Journal of
Mathematical
Inequalities
Volume 17, Number 3 (2023), 1113–1127 doi:10.7153/jmi-2023-17-72

ASYMPTOTIC EXPANSION FOR GENERALIZED MATHIEU SERIES

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(*Communicated by T. Burić*)

Abstract. The main result of this paper presents a series expansion of the generalized Mathieu series $S_m(r)$ with computable coefficients. This generalizes Elbert's work from $m=1$ to arbitrary $m \in \mathbb{Z}_{>0}$, and provides a method for calculating the asymptotic formula of $S_m(r)$ within a given error. Moreover, this paper revises an integral representation for $S_m(r)$ given by Cerone and Lenard.

1. Introduction

Let $\mu, r \in \mathbb{R}_{>0}$. The generalized Mathieu series is defined as

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^{\mu+1}}.$$

When $\mu = 1$, the classical Mathieu series $S_1(r)$ was first introduced by Mathieu in his work on elasticity of solid bodies [19]. Since then, Mathieu series have been extensively applied in mathematics and physics. The Mathieu series and their generalizations involve in the solution of the biharmonic equation in a rectangular plate [24], the solution of some linear ordinary differential equations [4] and Fredholm integral equation with nondegenerate kernel [11]. Meanwhile, the Mathieu series have close connection with Riemann zeta-function [6], Hurwitz zeta-function [7] and the Schlomilch series [14]. More applications of the Mathieu series can be found in reference [21, 22, 16, 15].

During the past decades, a number of research efforts are dedicated to the estimation of Mathieu series. When $\mu = 1$, Diananda [9] obtained that

$$\frac{1}{r^2} - \frac{5}{16r^4} < S_1(r) < \frac{1}{r^2} - \frac{1}{(2r^2+2r+1)(8r^2+5r+3)}. \tag{1}$$

This inequality improves the bound estimate of $S_1(r)$ given by [18, 3, 8], and implies that there exists the following asymptotic representation for $S_1(r)$ as r tends to infinity.

$$S_1(r) = \frac{1}{r^2} + \frac{c}{r^4} + O\left(\frac{1}{r^6}\right),$$

Mathematics subject classification (2020): 33E20, 41A60, 41A20, 11B68.
Keywords and phrases: Generalized Mathieu series, asymptotic expansions, Bernoulli numbers, special function.

where c is a constant with $-5/16 < c < -1/16$. Inspired by (1), Elbert [12] derived the asymptotic expansion for $S_1(r)$ as follow

$$S_1(r) \sim \sum_{i=0}^{\infty}(-1)^i \frac{B_{2i}}{r^{2i+2}}, \tag{2}$$

where $f(r) \sim g(r)$ denotes $\lim_{r\to\infty} \frac{f(r)}{g(r)} = 1$, and B_i denotes the Bernoulli numbers. $B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,\ \ldots,\ B_3 = B_5 = \ldots = 0$. Alzer et al. [1] obtained the bound estimate of $S_1(r)$ as follow

$$\frac{1}{r^2 + 1/2\zeta(3)} < S_1(r) < \frac{1}{r^2 + 1/6}, \tag{3}$$

and showed that $1/2\zeta(3)$ and $1/6$ are the best constants in the above kind of two-sided inequality, where $\zeta(k)$ denotes the Riemann zeta-function. Lin [17] considered the tails of $S_1(r)$ and obtained a slightly more accurate estimation than (3) when $k \geqslant r > 2$.

For $m \in \mathbb{Z}_{>0}$, Diananda [10] estimated that

$$\begin{cases} S_m(r) < \dfrac{1}{2m}\left\{\dfrac{1}{m} + \dfrac{m-2}{2(m+2)}\dfrac{1}{r^2} + \dfrac{m^2+5m+2}{8(m+2)}\dfrac{1}{r^4}\right\}, & \text{for } r > 0, \\ S_m(r) > \dfrac{1}{r^{2m+1}} \dfrac{\left(1-\frac{1}{4r^2}\right)^m}{\left(1+\frac{1}{2r}\right)^m - \left(1-\frac{1}{2r}\right)^m}, & \text{for } r > \frac{1}{2}, \end{cases} \tag{4}$$

which implies that S_m can be bounded by the linear combination of $r^{-i}(i \in \mathbb{Z}_{\geqslant 2})$.

Based on formula (4), researchers focus on the asymptotic expansions for $S_m(r)$ via r^{-i}. Cerone [5] and Tomovski [25] estimated bounds for $S_m(r)$ in terms of the Gamma function respectively. Zastavnyi [27] proved an asymptotic relation for a more general Mathieu series via Bernoulli polynomials and Gamma function. For $S_m(r)$, the above Bernoulli polynomial can be reduced to Bernoulli number as r tends to infinity, while the Gamma function still remains [27]. It should be noted that all of the above asymptotic results for $S_m(r)$ are based on the Gamma function.

When considering the expansions for $S_m(r)$ via r^i, it is easy to show that $S_m(r)$ has the Maclaurin expansion [6].

$$S_m(r) = 2\sum_{n\geqslant 0}(-1)^n \binom{m+n}{n}\zeta(2m+2n+1)r^{2n}, \quad \text{for } |r| < 1. \tag{5}$$

Although (5) gave an expansion for $S_m(r)$, the exact values of the coefficients are hard to calculate since there are infinite many $\zeta(s)$ be irrational number if s is odd [23].

