(t)\big)$

THEOREM 5. *For any initial value $\big(x(0), y(0)\big) \in R_+^2$, there exists a unique positive solution $\big(x(t), y(t)\big)^T \in \mathrm{R}_+^2 = \{(x,y) \in \mathrm{R}^2 \mid x > 0, y > 0\}$ for the model with $G_2\big(X(t)\big)$ on $t \geqslant 0$ and the solution will remain in R_+^2 with probability 1 i.e., $\big(x(t), y(t)\big)^T \in \mathrm{R}_+^2$ for all $t \geqslant 0$.*

Proof. Define a C^2-function $V : R_+^2 \to R_+$ by $V(x,y) = x^{\frac{1}{2}} - \frac{1}{2} - \frac{1}{2}\ln x + y^{\frac{1}{2}} - \frac{1}{2} - \frac{1}{2}\ln y$. By the famous Itô formula, we compute

$$\begin{aligned} \mathscr{L}V = \frac{1}{2}x^{-1}(x^{\frac{1}{2}} - 1)x(a_1 - b_1 x - c_1 y) + \frac{1}{2}x^{-2}\left(-\frac{1}{2}x^{\frac{1}{2}} + 1\right)\frac{\sigma_1^2 x^2 (x - x^*)^2}{2} \\ + \frac{1}{2}y^{-1}(y^{\frac{1}{2}} - 1)y(a_2 - b_2 y + c_2 x) + \frac{1}{2}y^{-2}\left(-\frac{1}{2}y^{\frac{1}{2}} + 1\right)\frac{\sigma_2^2 y^2 (y - y^*)^2}{2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(x^{\frac{1}{2}}-1)(a_1-b_1x-c_1y)+\frac{1}{4}\left(-\frac{1}{2}x^{\frac{1}{2}}+1\right)\sigma_1^2(x-x^*)^2\\
&\quad+\frac{1}{2}(y^{\frac{1}{2}}-1)(a_2-b_2y+c_2x)+\frac{1}{4}\left(-\frac{1}{2}y^{\frac{1}{2}}+1\right)\sigma_2^2(y-y^*)^2\\
&= -\frac{\sigma_1^2}{8}x^{\frac{5}{2}}+\frac{\sigma_1^2}{4}x^2+\left(\frac{\sigma_1^2x^*}{4}-\frac{b_1}{2}\right)x^{\frac{3}{2}}+\left(\frac{b_1}{2}-\frac{\sigma_1^2x^*}{2}-\frac{c_2}{2}\right)x\\
&\quad+\left(\frac{a_1}{2}-\frac{\sigma_1^2(x^*)^2}{8}\right)x^{\frac{1}{2}}-\frac{\sigma_2^2}{8}y^{\frac{5}{2}}+\frac{\sigma_2^2}{4}y^2+\left(\frac{\sigma_2^2y^*}{4}-\frac{b_2}{2}\right)y^{\frac{3}{2}}\\
&\quad+\left(\frac{b_2}{2}-\frac{\sigma_2^2y^*}{2}+\frac{c_2}{2}\right)y+\left(\frac{a_2}{2}-\frac{\sigma_2^2(y^*)^2}{8}\right)y^{\frac{1}{2}}\\
&\quad+\frac{c_2}{2}xy^{\frac{1}{2}}-\frac{c_1}{2}x^{\frac{1}{2}}y+\frac{\sigma_1^2(x^*)^2}{4}+\frac{\sigma_2^2(y^*)^2}{4}-\frac{a_1}{2}-\frac{a_2}{2}.
\end{aligned}
$$

The coefficient of the highest term of x is $-\frac{\sigma_1^2}{8}<0$ and the coefficient of the highest term of y is $-\frac{\sigma_2^2}{8}<0$, therefore there exists a constant K such that $\mathscr{L}V(x,y)\leqslant K$ for all $\big(x(t),y(t)\big)\in R_+^2$. By virtue of the similar proof of [Theorem 2.1] of [32], we complete the proof. □

THEOREM 6. *If $\sigma_1^2<\frac{2b_1}{x^*}$ and $\sigma_2^2<\frac{2b_2}{y^*}$, then the equilibrium solution (x^*,y^*) of the model with $G_2\big(X(t)\big)$ is globally asymptotically stable.*

Proof. Define a C^2-function $V:R_+^2\to R_+$ by $V(x,y)=x-x^*-x^*\ln\frac{x}{x^*}+(y-y^*-y^*\ln\frac{y}{y^*})\frac{c_1}{c_2}$. In view of Itô formula, we get

$$
\begin{aligned}
\mathscr{L}V &= (x-x^*)(a_1-b_1x-c_1y)+\frac{\sigma_1^2x^*}{2}(x-x^*)^2\\
&\quad+(y-y^*)(a_2-b_2y+c_2x)\frac{c_1}{c_2}+\frac{\sigma_2^2y^*}{2}(y-y^*)^2\frac{c_1}{c_2}\\
&=(x-x^*)\big[-b_1(x-x^*)-c_1(y-y^*)\big]+\frac{\sigma_1^2x^*}{2}(x-x^*)^2\\
&\quad+(y-y^*)\big[c_2(x-x^*)-b_2(y-y^*)\big]\frac{c_1}{c_2}+\frac{\sigma_2^2y^*}{2}(y-y^*)^2\frac{c_1}{c_2}\\
&=-\left(b_1-\frac{\sigma_1^2x^*}{2}\right)(x-x^*)^2-\frac{c_1}{c_2}\left(b_2-\frac{\sigma_2^2y^*}{2}\right)(y-y^*)^2.
\end{aligned}
$$

Therefore $\mathscr{L}V$ is negative definite. The equilibrium solution (x^*,y^*) of the model $G_2\big(X(t)\big)$ is globally asymptotically stable. □

THEOREM 7. *Suppose that $b_1-\frac{x^*\sigma_1^2}{2}>0$ and $b_2-\frac{y^*\sigma_2^2}{2}>0$ hold. Then the solution $\big(x(t),y(t)\big)$ of the system with $G_2\big(X(t)\big)$ is positively recurrent.*

Proof. Define $V(x,y) = x - x^* - x^* \ln \frac{x}{x^*} + (y - y^* - y^* \ln \frac{y}{y^*})\frac{c_1}{c_2}$. $D = \{(x,y) \in R_+^2 \mid \frac{1}{N} \leqslant x \leqslant N, \frac{1}{N} \leqslant y \leqslant N\}$. Applying Itô formula, we get

$$\begin{aligned}\mathscr{L}V &= (x-x^*)(a_1 - b_1 x - c_1 y) + \frac{\sigma_1^2 x^*}{2}(x-x^*)^2 \\ &\quad + (y-y^*)(a_2 - b_2 y + c_2 x)\frac{c_1}{c_2} + \frac{\sigma_2^2 y^*}{2}(y-y^*)^2\frac{c_1}{c_2} \\ &= (x-x^*)\big[-b_1(x-x^*) - c_1(y-y^*)\big] + \frac{\sigma_1^2 x^*}{2}(x-x^*)^2 \\ &\quad + (y-y^*)\big[c_2(x-x^*) - b_2(y-y^*)\big]\frac{c_1}{c_2} + \frac{\sigma_2^2 y^*}{2}(y-y^*)^2\frac{c_1}{c_2} \\ &= -\left(b_1 - \frac{\sigma_1^2 x^*}{2}\right)(x-x^*)^2 - \frac{c_1}{c_2}\left(b_2 - \frac{\sigma_2^2 y^*}{2}\right)(y-y^*)^2.\end{aligned}$$