Inspired by the previous works, the main purpose of this paper is to find the asymptotic expansion for $S_m(r)$ with computable coefficients.

THEOREM 1.1. *For $m \in \mathbb{Z}_{>0}$, we have*

$$S_m(r) \sim \frac{1}{m!}\sum_{i=0}^{\infty}\frac{(-1)^i(i+m-1)!B_{2i}}{i!}\cdot\frac{1}{r^{2m+2i}}.$$

The asymptotic expansion in Theorem 1.1 eliminates the Gamma function. Thus, it presents a explicit result for the series expansion of the generalized Mathieu series with numerical coefficients, since the Bernoulli numbers in even subscripts are computable. On the one hand, Theorem 1.1 generalizes Elbert's work [12] from $m=1$ to $m \in \mathbb{Z}_{>0}$, including (2) as a special case when $m=1$. On the other hand, Theorem 1.1 provides a method for calculating the asymptotic formula of $S_m(r)$ within a given error. For example, as r tends to infinity, we have the following asymptotic formulae.

$$
\begin{aligned}
S_1(r) &= \frac{1}{r^2} - \frac{1}{6r^4} - \frac{1}{30r^6} - \frac{1}{42r^8} - \frac{1}{30r^{10}} - \frac{5}{66r^{12}} + o\left(r^{-14}\right), \\
S_2(r) &= \frac{1}{2r^4} - \frac{1}{6r^6} - \frac{1}{20r^8} - \frac{1}{21r^{10}} - \frac{1}{12r^{12}} - \frac{5}{22r^{14}} + o\left(r^{-16}\right), \\
S_3(r) &= \frac{1}{3r^6} - \frac{1}{6r^8} - \frac{1}{15r^{10}} - \frac{5}{63r^{12}} - \frac{1}{6r^{14}} - \frac{35}{66r^{16}} + o\left(r^{-18}\right), \\
S_4(r) &= \frac{1}{4r^8} - \frac{1}{6r^{10}} - \frac{1}{12r^{12}} - \frac{5}{42r^{14}} - \frac{7}{24r^{16}} - \frac{35}{33r^{18}} + o\left(r^{-20}\right), \\
S_5(r) &= \frac{1}{5r^{10}} - \frac{1}{6r^{12}} - \frac{1}{10r^{14}} - \frac{1}{6r^{16}} - \frac{7}{15r^{18}} - \frac{21}{11r^{20}} + o\left(r^{-22}\right), \\
S_6(r) &= \frac{1}{6r^{12}} - \frac{1}{6r^{14}} - \frac{7}{60r^{16}} - \frac{2}{9r^{18}} - \frac{7}{10r^{20}} - \frac{35}{11r^{22}} + o\left(r^{-24}\right), \\
S_7(r) &= \frac{1}{7r^{14}} - \frac{1}{6r^{16}} - \frac{2}{15r^{18}} - \frac{2}{7r^{20}} - \frac{1}{r^{22}} - \frac{5}{r^{24}} + o\left(r^{-26}\right), \\
S_8(r) &= \frac{1}{8r^{16}} - \frac{1}{6r^{18}} - \frac{3}{20r^{20}} - \frac{5}{14r^{22}} - \frac{11}{8r^{24}} - \frac{15}{2r^{26}} + o\left(r^{-28}\right), \\
S_9(r) &= \frac{1}{9r^{18}} - \frac{1}{6r^{20}} - \frac{1}{6r^{22}} - \frac{55}{126r^{24}} - \frac{11}{6r^{26}} - \frac{65}{6r^{28}} + o\left(r^{-30}\right), \\
S_{10}(r) &= \frac{1}{10r^{20}} - \frac{1}{6r^{22}} - \frac{11}{60r^{24}} - \frac{11}{21r^{26}} - \frac{143}{60r^{28}} - \frac{91}{6r^{30}} + o\left(r^{-32}\right).
\end{aligned}
$$

For $m > 10$ or higher order of the remainder, similar asymptotic formulae for $S_m(r)$ can be deduced by Theorem 1.1.

Integral representation is another important research directions of Mathieu series. An integral representation for $S_1(r)$ was presented by Emersleben [13] as

$$
S_1(r) = \frac{1}{r}\int_0^\infty \frac{x}{e^x - 1}\sin(rx)dx. \tag{6}
$$

Inspired by (6), Tomovski and Trenčevski [26] obtained the integral representation for $S_m(r)$ as follow

$$
\begin{aligned}
S_m(r) = {} & \frac{2}{(2r)^m m!}\int_0^\infty \frac{t^m}{e^t-1}\cos\left(\frac{m\pi}{2} - rt\right)dt \\
& - 2\sum_{k=2}^{m}\left[\frac{(k-1)(2r)^{k-2m-1}}{k!(m-k+1)}\binom{-(m+1)}{m-k}\right. \\
& \left.\times \int_0^\infty \frac{t^k \cos\left[\frac{\pi}{2}(2m-k+1) - rt\right]}{e^t - 1}dt\right].
\end{aligned} \tag{7}
$$

Milovanović and Pogány [20] obtained that

$$S_m(r) = \frac{\pi}{m}\int_0^\infty \frac{\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m}{2j}\left(r^2-x^2+\frac{1}{4}\right)^{m-2j}x^{2j}}{\left[\left(x^2-r^2+\frac{1}{4}\right)^2+r^2\right]^m}\cdot\frac{\mathrm{d}x}{\cosh^2\pi x}. \tag{8}$$

Cerone and Lenard [6, p. 3, Thm. 2.1] investigated further integral representation for $S_\mu(r)$ in terms of Gamma function and Bessel function, where $\mu \in \mathbb{R}_{>0}$. Based on this result, they [6, p. 6, Thm. 2.5] gave an explicit representation for $S_m(r)$, where $m \in \mathbb{Z}_{>0}$. However, there is a mistake in the proof of the second result for $S_m(r)$.