When $(x,y) \to (0,0)$, we obtain

$$\mathscr{L}V = -\left(b_1 - \frac{\sigma_1^2 x^*}{2}\right)(x^*)^2 - \frac{c_1}{c_2}\left(b_2 - \frac{\sigma_2^2 y^*}{2}\right)(y^*)^2 < 0.$$

When $x \to 0$ and $y \to +\infty$, we have

$$\mathscr{L}V = -\left(b_1 - \frac{\sigma_1^2 x^*}{2}\right)(x^*)^2 - \frac{c_1}{c_2}\left(b_2 - \frac{\sigma_2^2 y^*}{2}\right)(y-y^*)^2 \to -\infty.$$

When $x \to +\infty$ and $y \to 0$, we deduce

$$\mathscr{L}V = -\left(b_1 - \frac{\sigma_1^2 x^*}{2}\right)(x-x^*)^2 - \frac{c_1}{c_2}\left(b_2 - \frac{\sigma_2^2 y^*}{2}\right)(y^*)^2 \to -\infty.$$

When $x \to +\infty$ and $y \to +\infty$, we attain

$$\mathscr{L}V = -\left(b_1 - \frac{\sigma_1^2 x^*}{2}\right)(x-x^*)^2 - \frac{c_1}{c_2}\left(b_2 - \frac{\sigma_2^2 y^*}{2}\right)(y-y^*)^2 \to -\infty.$$

Therefore, for $(x,y) \in D^c$, $\mathscr{L}V < 0$. Using the similar proof of [Theorem 3.26] of [31], it is a sufficient and necessary condition for positive recurrence. The proof is complete. □

THEOREM 8. *If* $a_1 - \frac{\sigma_1^2 (x^*)^2}{2} < 0$ *and* $b_1 - \sigma_1^2 x^* > 0$. *Then the species* $x(t)$ *of the model with* $G_2(X(t))$ *will go to extinction almost surely. In addition, if* $a_2 - \frac{\sigma_2^2 (y^*)^2}{2} < 0$ *and species* $y(t)$ *goes extinct, given that species* $x(t)$ *goes extinct.*

Proof. Applying Itô formula, we have

$$\begin{aligned}\mathrm{d}\ln x &= \left(a_1 - b_1 x - c_1 y - \frac{\sigma_1^2 (x-x^*)^2}{2}\right)\mathrm{d}t + \sigma_1(x-x^*)\mathrm{d}B_1(t) \\ &\leqslant \left(a_1 - b_1 x + \sigma_1^2 x^* x - \frac{\sigma_1^2 (x^*)^2}{2}\right)\mathrm{d}t - \frac{\sigma_1^2 x^2}{2}\mathrm{d}t + \sigma_1 x \mathrm{d}B_1(t) - \sigma_1 x^* \mathrm{d}B_1(t).\end{aligned}$$

That is to say,

$$\ln x(t) \leqslant \ln x(0) + \int_0^t \left(a_1 - b_1 x + \sigma_1^2 x^* x - \frac{\sigma_1^2 (x^*)^2}{2}\right) \mathrm{d}s - \int_0^t \frac{\sigma_1^2 x^2}{2} \mathrm{d}s + M - M_1 \quad (5)$$

where $M = \int_0^t \sigma_1 x \mathrm{d}B_1(s)$ and $M_1 = \int_0^t \sigma_1 x^* \mathrm{d}B_1(s)$, whose quadratic variation is

$$\langle M, M \rangle = \int_0^t \sigma_1^2 x^2 \mathrm{d}s.$$

By virtue of the exponential martingale inequality, for any positive constant T, α, β, we obtain

$$P\left\{ \sup_{0 \leqslant t \leqslant T} \left[M - \frac{\alpha}{2} \langle M, M \rangle \right] > \beta \right\} \leqslant e^{-\alpha\beta}.$$

Choose $T = n$, $\alpha = 1$, $\beta = 2\ln n$, we get

$$P\left\{ \sup_{0 \leqslant t \leqslant T} \left[M - \frac{1}{2} \langle M, M \rangle \right] > 2\ln n \right\} \leqslant \frac{1}{n^2}.$$

An application of Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega)$ such that for $n \geqslant n_0$,

$$\sup_{0 \leqslant t \leqslant T} \left[M - \frac{1}{2} \langle M, M \rangle \right] \leqslant 2\ln n.$$

That is to say

$$M \leqslant 2\ln n + \frac{1}{2} \langle M, M \rangle = 2\ln n + \frac{1}{2} \int_0^t \frac{\sigma_1^2 x^2}{2} \mathrm{d}s$$

for all $0 \leqslant t \leqslant n$, $n \geqslant n_0$. Substituting the above inequality into (5) leads to

$$\ln x(t) - \ln x(0) \leqslant \left(a_1 - \frac{\sigma_1^2 (x^*)^2}{2}\right) t - (b_1 - \sigma_1^2 x^*) \int_0^t x \mathrm{d}s + 2\ln n - M_1.$$

In other words, we have already shown that for $0 < n-1 \leqslant t \leqslant n$,

$$\frac{\ln x(t) - \ln x(0)}{t} \leqslant \left(a_1 - \frac{\sigma_1^2 (x^*)^2}{2}\right) - (b_1 - \sigma_1^2 x^*) \frac{1}{t} \int_0^t x \mathrm{d}s + \frac{2\ln n}{n-1} - \frac{M_1}{t}. \quad (6)$$

From the strong law of large number, we get

$$\lim_{t \to +\infty} \frac{M_1}{t} = 0. \quad (7)$$

Substituting (7) into (6), we deduce

$$\lim_{t \to +\infty} \sup \frac{\ln x}{t} \leqslant a_1 - \frac{\sigma_1^2 (x^*)^2}{2} - (b_1 - \sigma_1^2 x^*) \frac{1}{t} \int_0^t x \mathrm{d}s < 0.$$

That is to say

$$\lim_{t\to+\infty} x(t) = 0.$$

Using Itô formula on the second equation of the model with $G_2\big(X(t)\big)$, we have

$$\begin{aligned}
\mathrm{d}\ln y &= \left(a_2 - b_2 y + c_2 x - \frac{\sigma_2^2 (y-y^*)^2}{2}\right)\mathrm{d}t + \sigma_2 (y-y^*)\mathrm{d}B_2(t) \\
&\leqslant \left(a_2 + c_2 x - \frac{\sigma_2^2 (y^*)^2}{2}\right)\mathrm{d}t - \frac{\sigma_2^2 y^2}{2}\mathrm{d}t + \sigma_2 y \mathrm{d}B_2(t) - \sigma_2 y^* \mathrm{d}B_2(t).
\end{aligned}$$

Using the similar method above, we can get

$$\frac{\ln y(t) - \ln y(0)}{t} \leqslant \left(a_2 - \frac{\sigma_2^2 (y^*)^2}{2}\right) + \frac{1}{t}\int_0^t c_2 x \mathrm{d}s - \frac{M_2}{t} \tag{8}$$

where $M_2 = \int_0^t \sigma_2 y^* \mathrm{d}B_2(s)$. From the strong law of large number, we obtain

$$\lim_{t\to+\infty} \frac{M_2}{t} = 0. \tag{9}$$

Substituting (9) into (8) and using the fact that $\lim_{t\to+\infty} x(t) = 0$, we attain

$$\lim_{t\to+\infty} \sup \frac{\ln y}{t} \leqslant a_2 - \frac{\sigma_2^2 (y^*)^2}{2} < 0.$$

That is to say, under the premise of species $x(t)$ extinction, species $y(t)$ is also extinct if $a_2 - \frac{\sigma_2^2 (y^*)^2}{2} < 0$. We complete the proof. □

THEOREM 9. *The system with $G_2\big(X(t)\big)$ is stochastically ultimately bounded for any initial value $\big(x(0), y(0)\big) \in R_+^2$.*

Proof. We first claim that there is a positive constant $K = K(\theta)$, which is independent of the initial value $\big(x(0), y(0)\big)$, such that the solution $X = (x, y)$ of the system with $G_2\big(X(t)\big)$ has the property that

$$\lim_{t\to+\infty} \sup E\,|x|^\theta \leqslant K.$$

Define $V = x^\theta + y^\theta$. It follows from Itô formula that

$$\begin{aligned}
\mathrm{d}V = \Bigg[&\theta x^\theta (a_1 - b_1 x - c_1 y) - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^\theta (x-x^*)^2 + \theta y^\theta (a_2 - b_2 y + c_2 x) \\
&- \frac{\theta(1-\theta)}{2}\sigma_2^2 y^\theta (y-y^*)^2\Bigg]\mathrm{d}t + \theta\sigma_1 (x-x^*) x^\theta \mathrm{d}B_1(t) + \theta\sigma_2 (y-y^*) y^\theta \mathrm{d}B_2(t).
\end{aligned} \tag{10}$$

Denote

$$\begin{aligned}\mathscr{L}V &= \theta x^{\theta}(a_1 - b_1x - c_1y) - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^{\theta}(x-x^*)^2 \\ &\quad + \theta y^{\theta}(a_2 - b_2y + c_2x) - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^{\theta}(y-y^*)^2 \\ &\leqslant \theta a_1 x^{\theta} + \theta a_2 y^{\theta} + \theta c_2 x^{\theta} y - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^{\theta}(x-x^*)^2 - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^{\theta}(y-y^*)^2 \\ &\triangleq F - V\end{aligned}$$

where

$$\begin{aligned}F = \theta a_1 x^{\theta} + \theta a_2 y^{\theta} + \theta c_2 x^{\theta} y + x^{\theta} + y^{\theta} \\ - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^{\theta}(x-x^*)^2 - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^{\theta}(y-y^*)^2.\end{aligned}$$

Since the coefficient of the highest of x is $-\frac{\theta(1-\theta)}{2} < 0$ and the coefficient of the highest of y is $-\frac{\theta(1-\theta)}{2} < 0$, then there exists a constant K' such $F \leqslant K'$ in R_+^2. Hence we have $\mathscr{L}V \leqslant K' - V$. Substituting this into (10) yields

$$dV \leqslant (K'-V)dt + \theta\sigma_1(x-x^*)x^{\theta}dB_1(t) + \theta\sigma_2(y-y^*)y^{\theta}dB_2(t). \tag{11}$$

From (11) and once again by Itô formula, we get

$$d[e^tV] = e^t(Vdt + dV) \leqslant K'e^tdt + e^t\theta\sigma_1(x-x^*)x^{\theta}dB_1(t) + e^t\theta\sigma_2(y-y^*)y^{\theta}dB_2(t).$$

Taking expectation of both side of the above inequality, we get $e^tEV \leqslant V\big(x(0),y(0)\big) + K'e^t$. This implies that

$$\lim_{t\to+\infty} \sup EV \leqslant K'.$$

On the other hand, we deduce $|X|^2 \leqslant 2\max\{x,y\}$. Thus $|X|^{\theta} \leqslant 2^{\frac{\theta}{2}}\max\{x^{\theta},y^{\theta}\} \leqslant 2^{\frac{\theta}{2}}V$. We have

$$\lim_{t\to+\infty} \sup E|X|^{\theta} \leqslant 2^{\frac{\theta}{2}}K' \triangleq K.$$

Then, for any $\varepsilon > 0$, let $H = \frac{K^2}{\varepsilon^2}$. By Chebyshev's inequality, we attain $P\{|X| > H\} \leqslant \frac{E(\sqrt{|X|})}{\sqrt{H}}$. Hence

$$\lim_{t\to+\infty} \sup P\{|X| > H\} \leqslant \frac{K}{\sqrt{H}} = \varepsilon.$$

This means

$$\lim_{t\to+\infty} \sup P\{|X| \leqslant H\} \geqslant 1 - \varepsilon. \quad \square$$

4. Analysis of model with $G_3\big(X(t)\big)$

THEOREM 10. *For any initial value $\big(x(0),y(0)\big) \in R^2_+$, there exists a unique positive solution $\big(x(t),y(t)\big)^T \in \mathrm{R}^2_+ = \big\{(x,y) \in \mathrm{R}^2 \mid x>0, y>0\big\}$ for the model with $G_3\big(X(t)\big)$ on $t \geqslant 0$ and the solution will remain in R^2_+ with probability 1 i.e., $\big(x(t),y(t)\big)^T \in \mathrm{R}^2_+$ for all $t \geqslant 0$ a.s.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous, then for any given initial value $\big(x(0),y(0)\big) \in R^2_+$, there is a unique local solution $\big(x(t),y(t)\big)$ on $t \in [0,\tau_e]$, where τ_e is the explosion time. To show this solution is global, we only need to show that $\tau_e = \infty$. Define the stopping times

$$\tau^+ = \inf\big\{t \in [0,\tau_e] : P(x(t)) \leqslant 0 \ \text{ or } \ P(y(t)) \leqslant 0\big\}.$$

We have $\tau^+ \leqslant \tau_e$. By the way, if $\tau^+ = \infty$ a.s., then $\tau_e = \infty$ a.s.