In this paper, we will explain the mistake in the proof of [6, p. 6, Thm. 2.5] and present the modified explicit integral representation for $S_m(r)$.

THEOREM 1.2. *For $m \in \mathbb{Z}_{>0}$, we have*

$$S_m(r) = \frac{1}{2^{2m-2}r^{2m-1}m}\sum_{k=0}^{m-1}\frac{(-1)^{\left[\frac{k+1}{2}\right]}\cdot 2^k\cdot\binom{2m-k-2}{m-1}\cdot r^k}{k!}A_k(r),$$

where

$$A_k(r) = \begin{cases} \displaystyle\int_0^\infty \frac{x^{k+1}}{e^x-1}\cos(rx)dx, & \text{for } k \text{ is odd,} \\ \displaystyle\int_0^\infty \frac{x^{k+1}}{e^x-1}\sin(rx)dx, & \text{for } k \text{ is even.} \end{cases}$$

Theorem 1.2 is a generalization of (6) given by Emersleben [13]. This result simplifies the integral representation (7) given by Tomovski and Trenčevski [26], and (8) given by Milovanović and Pogány [20].

2. Preliminaries

Let k be a non-negative integer. For the simplicity of description, we denote

$$\begin{cases} f_k(x) = \dfrac{x^k}{e^x-1}, & \text{for } k>0, \\ F_k(r) = \displaystyle\int_0^\infty \frac{x}{e^x-1}(rx)^k\cos(rx)dx, & \text{for } k>0, \\ G_k(r) = \displaystyle\int_0^\infty \frac{x}{e^x-1}(rx)^k\sin(rx)dx, & \text{for } k\geqslant 0. \end{cases} \tag{9}$$

Bernoulli numbers satisfy the series expansion

$$f_k(x) = \sum_{n=0}^\infty B_n\frac{x^{n+k-1}}{n!}, \quad \text{for } |x|<2\pi. \tag{10}$$

By equation (10) and L'Hospital's rule, we have

$$\lim_{x\to 0} f_k(x) = \begin{cases} B_0, & \text{for } k=1, \\ 0, & \text{for } k>1, \end{cases} \qquad \text{and} \qquad \lim_{x\to\infty} f_k(x) = 0.$$

For $i,k \in \mathbb{Z}_{>0}$, we have

$$\lim_{x\to 0} f_k^{(i)}(x) = \begin{cases} 0, & \text{for } 0 \leqslant i < k-1, \\ \frac{i!B_{i-k+1}}{(i-k+1)!}, & \text{for } k-1 \leqslant i, \end{cases} \tag{11}$$

and

$$\lim_{x\to\infty} f_k^{(i)}(x) = 0, \tag{12}$$

where $f_k^{(i)}(x)$ denotes the i-th derivative of $f_k(x)$.

Based on the notations above, we give the relationship between Bernoulli numbers and integral representations involving trigonometric function.

LEMMA 2.1. *For $i,k \in \mathbb{Z}_{>0}$, we have*

$$\lim_{r\to\infty} \int_0^\infty f_k^{(i)}(x)\sin(rx)dx = 0,$$
$$\lim_{r\to\infty} \int_0^\infty f_k^{(i)}(x)\cos(rx)dx = 0.$$

Proof. Equation (11) and (12) implies that

$$\int_0^\infty f_k^{(i)}(x)dx < \infty, \qquad \text{for } i,k \in \mathbb{Z}_{>0}.$$

Then, Lemma 2.1 holds by the Riemann-Lebesgue lemma. □

LEMMA 2.2. *Let $k \in \mathbb{Z}_{>0}$ and $s = \left[\frac{k+2}{2}\right]$. The integration $F_k(r)$ has the asymptotic expansion*

$$F_k(r) \sim \begin{cases} (-1)^s \sum_{i=0}^\infty (-1)^i \frac{(k+2i)!B_{2i}}{(2i)!r^{2i+1}}, & \textit{for } k \textit{ is odd,} \\ (-1)^s \sum_{i=0}^\infty (-1)^i \frac{(k+2i+1)!B_{2i+1}}{(2i+1)!r^{2i+2}}, & \textit{for } k \textit{ is even.} \end{cases}$$

Proof. Recall the notation in (9), we have

$$F_k(r) = r^k \int_0^\infty f_{k+1}(x)\cos(rx)dx. \tag{13}$$

Integrating by parts in (13), we have

$$F_k(r) = -r^{k-1} \int_0^\infty f'_{k+1}(x)\sin(rx)dx.$$

By Lemma 2.1, denote

$$I_{k,1}(r) = r^{3-k}F_k(r) = -r^2 \int_0^\infty f'_{k+1}(x)\sin(rx)dx.$$

Integrating twice by parts, we obtain

$$\begin{aligned} I_{k,1}(r) &= \left[rf'_{k+1}(x)\cos(rx)\right]_0^\infty - r\int_0^\infty f''_{k+1}(x)\cos(rx)dx \\ &= \begin{cases} -rB_0 + \int_0^\infty f'''_2(x)\sin(rx)dx, & \text{for } k=1, \\ \int_0^\infty f'''_{k+1}(x)\sin(rx)dx, & \text{for } k>1. \end{cases} \end{aligned}$$

Lemma 2.1 implies that

$$\lim_{r\to\infty} I_{k,1}(r) = \begin{cases} -r, & \text{for } k=1, \\ 0, & \text{for } k>1. \end{cases}$$

First, we suppose $k>1$ and let $s = \left[\frac{k+2}{2}\right] \in \mathbb{Z}_{>0}$. Note that s is the largest integer equal or less than $\frac{k+2}{2}$. For $1 \leqslant t \leqslant s$, we have

$$I_{k,t}(r) = r^{2t-k+1}F_k(r) = (-1)^t r^2 \int_0^\infty f_{k+1}^{(2t-1)}(x)\sin(rx)dx.$$

The second equality holds by (11) and (12). For $t=s$, we obtain

$$\begin{aligned} I_{k,s}(r) &= (-1)^{s-1}\left[rf_{k+1}^{(2s-1)}(x)\cos(rx)\right]_0^\infty - (-1)^{s-1}r\int_0^\infty f_{k+1}^{(2s)}(x)\cos(rx)dx \\ &= \begin{cases} (-1)^s\left[r(2s-1)!B_0 - \int_0^\infty f_{k+1}^{(2s+1)}(x)\sin(rx)dx\right], & \text{for } k \text{ is odd,} \\ (-1)^s\left[r(2s-1)!B_1 - \int_0^\infty f_{k+1}^{(2s+1)}(x)\sin(rx)dx\right], & \text{for } k \text{ is even.} \end{cases} \end{aligned}$$

This implies,

$$\begin{aligned} I_{k,s+1}(r) &= \begin{cases} r^{2s-k+3}F_k(r) + (-1)^{s+1}\cdot r^3 k! B_0, & \text{for } k \text{ is odd,} \\ r^{2s-k+3}F_k(r) + (-1)^{s+1}\cdot r^3 (k+1)! B_1, & \text{for } k \text{ is even,} \end{cases} \\ &= (-1)^{s+1} r^2 \int_0^\infty f_{k+1}^{(2s+1)}(x)\sin(rx)dx. \end{aligned}$$

For $t \geqslant s+1$, we have

$$\begin{aligned} I_{k,t}(r) &= r^{2t-k+1}F_k(r) + (-1)^{s+1} r^{2t-2s+1} g_{k,t}(r) \\ &= (-1)^t r^2 \int_0^\infty f_{k+1}^{(2t-1)}(x)\sin(rx)dx, \end{aligned}$$

where

$$g_{k,t}(r)=\begin{cases} k!B_0-\frac{(k+2)!B_2}{2!r^2}+\cdots+(-1)^{t-s-1}\frac{(2t-3)!B_{(2t-k-3)}}{(2t-k-3)!r^{2t-k-3}}, & \text{for } k \text{ is odd,}\\ (k+1)!B_1-\frac{(k+3)!B_3}{3!r^2}+\cdots+(-1)^{t-s-1}\frac{(2t-3)!B_{(2t-k-3)}}{(2t-k-3)!r^{2t-k-4}}, & \text{for } k \text{ is even.}\end{cases}$$

For $t \geqslant s+1$, we have

$$\begin{aligned} I_{k,t}(r) &= (-1)^t r^2 \int_0^\infty f_{k+1}^{(2t-1)}(x)\sin(rx)dx \\ &= (-1)^{t-1}\left[rf_{k+1}^{(2t-1)}(x)\cos(rx)\right]_0^\infty-(-1)^{t-1}r\int_0^\infty f_{k+1}^{(2t)}(x)\cos(rx)dx \\ &= (-1)^t\cdot r\cdot\frac{(2t-1)!B_{2t-k-1}}{(2t-k-1)!}+(-1)^{t+1}\int_0^\infty f_{k+1}^{(2t+1)}(x)\sin(rx)dx. \end{aligned} \tag{14}$$

Equation (14) implies the error estimates

$$\begin{aligned} |I_{k,t}(r)| &< \left|r\cdot\frac{(2t-1)!B_{2t-k-1}}{(2t-k-1)!}\right|+\int_0^\infty\left|f_{k+1}^{(2t+1)}(x)\right|dx \\ &= \left|r\cdot\frac{(2t-1)!B_{2t-k-1}}{(2t-k-1)!}\right|+M_{k,t}, \end{aligned}$$

where $M_{k,t}$ is a constant with respect to k and t. Hence, for $t \geqslant s+1$, we have

$$\left|r^{2s-k}F_k(r)+(-1)^{s+1}g_{k,t}(r)\right|<\frac{\left|r\cdot\frac{(2t-1)!B_{2t-k-1}}{(2t-k-1)!}\right|+M_{k,t}}{r^{2t-2s+1}},$$

which implies

$$r^{2s-k}F_k(r)\sim(-1)^s g_{k,t}(r).$$

When k is odd,

$$rF_k(r)\sim(-1)^s\sum_{i=0}^\infty(-1)^i\frac{(k+2i)!B_{2i}}{(2i)!r^{2i}}.$$

When k is even,

$$r^2F_k(r)\sim(-1)^s\sum_{i=0}^\infty(-1)^i\frac{(k+2i+1)!B_{2i+1}}{(2i+1)!r^{2i}}.$$

This proves Lemma 2.2 for $k>1$.