Assume that $\tau^+ < \infty$, then there exists a $T > 0$ such that $P(\tau^+ < T) > 0$. Define a C^2-function $V\colon\ R^2_+ \to R_+$ by $V(x,y) = x - x^* - x^* \ln \frac{x}{x^*} + (y - y^* - y^* \ln \frac{y}{y^*})\frac{c_1}{c_2}$. It follows from Itô formula that

$$\begin{aligned} \dot V &= \left(1-\frac{x^*}{x}\right)(a_1 - b_1 x - c_1 y)x + \theta\left(1-\frac{y^*}{y}\right)\big(a_2 - b_2 y + c_2 x\big)y \\ &= (x-x^*)(a_1 - b_1 x - c_1 y) + \theta(y-y^*)\big(a_2 - b_2 y + c_2 x\big) \end{aligned} \tag{12}$$

where $\theta = \frac{c_1}{c_2}$.

At the equilibrium point we have

$$\begin{cases} a_1 - b_1 x^* - c_1 y^* = 0, \\ a_2 - b_2 y^* + c_2 x^* = 0. \end{cases} \tag{13}$$

Substituting (13) into (12), we get

$$\begin{aligned} \dot V &= (x-x^*)\big[-b_1(x-x^*) - c_1(y-y^*)\big] + \frac{c_1}{c_2}(y-y^*)\big[-b_2(y-y^*) + c_2(x-x^*)\big] \\ &= -b_1(x-x^*)^2 - b_2\frac{c_1}{c_2}(y-y^*)^2 \end{aligned}$$

which follows that $\dot V \leqslant 0$. Then $\dot V = 0$ holds, only, if $x = x^*, y = y^*$. Therefore

$$\begin{aligned} \mathrm{d}V = &\left[\dot V + \frac{1}{2}\sigma_1^2 x^*(y-y^*)^2 + \frac{1}{2}\sigma_2^2 y^*(x-x^*)^2\frac{c_1}{c_2}\right]\mathrm{d}t \\ &+ \sigma_1(x-x^*)(y-y^*)\mathrm{d}B_1(t) + \sigma(y-y^*)(x-x^*)\frac{c_1}{c_2}\mathrm{d}B_2(t). \end{aligned}$$

Integrating and using the fact that $\dot V \leqslant 0$, we obtain

$$\begin{aligned} V\big(x(t),y(t)\big) \leqslant{}& V\big(x(0),y(0)\big) + \frac{1}{2}\int_0^t \big[x^*\sigma_1^2(y-y^*)^2 + y^*\sigma_2^2(x-x^*)^2\frac{c_1}{c_2}\big]\mathrm{d}s \\ &+ \int_0^t \sigma_1(x-x^*)(y-y^*)\mathrm{d}B_1(s) + \int_0^t \sigma(y-y^*)(x-x^*)\frac{c_1}{c_2}\mathrm{d}B_2(s). \end{aligned} \tag{14}$$

Note that some components of $\left(x(\tau^+), y(\tau^+)\right)$ equal 0. Thereby

$$\lim_{t\to\tau^+} V\left(x(t), y(t)\right) = \infty.$$

Extending t to τ^+ in (14), we get

$$\begin{aligned}\infty \leqslant V\left(x(0), y(0)\right) + \frac{1}{2}\int_0^{\tau^+} \left[x^*\sigma_1^2(y-y^*)^2 + y^*\sigma_2^2(x-x^*)^2\frac{c_1}{c_2}\right]\mathrm{d}s\\ + \int_0^{\tau^+} \sigma_1(x-x^*)(y-y^*)\mathrm{d}B_1(s) + \int_0^{\tau^+} \sigma(y-y^*)(x-x^*)\mathrm{d}B_2(s) < \infty.\end{aligned}$$

That contradicts our assumption, So $\tau^+ = \infty$ a.s. By virtue of the similar proof of [Theorem 3.3] of [23], we complete the proof. □

THEOREM 11. *If* $b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2} > 0$ *and* $b_2\frac{c_1}{c_2} - \frac{x^*\sigma_1^2}{2} > 0$, *then the equilibrium solution* (x^*, y^*) *of the model with* $G_2\left(X(t)\right)$ *is globally asymptotically stable.*

Proof. Define a C^2-function $V : R_+^2 \to R_+$ by $V(x,y) = x - x^* - x^*\ln\frac{x}{x^*} + (y - y^* - y^*\ln\frac{y}{y^*})\frac{c_1}{c_2}$. In view of Itô formula, we get

$$\begin{aligned}\mathscr{L}V &= (x-x^*)(a_1 - b_1 x - c_1 y) + \frac{\sigma_2^2 y^*}{2}(x-x^*)^2\frac{c_1}{c_2}\\ &\quad + (y-y^*)(a_2 - b_2 y + c_2 x)\frac{c_1}{c_2} + \frac{\sigma_1^2 x^*}{2}(y-y^*)^2\\ &= (x-x^*)\left[-b_1(x-x^*) - c_1(y-y^*)\right] + \frac{\sigma_2^2 y^*}{2}(x-x^*)^2\frac{c_1}{c_2}\\ &\quad + \frac{\sigma_1^2 x^*}{2}(y-y^*)^2 + (y-y^*)\left[c_2(x-x^*) - b_2(y-y^*)\right]\frac{c_1}{c_2}\\ &= -\left(b_1 - \frac{c_1 y^*\sigma_2^2}{2c_2}\right)(x-x^*)^2 - \left(b_2\frac{c_1}{c_2} - \frac{x^*\sigma_1^2}{2}\right)(y-y^*)^2.\end{aligned}$$

Therefore $\mathscr{L}V$ is negative definite. The equilibrium solution (x^*, y^*) of the model $G_3\left(X(t)\right)$ is globally asymptotically stable. □

THEOREM 12. *If* $a_1 - \frac{\sigma_1^2 (y^*)^2}{2} < 0$ *and* $c_1 - y^*\sigma_1^2 > 0$. *Then the species* $x(t)$ *of the model with* $G_3\left(X(t)\right)$ *will go to extinction almost surely. In addition, under the premise of species* $x(t)$ *extinction, species* $y(t)$ *is also extinct if* $a_2 - \frac{\sigma_2^2 (x^*)^2}{2} < 0$.