For $k=1$, suppose we have the relation

$$\begin{aligned} I_{1,t}(r) &= r^{2t-1}\left[rF_1(r)+B_0-\frac{3B_2}{r^2}+\cdots+(-1)^t\frac{(2t-3)B_{2t-4}}{r^{2t-4}}\right] \\ &= (-1)^t r^2\int_0^\infty f_2^{(2t-1)}(x)\sin(rx)dx, \end{aligned} \tag{15}$$

for some $t \geqslant 2$. Then integration by parts yields again

$$I_{1,t}(r) = (-1)^t(2t-1)rB_{2t-2} + (-1)^{t+1}\int_0^\infty f_2^{(2t+1)}(x)\sin(rx)dx. \tag{16}$$

Therefore

$$\begin{aligned} I_{1,t+1}(r) &= r^{2t+1}\left[rF_1(r) + B_0 - \frac{3B_2}{r^2} + \cdots + (-1)^{t+1}\frac{(2t-1)B_{2t-2}}{r^{2t-2}}\right] \\ &= (-1)^{t+1}r^2\int_0^\infty f_2^{(2t+1)}(x)\sin(rx)dx. \end{aligned}$$

Hence (15) is true for all $t \geqslant 2$. Relation (16) implies also the error estimate

$$|I_{1,t}(r)| < |(2t-1)rB_{2t-2}| + M_{1,t},$$

where $M_{1,t}$ is a constant with respect to t. Hence

$$\left|F_1(r) + \frac{B_0}{r} - \frac{3B_2}{r^3} + \cdots + (-1)^{t-2}\frac{(2t-3)rB_{2t-4}}{r^{2t-3}}\right| < \frac{|(2t-1)B_{2t-2}| + M_{1,t}}{r^{2t}}.$$

This proves Lemma 2.2 for $k = 1$. □

Using a similar method, we can prove

LEMMA 2.3. *Let $k \in \mathbb{Z}_{\geqslant 0}$ and $s = \left[\frac{k+1}{2}\right]$. The integration $G_k(r)$ has the asymptotic expansion*

$$G_k(r) \sim \begin{cases} (-1)^s\sum_{i=0}^\infty(-1)^i\frac{(k+2i+1)!B_{2i+1}}{(2i+1)!r^{2i+2}}, & \text{for } k \text{ is odd,} \\ (-1)^s\sum_{i=0}^\infty(-1)^i\frac{(k+2i)!B_{2i}}{(2i)!r^{2i+1}}, & \text{for } k \text{ is even.} \end{cases}$$

Proof. Integrating $G_k(r)$ by parts, we have

$$G_k(r) = r^{k-1}\int_0^\infty f'_{k+1}(x)\cos(rx)dx.$$

Combining with Lemma 2.1 and integrating twice by parts, we have

$$\begin{aligned} J_{k,1}(r) = r^{3-k}G_k(r) &= r^2\int_0^\infty f'_{k+1}(x)\cos(rx)dx \\ &= \left[f''_{k+1}(x)\cos(rx)\right]_0^\infty - \int_0^\infty f'''_{k+1}(x)\cos(rx)dx, \end{aligned}$$

and

$$\lim_{r\to\infty} J_{k,1}(r) = -\lim_{x\to 0} f''_{k+1}(x) = \begin{cases} -\frac{2B_{2-k}}{(2-k)!}, & \text{for } 0 \leqslant k \leqslant 2, \\ 0, & \text{for } 2 < k. \end{cases}$$

First, we suppose $k > 2$. Let $s = \left[\frac{k+1}{2}\right] \in \mathbb{Z}_{>0}$. For $1 \leqslant t \leqslant s$, we obtain

$$J_{k,t}(r) = r^{2t-k+1} G_k(r) = (-1)^{t+1} r^2 \int_0^\infty f_{k+1}^{(2t-1)}(x)\cos(rx)dx. \tag{17}$$

Equation (17) follows by (11) and (12). For $t = s$, we obtain

$$\begin{aligned}
J_{k,s}(r) &= (-1)^{s-1}\left[f_{k+1}^{(2s)}(x)\cos(rx)\right]_0^\infty - (-1)^{s-1}\int_0^\infty f_{k+1}^{(2s+1)}(x)\cos(rx)dx \\
&= \begin{cases} (-1)^s\left[(k+1)!B_1 + \int_0^\infty f_{k+1}^{(2s+1)}(x)\cos(rx)dx\right], & \text{for } k \text{ is odd,} \\ (-1)^s\left[k!B_0 + \int_0^\infty f_{k+1}^{(2s+1)}(x)\cos(rx)dx\right], & \text{for } k \text{ is even.} \end{cases}
\end{aligned}$$

This implies,

$$\begin{aligned}
J_{k,s+1}(r) &= \begin{cases} r^{2s-k+3}G_k(r) + (-1)^{s+1}\cdot r^2(k+1)!B_1, & \text{for } k \text{ is odd,} \\ r^{2s-k+3}G_k(r) + (-1)^{s+1}\cdot r^2 k!B_0, & \text{for } k \text{ is even,} \end{cases} \\
&= (-1)^s r^2 \int_0^\infty f_{k+1}^{(2s+1)}(x)\cos(rx)dx.
\end{aligned}$$

For $t \geqslant s+1$, we have

$$\begin{aligned}
J_{k,t}(r) &= r^{2t-k+1}G_k(r) + (-1)^{s+1} r^{2t-2s} h_{k,t}(r) \\
&= (-1)^{t+1} r^2 \int_0^\infty f_{k+1}^{(2t-1)}(x)\cos(rx)dx \\
&= (-1)^{t-1}\left[f_{k+1}^{(2t)}(x)\cos(rx)\right]_0^\infty - (-1)^{t-1}\int_0^\infty f_{k+1}^{(2t+1)}(x)\cos(rx)dx \\
&= (-1)^t \frac{(2t)!B_{2t-k}}{(2t-k)!} + (-1)^{t+2}\int_0^\infty f_{k+1}^{(2t+1)}(x)\cos(rx)dx,
\end{aligned} \tag{18}$$

where

$$h_{k,t}(r) = \begin{cases} (k+1)!B_1 - \frac{(k+3)!B_3}{3!r^2} + \cdots + (-1)^{t-s-1}\frac{(2t-2)!B_{(2t-k-2)}}{(2t-k-2)!r^{2t-k-3}}, & \text{for } k \text{ is odd,} \\ k!B_0 - \frac{(k+2)!B_2}{2!r^2} + \cdots + (-1)^{t-s-1}\frac{(2t-2)!B_{(2t-k-2)}}{(2t-k-2)!r^{2t-k-2}}, & \text{for } k \text{ is even.} \end{cases}$$

Equation (18) implies the error estimates

$$|J_{k,t}(r)| < \left|\frac{(2t)!B_{2t-k}}{(2t-k)!}\right| + \int_0^\infty f_{k+1}^{(2t+1)}(x)\sin(rx)dx = \left|\frac{(2t)!B_{2t-k}}{(2t-k)!}\right| + N_{k,t},$$

where $N_{k,t}$ is a constant with respect to k and t. Hence, for $t \geqslant s+1$, we have

$$\left|r^{2s-k+1}G_k(r) + (-1)^{s+1}h_{k,t}(r)\right| < \frac{\left|\frac{(2t)!B_{2t-k}}{(2t-k)!}\right| + N_{k,t}}{r^{2t-2s}},$$

which implies

$$r^{2s-k+1}G_k(r) \sim (-1)^s h_{k,t}(r).$$

When k is odd,

$$r^2 G_k(r) \sim (-1)^s \sum_{i=0}^{\infty} (-1)^i \frac{(k+2i+1)!B_{2i+1}}{(2i+1)!r^{2i}}.$$

When k is even,

$$rG_k(r) \sim (-1)^s \sum_{i=0}^{\infty} (-1)^i \frac{(k+2i)!B_{2i}}{(2i)!r^{2i}}.$$

This proves Lemma 2.3 for $k > 2$.

When $k = 0$, Elbert proved Lemma 2.3 in reference [12]. For $k = 1$, suppose we have the relation

$$\begin{aligned} J_{1,t}(r) &= r^{2t}\left[G_1(r) + \frac{2B_1}{r^2} + \cdots + (-1)^t \frac{(2t-2)B_{2t-3}}{r^{2t-2}}\right] \\ &= (-1)^{t-1} r^2 \int_0^\infty f_2^{(2t-1)}(x)\cos(rx)dx \end{aligned} \tag{19}$$

for some $t \geqslant 2$. Then integration by parts yields again

$$J_{1,t}(r) = (-1)^t(2t)B_{2t-1} + (-1)^t \int_0^\infty f_2^{(2t+1)}(x)\cos(rx)dx. \tag{20}$$

Therefore

$$\begin{aligned} J_{1,t+1}(r) &= r^{2t+2}\left[G_1(r) + \frac{2B_1}{r^2} + \cdots + (-1)^{t+1} \frac{2tB_{2t-1}}{r^{2t}}\right] \\ &= (-1)^t r^2 \int_0^\infty f_2^{(2t+1)}(x)\cos(rx)dx. \end{aligned}$$

Hence (19) is true for all $t \geqslant 2$. Relation (20) implies also the error estimate

$$|J_{1,t}(r)| < |2tB_{2t-1}| + N_{1,t},$$

where $N_{1,t}$ is a constant with respect to t. Hence

$$\left| G_1(r) + \frac{2B_1}{r^2} + \cdots + (-1)^t \frac{(2t-2)B_{2t-3}}{r^{2t-2}} \right| < \frac{|2tB_{2t-1}| + N_{1,t}}{r^{2t}}.$$

This proves Lemma 2.3 for $k = 1$.