Proof. Applying Itô formula, we have

$$\begin{aligned}\mathrm{d}\ln x &= \left(a_1 - b_1 x - c_1 y - \frac{\sigma_1^2 (y-y^*)^2}{2}\right)\mathrm{d}t + \sigma_1(y-y^*)\mathrm{d}B_1(t)\\ &\leqslant \left(a_1 - c_1 y + \sigma_1^2 y^* y - \frac{\sigma_1^2 (y^*)^2}{2}\right)\mathrm{d}t - \frac{\sigma_1^2 y^2}{2}\mathrm{d}t + \sigma_1 y\mathrm{d}B_1(t) - \sigma_1 y^*\mathrm{d}B_1(t).\end{aligned}$$

That is to say,

$$\ln x(t) \leqslant \ln x(0) + \int_0^t \left(a_1 - c_1 y + \sigma_1^2 y^* y - \frac{\sigma_1^2 (y^*)^2}{2}\right) \mathrm{d}s - \int_0^t \frac{\sigma_1^2 x y^2}{2} \mathrm{d}s + M - M_1 \quad (15)$$

where $M = \int_0^t \sigma_1 y \mathrm{d}B_1(s)$ and $M_1 = \int_0^t \sigma_1 y^* \mathrm{d}B_1(s)$, whose quadratic variation is

$$\langle M, M \rangle = \int_0^t \sigma_1^2 y^2 \mathrm{d}s.$$

By virtue of the exponential martingale inequality, for any positive constant T, α, β, we get

$$P\left\{ \sup_{0 \leqslant t \leqslant T} \left[M - \frac{\alpha}{2} \langle M, M \rangle \right] > \beta \right\} \leqslant e^{-\alpha\beta}.$$

Choose $T = n$, $\alpha = 1$, $\beta = 2\ln n$, we obtain

$$P\left\{ \sup_{0 \leqslant t \leqslant T} \left[M - \frac{1}{2} \langle M, M \rangle \right] > 2\ln n \right\} \leqslant \frac{1}{n^2}.$$

An application of Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega)$ such that for $n \geqslant n_0$,

$$\sup_{0 \leqslant t \leqslant T} \left[M - \frac{1}{2} \langle M, M \rangle \right] \leqslant 2\ln n.$$

That is to say

$$M \leqslant 2\ln n + \frac{1}{2} \langle M, M \rangle = 2\ln n + \frac{1}{2} \int_0^t \frac{\sigma_1^2 y^2}{2} \mathrm{d}s$$

for all $0 \leqslant t \leqslant n$, $n \geqslant n_0$. Substituting the above inequality into (15) leads to

$$\ln x(t) - \ln x(0) \leqslant \left(a_1 - \frac{\sigma_1^2 (y^*)^2}{2}\right) t - (c_1 - \sigma_1^2 y^*) \int_0^t y \mathrm{d}s + 2\ln n - M_1.$$

In other words, we have already shown that for $0 < n-1 \leqslant t \leqslant n$,

$$\frac{\ln x(t) - \ln x(0)}{t} \leqslant \left(a_1 - \frac{\sigma_1^2 (y^*)^2}{2}\right) - (c_1 - \sigma_1^2 y^*) \frac{1}{t} \int_0^t y \mathrm{d}s + \frac{2\ln n}{n-1} - \frac{M_1}{t}. \quad (16)$$

From the strong law of large number, we deduce

$$\lim_{t \to +\infty} \frac{M_1}{t} = 0. \quad (17)$$

Substituting (17) into (16), we attain

$$\lim_{t \to +\infty} \sup \frac{\ln x}{t} \leqslant \left(a_1 - \frac{\sigma_1^2 (y^*)^2}{2}\right) - (c_1 - \sigma_1^2 y^*) \frac{1}{t} \int_0^t y \mathrm{d}s < 0.$$

That is to say

$$\lim_{t\to+\infty} x(t) = 0.$$

Applying Itô formula to the second formula of the model with $G_3\big(X(t)\big)$, we have

$$\begin{aligned}
\mathrm{d}\ln y &= \left(a_2 - b_2 y + c_2 x - \frac{\sigma_2^2(x-x^*)^2}{2}\right)\mathrm{d}t + \sigma_2(x-x^*)\mathrm{d}B_2(t) \\
&\leqslant \left(a_2 + c_2 x - x^*\sigma_2^2 x - \frac{\sigma_2^2(x^*)^2}{2}\right)\mathrm{d}t - \frac{\sigma_2^2 x^2}{2}\mathrm{d}t + \sigma_2 x \mathrm{d}B_2(t) - \sigma_2 x^* \mathrm{d}B_2(t).
\end{aligned}$$

Using the similar method above, we can get

$$\frac{\ln y(t) - \ln y(0)}{t} \leqslant \left(a_2 - \frac{\sigma_2^2(x^*)^2}{2}\right) + \frac{1}{t}\int_0^t (c_2 - x^*\sigma_2^2)x\mathrm{d}s - \frac{M_2}{t} \tag{18}$$

where $M_2 = \int_0^t \sigma_2 x^* \mathrm{d}B_2(s)$. From the strong law of large number, we attain

$$\lim_{t\to+\infty} \frac{M_2}{t} = 0. \tag{19}$$

Substituting (19) into (18) and using the fact that $\lim_{t\to+\infty} x(t) = 0$, we deduce

$$\lim_{t\to+\infty} \sup \frac{\ln y}{t} \leqslant a_2 - \frac{\sigma_2^2(x^*)^2}{2} < 0.$$

That is to say, if $a_2 - \frac{\sigma_2^2(x^*)^2}{2} < 0$ and species $y(t)$ goes extinct, given that species $x(t)$ goes extinct. □

THEOREM 13. *Suppose that* $b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2} > 0$ *and* $\frac{b_2 c_1}{c_2} - \frac{x^*\sigma_1^2}{2} > 0$ *hold. Then the solution* $\big(x(t), y(t)\big)$ *of the system with* $G_2\big(X(t)\big)$ *is positively recurrent.*

Proof. Define $V(x,y) = x - x^* - x^*\ln\frac{x}{x^*} + (y - y^* - y^*\ln\frac{y}{y^*})\frac{c_1}{c_2}$. $D = \big\{(x,y)\in R_+^2 \mid \frac{1}{N}\leqslant x\leqslant N, \frac{1}{N}\leqslant y\leqslant N\big\}$. Applying Itô formula, we have

$$\begin{aligned}
\mathscr{L}V &= (x-x^*)(a_1 - b_1 x - c_1 y) + \frac{\sigma_2^2 y^*}{2}(x-x^*)^2\frac{c_1}{c_2} \\
&\quad + (y-y^*)(a_2 - b_2 y + c_2 x)\frac{c_1}{c_2} + \frac{\sigma_1^2 x^*}{2}(y-y^*)^2 \\
&= (x-x^*)\big[-b_1(x-x^*) - c_1(y-y^*)\big] + \frac{\sigma_2^2 y^*}{2}(x-x^*)^2\frac{c_1}{c_2} \\
&\quad + \frac{\sigma_1^2 x^*}{2}(y-y^*)^2 + (y-y^*)\big[c_2(x-x^*) - b_2(y-y^*)\big]\frac{c_1}{c_2} \\
&= -\left(b_1 - \frac{c_1 y^*\sigma_2^2}{2c_2}\right)(x-x^*)^2 - \left(\frac{b_2 c_1}{c_2} - \frac{x^*\sigma_1^2}{2}\right)(y-y^*)^2.
\end{aligned}$$