For $k = 2$, suppose we have the relation

$$\begin{aligned} J_{2,t}(r) &= r^{2t-1}\left[G_2(r) + \frac{2B_0}{r} + \cdots + (-1)^t \frac{(2t-2)(2t-3)B_{2t-4}}{r^{2t-3}}\right] \\ &= (-1)^{t-1} r^2 \int_0^\infty f_3^{(2t-1)}(x)\cos(rx)dx, \end{aligned} \tag{21}$$

for some $t \geqslant 2$. Then integration by parts yields again

$$J_{2,t}(r) = (-1)^t(2t)(2t-1)B_{2t-2} + (-1)^t \int_0^\infty f_3^{(2t+1)}(x)\cos(rx)dx. \tag{22}$$

Therefore

$$\begin{aligned} J_{2,t+1}(r) &= r^{2t+1}\left[G_2(r) + \frac{2B_0}{r} + \cdots + (-1)^{t+1}\frac{(2t)(2t-1)B_{2t-2}}{r^{2t-1}}\right] \\ &= (-1)^t r^2 \int_0^\infty f_3^{(2t+1)}(x)\cos(rx)dx. \end{aligned}$$

Hence (21) is true for all $t \geqslant 2$. Relation (22) implies also the error estimate

$$|J_{2,t}(r)| < |2t(2t-1)B_{2t-1}| + N_{2,t},$$

where $N_{2,t}$ is a constant with respect to t. Hence

$$\left|G_2(r) + \frac{2B_0}{r} + \cdots + (-1)^t\frac{(2t-2)(2t-3)B_{2t-4}}{r^{2t-3}}\right| < \frac{|2t(2t-1)B_{2t-1}| + N_{2,t}}{r^{2t-1}}.$$

This proves Lemma 2.3 for $k = 2$. $\square$

LEMMA 2.4. *For $m, i \in \mathbb{Z}_{>0}$, we have*

$$\binom{i+m}{m} = \frac{1}{4^m}\sum_{k=0}^{m} 2^k \binom{k+2i}{k}\binom{2m-k}{m}.$$

Proof. We shall prove Lemma 2.4 using the Gauss hypergeometric function, which is defined by

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, x\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial):

$$(a)_\mu := \frac{\Gamma(a+\mu)}{\Gamma(a)} = \begin{cases} 1, & \text{if } \mu = 0;\ a \in \mathbb{C} \setminus \{0\}, \\ a(a+1)\cdots(a+n-1), & \text{if } \mu = n \in \mathbb{N};\ a \in \mathbb{C}. \end{cases}$$

Note that $(0)_0 := 1$. Then we have

$$\sum_{k=0}^{m} 2^k \binom{k+2i}{k}\binom{2m-k}{m} = \binom{2m}{m}\, {}_2F_1\left[\begin{matrix} -m, 1+2i \\ -2m \end{matrix}\,\middle|\, 2\right]. \tag{23}$$

By Pfaff transform [2, p. 79, Eq. (2.3.14)], we have

$${}_2F_1\left[\begin{matrix} -m, 1+2i \\ -2m \end{matrix}\,\middle|\, 2\right] = \frac{(-2m-2i-1)_m}{(-2m)_m}\, {}_2F_1\left[\begin{matrix} -m, 1+2i \\ m+2i+2 \end{matrix}\,\middle|\, -1\right]. \tag{24}$$

By Euler transform [2, p. 68, Eq. (2.2.7)] and Kummer transform [2, p. 126, Cor. 3.1.2], we have

$$\begin{aligned} {}_2F_1\left[\begin{matrix}-m, 1+2i\\ m+2i+2\end{matrix}\,\middle|\,-1\right] &= 2^{2m+1}\,{}_2F_1\left[\begin{matrix}2m+2i+2, 1+m\\ m+2i+2\end{matrix}\,\middle|\,-1\right] \\ &= 2^{2m+1}\frac{\Gamma(m+2i+2)\Gamma(m+i+2)}{\Gamma(2m+2i+3)\Gamma(i+1)}. \end{aligned} \tag{25}$$

Combining equation (23)-(25), we have

$$\begin{aligned} \sum_{k=0}^{m} 2^k \binom{k+2i}{k}\binom{2m-k}{m} &= 2^{2m+1}\binom{2m}{m}\frac{(-2m-2i-1)_m(i+1)_{m+1}}{(-2m)_m(m+2i+2)_{m+1}} \\ &= 2^{2m}\binom{i+m}{m}. \end{aligned}$$

This completes the proof of Lemma 2.4. □

3. Proof of Theorems

In this section, we prove Theorem 1.1 and 1.2.

Proof of Theorem 1.1. For $m \in \mathbb{Z}_{>0}$, Cerone and Lenard [6, p. 5, Eq. (2.15)] proved the integral representation as follow

$$S_m(r) = \frac{1}{2^{m-1}r^{2m-1}m!}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{x}{e^x-1}(rx)^{m-\frac{1}{2}}J_{m-\frac{1}{2}}(rx)dx,$$

where $J_m(z)$ is the m-th order Bessel function of the first kind. By the properties of $J_m(z)$, we can deduce that, for m is odd,

$$\begin{aligned} S_m(r) = &\frac{(-1)^{\frac{m-1}{2}}}{2^{m-1}r^{2m-1}m!}\left\{\sum_{k=0}^{\frac{m-1}{2}}\frac{(-1)^k(m+2k-1)!}{(2k)!(m-2k-1)!2^{2k}}G_{m-2k-1}(r)\right. \\ &\left. + \sum_{k=0}^{\frac{m-3}{2}}\frac{(-1)^k(m+2k)!}{(2k+1)!(m-2k-2)!2^{2k+1}}F_{m-2k-2}(r)\right\} \\ = &\frac{1}{2^{2m-2}r^{2m-1}m!}\left\{\sum_{k=0}^{\frac{m-1}{2}}\frac{(-1)^k(2m-2k-2)!\cdot 2^{2k}}{(m-2k-1)!(2k)!}G_{2k}(r)\right. \\ &\left. - \sum_{k=0}^{\frac{m-3}{2}}\frac{(-1)^k(2m-2k-3)!\cdot 2^{2k+1}}{(m-2k-2)!(2k+1)!}F_{2k+1}(r)\right\}. \end{aligned} \tag{26}$$