When $(x,y) \to (0,0)$, we have

$$\mathscr{L}V = -\left(b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2}\right)(x^*)^2 - \left(\frac{b_2 c_1}{c_2} - \frac{x^* \sigma_1^2}{2}\right)(y^*)^2 < 0.$$

When $x \to 0$ and $y \to +\infty$, we get

$$\mathscr{L}V = -\left(b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2}\right)(x^*)^2 - \left(\frac{b_2 c_1}{c_2} - \frac{x^* \sigma_1^2}{2}\right)(y-y^*)^2 \to -\infty.$$

When $x \to +\infty$ and $y \to 0$, we obtain

$$\mathscr{L}V = -\left(b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2}\right)(x-x^*)^2 - \left(\frac{b_2 c_1}{c_2} - \frac{x^* \sigma_1^2}{2}\right)(y^*)^2 \to -\infty.$$

When $x \to +\infty$ and $y \to +\infty$, we deduce

$$\mathscr{L}V = -\left(b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2}\right)(x-x^*)^2 - \left(\frac{b_2 c_1}{c_2} - \frac{x^* \sigma_1^2}{2}\right)(y-y^*)^2 \to -\infty.$$

Therefore, for $(x,y) \in D^c$, $\mathscr{L}V < 0$. Using the similar proof of [Theorem 3.26] of [31], it is a sufficient and necessary condition for positive recurrence. The proof is complete. □

THEOREM 14. *The system with $G_3\big(X(t)\big)$ is stochastically ultimately bounded for any initial value $\big(x(0),y(0)\big) \in R_+^2$.*

Proof. We first claim that there is a positive constant $K = K(\theta)$, which is independent of the initial value $\big(x(0),y(0)\big)$, such that the solution $X = (x,y)$ of the system with $G_3\big(X(t)\big)$ has the property that

$$\lim_{t \to +\infty} \sup E\,|x|^{\theta} \leqslant K.$$

Define $V = x^{\theta} + y^{\theta}$. It follows from Itô formula that

$$\begin{aligned} \mathrm{d}V = \bigg[&\theta x^{\theta}(a_1 - b_1 x - c_1 y) - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^{\theta}(y-y^*)^2 \\ &+ \theta y^{\theta}(a_2 - b_2 y + c_2 x) - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^{\theta}(x-x^*)^2\bigg]\mathrm{d}t \\ &+ \theta\sigma_1(y-y^*)x^{\theta}\mathrm{d}B_1(t) + \theta\sigma_2(x-x^*)y^{\theta}\mathrm{d}B_2(t). \end{aligned} \tag{20}$$

Denote

$$\begin{aligned} \mathscr{L}V &= \theta x^{\theta}(a_1 - b_1 x - c_1 y) - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^{\theta}(y-y^*)^2 \\ &\quad + \theta y^{\theta}(a_2 - b_2 y + c_2 x) - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^{\theta}(x-x^*)^2 \\ &\leqslant \theta a_1 x^{\theta} + \theta a_2 y^{\theta} + \theta c_2 x y^{\theta} - \frac{\theta(1-\theta)}{2}\sigma_1^2 x^{\theta}(y-y^*)^2 - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^{\theta}(x-x^*)^2 \\ &\triangleq F - V \end{aligned}$$

where

$$F = (\theta a_1 + 1)x^\theta + (\theta a_2 + \theta c_2 x + 1)y^\theta$$
$$- \frac{\theta(1-\theta)}{2}\sigma_1^2 x^\theta (y-y^*)^2 - \frac{\theta(1-\theta)}{2}\sigma_2^2 y^\theta (x-x^*)^2.$$

Note that F is bounded in R_+^2, then there exists a constant K' such $F \leqslant K'$ Hence we have

$$\mathscr{L}V \leqslant K' - V.$$

Substituting this into model (20) yields

$$dV \leqslant (K'-V)dt + \theta\sigma_1(y-y^*)x^\theta dB_1(t) + \theta\sigma_2(x-x^*)y^\theta dB_2(t). \tag{21}$$

From (21) and once again by Itô formula, we get

$$d[e^t V] = e^t(V dt + dV) \leqslant K' e^t dt + e^t \theta\sigma_1(y-y^*)x^\theta dB_1(t) + e^t\theta\sigma_2(x-x^*)y^\theta dB_2(t).$$

Taking expectation of both side of the above inequality, we get $e^t EV \leqslant V\big(x(0), y(0)\big) + K'e^t$. This implies that

$$\lim_{t\to+\infty} \sup EV \leqslant K'.$$

On the other hand, we deduce $|X|^2 \leqslant 2\max\{x,y\}$. Thus $|X|^\theta \leqslant 2^{\frac{\theta}{2}} \max\{x^\theta, y^\theta\} \leqslant 2^{\frac{\theta}{2}} V$. We have

$$\lim_{t\to+\infty} \sup E|X|^\theta \leqslant 2^{\frac{\theta}{2}} K' \triangleq K.$$

Then, for any $\varepsilon > 0$, let $H = \frac{K^2}{\varepsilon^2}$. By Chebyshev's inequality, we attain $P\{|X| > H\} \leqslant \frac{E(\sqrt{|X|})}{\sqrt{H}}$. Hence

$$\lim_{t\to+\infty} \sup P\{|X| > H\} \leqslant \frac{K}{\sqrt{H}} = \varepsilon.$$

This means

$$\lim_{t\to+\infty} \sup P\{|X| \leqslant H\} \geqslant 1 - \varepsilon. \quad \square$$

5. Numerical simulations

In this section we provide numerical simulation results to substantiate the analytical findings for the stochastic model system reported in the previous sections. We will use Milstein's Method to illustrate our results [33].

For the stochastic model with $G_1\big(X(t)\big)$, we consider the following discretized equations:

$$\begin{cases} x_{k+1} = x_k + x_k\big(a_1 - b_1 x_k - c_1 y_k\big)\Delta t + \sigma_1(x_k - x^*)\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}(x_k - x^*)^2(\xi_k^2 - 1)\Delta t, \\ y_{k+1} = y_k + y_k\big(a_2 - b_2 y_k + c_2 x_k\big)\Delta t + \sigma_2(y_k - y^*)\sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2}(y_k - y^*)^2(\eta_k^2 - 1)\Delta t. \end{cases}$$

where ξ_k and η_k, $k = 1,2,\ldots,n$ are two independent Gaussian random variables that follow $N(0,1)$.

In fig: 1, to ensure that $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ and $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$ hold, we select $\sigma_1 = 0.02$ and $\sigma_2 = 0.02$, then we obtain the existence and uniqueness of the solution for the model with $G_1(X(t))$. In Theorem 2, after computing the bounds of the noise intensities $\sqrt{-2\hat{A}} = 0.3216$ and $\sqrt{-2\hat{D}} = 0.3613$, we choose $\sigma_1 = 0.3$ and $\sigma_2 = 0.3$. Then the stability of equilibrium points E^2 and E^* are drawn in fig: 3 and fig: 4 respectively. For the axial equilibrium point E^1, we choose $\sigma_1 = 0.15$ and $\sigma_2 = 0.15$, the fig: 2 shows that the axial equilibrium point E^1 is stable. The positive recurrence conditions prescribed in Theorem 3 are sufficient conditions, we choose $\sigma_1 = 0.2$ and $\sigma_2 = 0.2$ to ensure that $\sigma_1 \leqslant \sqrt{\frac{2b_1}{x^*}}$ and $\sigma_2 \leqslant \sqrt{\frac{2b_2}{y^*}}$ are established, which can be seen in the fig: 5. fig: 6 shows that the system with $G_1(X(t))$ is stochastically ultimately bounded.

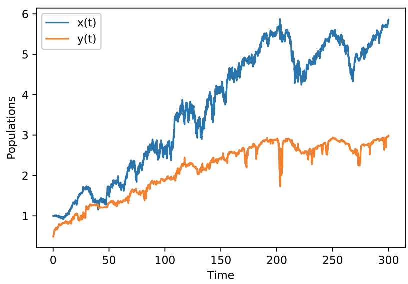

Figure 1: *The existence and uniqueness of solution of the system with $G_1(X(t))$*

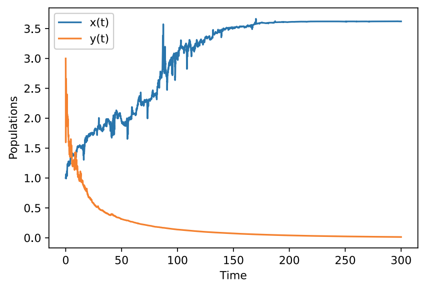

Figure 2: *Globally asymptotically stability of the system with $G_1(X(t))$ at point E^1*

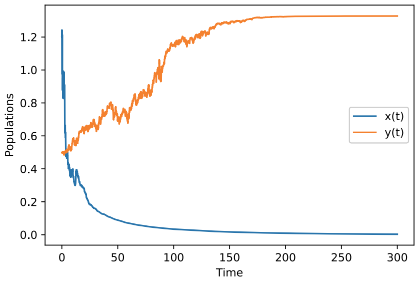

Figure 3: *Globally asymptotic stability of the system with $G_1(X(t))$ at point E^2*

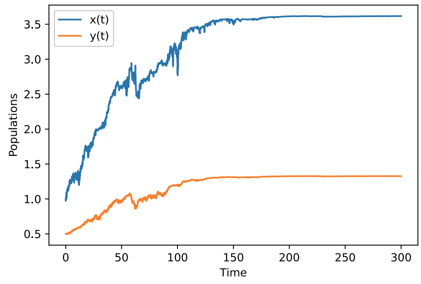

Figure 4: *Globally asymptotic stability of the system with $G_1(X(t))$ at point E^**

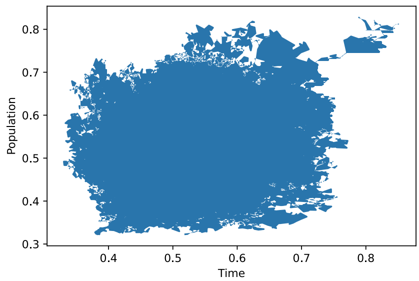

Figure 5: *The positive recurrence of the system with* $G_1\big(X(t)\big)$

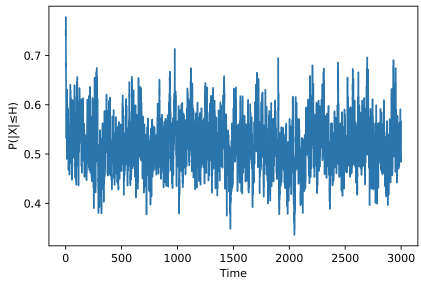

Figure 6: *The stochastic ultimate boundedness of the system with* $G_1\big(X(t)\big)$

Consider the discretized equations for model with $G_2\big(X(t)\big)$:

$$\begin{cases} x_{k+1} = x_k + x_k\big(a_1 - b_1x_k - c_1y_k\big)\Delta t \\ \qquad\qquad +\sigma_1 x_k(x_k - x^*)\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}[x_k(x_k - x^*)]^2(\xi_k^2 - 1)\Delta t, \\ y_{k+1} = y_k + y_k\big(a_2 - b_2y_k + c_2x_k\big)\Delta t \\ \qquad\qquad +\sigma_2 y_k(y_k - y^*)\sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2}[y_k(y_k - y^*)]^2(\eta_k^2 - 1)\Delta t \end{cases}$$

where ξ_k and η_k, $k = 1,2,\dots,n$ are two independent Gaussian random variables that follow $N(0,1)$.

In view of Theorem 5, if we choose $\sigma_1 = 0.8$, $\sigma_2 = 0.8$, then the solution of the model with $G_2\big(X(t)\big)$ exists and is unique. To guarantee the conditions $\sigma_1^2 < 2b_1x^*$ and $\sigma_2^2 < 2b_2y^*$ are satisfied, we let $\sigma_1 = 0.25$ and $\sigma_2 = 0.25$. Hence the asymptotic stability are obtained. fig: 8 confirms the conclusion. In fig: 9, we choose $\sigma_1 = 0.15$, $\sigma_2 = 0.1$. Then it is easy to obtain $b_1 - \frac{x^*\sigma_1^2}{2} > 0$ and $b_2 - \frac{y^*\sigma_2^2}{2} > 0$. If we choose $\sigma_1 = 1.6$, then the first conditions $a_1 - \frac{\sigma_1^2(x^*)^2}{2} < 0$ and $b_1 - \sigma_1^2x^* > 0$ of Theorem 8 will be valid. As a result, prey population goes to extinction and extinction time for this simulation is 125. Again for $\sigma_2 = 1.4$, second condition $a_2 - \frac{\sigma_2^2(y^*)^2}{2} < 0$ of Theorem 8 is valid and as a result predator population goes to extinction as depicted in fig: 10. Fig: 11 shows that the system with $G_2\big(X(t)\big)$ is stochastically ultimately bounded.

Let us now turn to model with $G_3\big(X(t)\big)$, consider the discretized equations:

$$\begin{cases} x_{k+1} = x_k + x_k\big(a_1 - b_1x_k - c_1y_k\big)\Delta t \\ \qquad\qquad +\sigma_1 x_k(y_k - y^*)\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}[x_k(y_k - y^*)]^2(\xi_k^2 - 1)\Delta t, \\ y_{k+1} = y_k + y_k\big(a_2 - b_2y_k + c_2x_k\big)\Delta t \\ \qquad\qquad +\sigma_2 y_k(x_k - x^*)\sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2}[y_k(x_k - x^*)]^2(\eta_k^2 - 1)\Delta t \end{cases}$$

where ξ_k and η_k, $k = 1,2,\dots,n$ are two independent Gaussian random variables that follow $N(0,1)$.