For m is even, we have

$$\begin{aligned} S_m(r) = & \frac{(-1)^{\frac{m}{2}}}{2^{m-1}r^{2m-1}m!}\left\{\sum_{k=0}^{\frac{m-2}{2}} \frac{(-1)^k(m+2k-1)!}{(2k)!(m-2k-1)!2^{2k}}F_{m-2k-1}(r)\right. \\ & \left. -\sum_{k=0}^{\frac{m-2}{2}} \frac{(-1)^k(m+2k)!}{(2k+1)!(m-2k-2)!2^{2k+1}}G_{m-2k-2}(r)\right\} \\ = & \frac{-1}{2^{2m-2}r^{2m-1}m!}\left\{\sum_{k=0}^{\frac{m-2}{2}} \frac{(-1)^k(2m-2k-3)!\cdot 2^{2k+1}}{(m-2k-2)!(2k+1)!}F_{2k+1}(r)\right. \\ & \left. -\sum_{k=0}^{\frac{m-2}{2}} \frac{(-1)^k(2m-2k-2)!\cdot 2^{2k}}{(m-2k-1)!(2k)!}G_{2k}(r)\right\}. \end{aligned} \tag{27}$$

Combining equation (26) and (27), for $m\in\mathbb{Z}_{>0}$, we have

$$S_m(r) = \frac{1}{2^{2m-2}r^{2m-1}m!}\sum_{k=0}^{m-1} \frac{(-1)^{\left[\frac{k+1}{2}\right]}(2m-k-2)!\cdot 2^k}{(m-k-1)!\cdot k!}A_k(r), \tag{28}$$

where

$$A_k(r) = \begin{cases} F_k(r), & \text{for } k \text{ is odd,} \\ G_k(r), & \text{for } k \text{ is even.} \end{cases}$$

By Lemma 2.2, for $k\geqslant 1$ is odd, we have

$$F_k(r) \sim (-1)^{\frac{k+1}{2}}\sum_{i=0}^{\infty}(-1)^i\frac{(k+2i)!B_{2i}}{(2i)!r^{2i+1}}.$$

By Lemma 2.3, for $k\geqslant 0$ is even, we have

$$G_k(r) \sim (-1)^{\frac{k}{2}}\sum_{i=0}^{\infty}(-1)^i\frac{(k+2i)!B_{2i}}{(2i)!r^{2i+1}}.$$

By Lemma 2.4, for $m\in\mathbb{Z}_{>0}$, we have

$$\begin{aligned} S_m(r) \sim & \frac{1}{2^{2m-2}r^{2m-1}m!}\sum_{k=0}^{m-1}\sum_{i=0}^{\infty} \frac{(-1)^i\cdot 2^k\cdot(2m-k-2)!\cdot(k+2i)!\cdot B_{2i}}{(m-k-1)!\cdot k!\cdot(2i)!\cdot r^{2i+1}} \\ \sim & \frac{1}{2^{2m-2}\cdot m}\sum_{k=0}^{m-1}\sum_{i=0}^{\infty} \frac{(-1)^i\cdot 2^k\cdot\binom{k+2i}{k}\cdot\binom{2m-k-2}{m-1}\cdot B_{2i}}{r^{2m+2i}} \\ \sim & \frac{1}{m}\sum_{i=0}^{\infty}\frac{(-1)^i\binom{i+m-1}{m-1}B_{2i}}{r^{2m+2i}}. \end{aligned}$$

This proves Theorem 1.1. □

Proof of Theorem 1.2. Theorem 1.2 follows from equation (9) and (28). □

REMARK. Theorem 1.2 modified the integral representation given by [6, p. 6, Thm. 2.5]. In reference [6, p. 5, Eq. (2.15)], Cerone and Lenard present an integral representation for $S_m(r)$ in terms of Bessel function, where $m \in \mathbb{Z}_{>0}$. When further simplified [6, p. 5, Eq. (2.15)] to [6, p. 6, Thm. 2.5], a mistake occurred in [6, p. 6, Eq. (2.20)], which says

$$S_1^{(m-1)}(r) = (-1)^{m-1} m! (2r)^{m-1} S_m(r). \tag{29}$$

Equation (29) is wrong when $m \geqslant 3$, since $S_1''(r) = -4S_2(r) + 24r^2 S_3(r)$.

Acknowledgement. The authors would like to thank Chen Wang for his detailed comments in the proof of Lemma 2.4. The authors would also like to thank the referee for helpful corrections. X. Lin is supported by the National Natural Science Foundation of China (No. 12301010).

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(Received January 12, 2023)

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