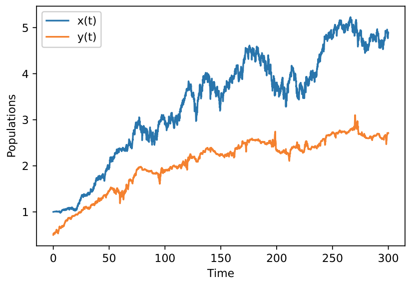

Figure 7: *The existence and uniqueness of solution of the system with* $G_2(X(t))$

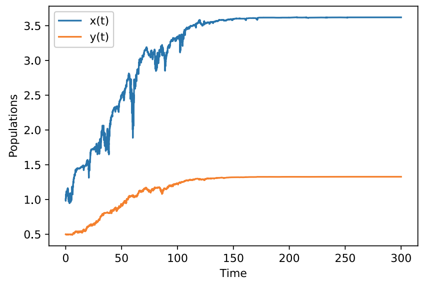

Figure 8: *Globally asymptotic stability of the system with* $G_2(X(t))$

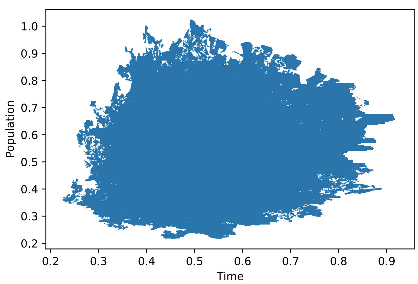

Figure 9: *The positive recurrence of the system with* $G_2(X(t))$

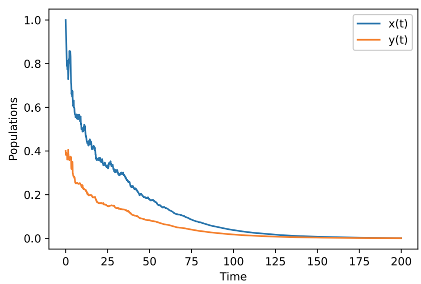

Figure 10: *The extinction of the system with* $G_2(X(t))$

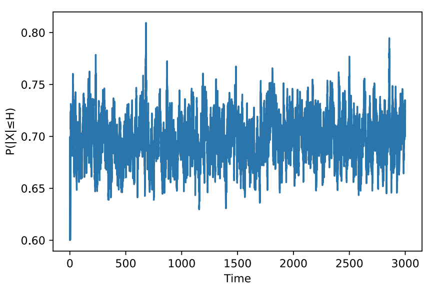

Figure 11: *The stochastic ultimate boundedness of the system with* $G_2(X(t))$

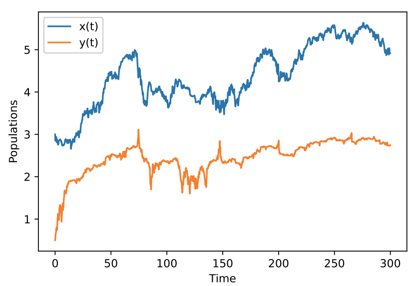

Figure 12: *The existence and uniqueness of solution of the system with* $G_3(X(t))$

For numerical simulation of the Theorem 10, we choose the parameters $\sigma_1 = 0.04$ and $\sigma_2 = 0.035$. We use different values of σ_1 and σ_2 in order to understand their role on the dynamics. fig: 12 shows that the solution of the model with $G_3(X(t))$

exists and is unique. To demonstrate the the environmental effect on the equilibrium (x^*, y^*) of the model with $G_3(X(t))$, we consider the environmental forcing intensities as $\sigma_1 = 0.9$ and $\sigma_2 = 1.2$ so that this values satisfy the conditions $b_1 - \frac{c_1 y^* \sigma_2^2}{2c_2} > 0$ and $b_2 \frac{c_1}{c_2} - \frac{x^* \sigma_1^2}{2} > 0$. fig: 13 confirms this. fig: 16 shows that the system with $G_3(X(t))$ is stochastically ultimately bounded. If we choose $\sigma_1 = 1.25$, then the first conditions $a_1 - \frac{\sigma_1^2 (y^*)^2}{2} < 0$ and $c_1 - y^* \sigma_1^2 > 0$ of Theorem 12 will be valid. As a result, prey population goes to extinction and extinction time for this simulation is 125. Again for $\sigma_2 = 1.6$, second condition $a_2 - \frac{\sigma_2^2 (y^*)^2}{2} < 0$ of Theorem 12 is valid and as a result predator population goes to extinction as depicted in fig: 14. Finally, we choose $\sigma_1 = 1.1$ and $\sigma_2 = 1.5$ such that both conditions required for positive recurrence are satisfied.

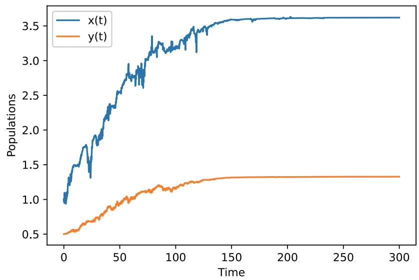

Figure 13: *Globally asymptotic stability of the system with* $G_3(X(t))$

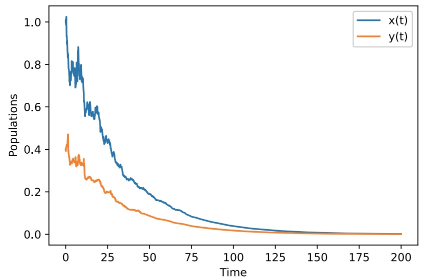

Figure 14: *The extinction of the system with* $G_3(X(t))$

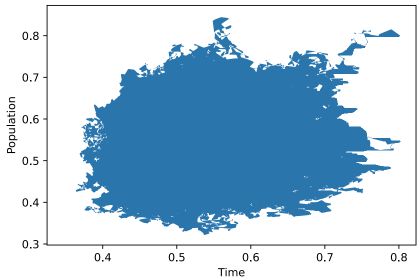

Figure 15: *The positive recurrence of the system with* $G_3(X(t))$

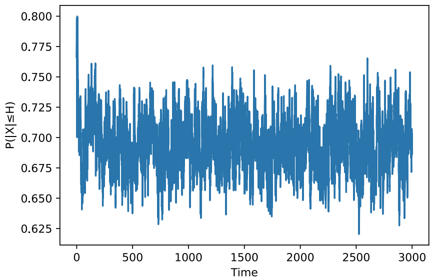

Figure 16: *The stochastic ultimate boundedness of the system with* $G_3(X(t))$

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