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# SHARP BOUNDS ON THE HANKEL DETERMINANT OF THE INVERSE FUNCTIONS FOR CERTAIN ANALYTIC FUNCTIONS

LEI SHI, MUHAMMAD ARIF*, H. M. SRIVASTAVA AND MUHAMMAD IHSAN

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*Abstract.* In most cases, the problem of finding bounds for the inverse function is much more difficult than finding bounds for the function itself. Thus, there are relatively little sharp bounds of Hankel determinant on the inverse functions. In the present paper, we introduce a subclass of bounded turning function $\mathscr{R}_{car}$ associated with a cardioid domain. The purpose of this article is to investigate certain coefficient related problems on the inverse functions for $f \in \mathscr{R}_{car}$. The bounds of some initial coefficients, the Fekete-Szegö type inequality and the estimation of Hankel determinants of second and third order are obtained. All of these bounds are proved to be sharp.

## 1. Introduction and definitions

Before starting to investigate the main problems, we provide some elementary function theories in literature. In this paper, the letters $\mathscr{A}$ and $\mathscr{S}$ are represented for the classes of normalised analytic and univalent functions, respectively. These classes are defined in the set-builder form of

$$\mathscr{A} := \left\{ f \in \Pi(\mathbb{D}) : f(0) = f'(0) - 1 = 0, \quad z \in \mathbb{D} \right\}$$

and

$$\mathscr{S} := \{ f \in \mathscr{A} : f \text{ is univalent in } \mathbb{D} \}.$$

Here, $\Pi(\mathbb{D})$ stands for the set of analytic functions defined in the region

$$\mathbb{D} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

That is to say, if $f \in \mathscr{A}$, then it can be expressed in the series expansion of

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

In 1916, the famous coefficient problem stated by Bieberbach in [5] contributed to this field's development as a viable new study subject. De Branges [7] solved this renowned

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* Corresponding author.


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conjecture in 1985 by establishing that if $f \in \mathscr{S}$, then $|a_n| \leqslant n$ for $n \geqslant 2$, with the equality holds if $f$ is a Koebe function or its rotation. The Koebe function is given by

$$K(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n.$$

From 1916 through 1985, several of the world's most eminent intellectuals attempted to validate or refute this claim. As a response, they found many sub-collections of $\mathscr{S}$ that are linked to different image domains. The most fundamental, well-studied, and elegant geometric interpretations of these subfamilies are the families of starlike $\mathscr{S}^*$ and convex $\mathscr{K}$ functions, which are stated as

$$\mathscr{S}^* := \left\{ f \in \mathscr{A} : \Re \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D} \right\},$$
$$\mathscr{K} := \left\{ f \in \mathscr{A} : \Re \frac{(zf'(z))'}{f'(z)} > 0, \quad z \in \mathbb{D} \right\}.$$

These functions are closely related to the class $\mathscr{P}$ defined in term of set-builder notation of

$$\mathscr{P} := \{ p \in \mathscr{A} : \quad \Re p(z) > 0, \quad z \in \mathbb{D} \},$$

where the function $p$ has the series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

Let $\alpha \in [0,1)$. We denote by $\mathscr{R}(\alpha)$ the subclass of functions $f \in \mathscr{A}$ such that

$$\Re f'(z) > \alpha, \quad z \in \mathbb{D}. \tag{3}$$

Functions in $\mathscr{R}(\alpha)$ are called of bounded turning of order $\alpha$ and in $\mathscr{R} := \mathscr{R}(0)$ of bounded turning, see [10].

The theory of univalent functions with a firm basis from the family $\mathscr{S}$ is interesting when geometric and analytic concerns are both taken into account. The $1/4$-theorem of Koebe ensures that for any univalent function $f$ in $\mathbb{D}$, its inverse $f^{-1}$ exists at least on a disc of radius $1/4$ with the Taylor's series representation

$$f^{-1}(w) := w + \sum_{n=2}^{\infty} B_n w^n, \quad |w| < 1/4. \tag{4}$$

Utilizing the representation $f\left(f^{-1}(w)\right) = w$, we obtain

$$B_2 = -a_2, \tag{5}$$
$$B_3 = -a_3 + 2a_2^2, \tag{6}$$
$$B_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{7}$$
$$B_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \tag{8}$$

In 1923, Löwner [24] developed the renowned parametric approach to obtain the Bieberbach conjecture for the third coefficient. In recent years, a great deal of interest had been shown on the inverse function, where the relevant function $f$ belongs to some specific subfamilies of univalent functions. For instance, Krzyz *et al.* [16] determined the upper bounds of the initial coefficient contained in the inverse function $f^{-1}$ when $f \in \mathscr{S}^*(\alpha)$ with $0 \leqslant \alpha < 1$. These findings were improved later by Kapoor and Mishra in [13]. Also, for the class $\mathscr{S}\mathscr{S}^*(\xi)$ $(0 < \xi \leqslant 1)$ of strongly starlike function, Ali [2] investigated the sharp bounds of the first four initial coefficient along with sharp estimate of Fekete-Szegö coefficient functional of the inverse function. For more contributions in this specific direction, see the articles by Juneja and Rajasekaran [12], Ponnusamy *et al.* [28], Silverman [33], and Sim and Thomas [34].

The Hankel determinant $\mathscr{H}_{q,n}(f)$, for $q,n \in \mathbb{N} = \{1,2,\cdots\}$, containing coefficients of the function $f \in \mathscr{S}$

$$\mathscr{H}_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix},$$

was examined by Pommerenke [26, 27]. By varying the parameters $q$ and $n$, we get the determinants given by

$$\mathscr{H}_{2,1}(f) = a_3 - a_2^2, \tag{9}$$

$$\mathscr{H}_{2,2}(f) = a_2a_4 - a_3^2, \tag{10}$$

$$\mathscr{H}_{3,1}(f) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5. \tag{11}$$

They are referred as first, second and third order Hankel determinants, respectively. There are indeed a little works in the literature that address and investigate the sharp bounds of Hankel determinants for functions in the general family $\mathscr{S}$. The first contributed sharp inequality for the function $f \in \mathscr{S}$ is $|\mathscr{H}_{2,n}(f)| \leqslant |\nu|\sqrt{n}$, where $\nu$ is constant. This result is due to Hayman [11]. Further for the same class $\mathscr{S}$, it was obtained in [25] that

$$|\mathscr{H}_{2,2}(f)| \leqslant \lambda, \quad 1 \leqslant \lambda \leqslant \frac{11}{3},$$

$$|\mathscr{H}_{3,1}(f)| \leqslant \mu, \quad \frac{4}{9} \leqslant \mu \leqslant \frac{32+\sqrt{285}}{15}.$$

The problems of researching the Hankel determinants sharp bounds for a certain class of complex valued functions has piqued the interest of many field specialists. The exact bound of second Hankel determinant for the collection $\mathscr{S}^*(\phi)$ of starlike functions (Ma-Minda) was found in [23], and further studied in [8].

To obtain the bounds of $|\mathscr{H}_{3,1}(f)|$ is significantly more difficult. Babalola [4] studied third Hankel determinant for the families of $\mathscr{K}$, $\mathscr{S}^*$ and $\mathscr{R}$. For more references in this field, see [6, 19, 29, 31, 35, 36, 37, 38]. In 2018, Kowalczyk *et al.* [17]

and Lecko *et al.* [21] achieved sharp bounds of $|\mathscr{H}_{3,1}(f)|$ for the collections $\mathscr{K}$ and $\mathscr{S}^*\left(\frac{1}{2}\right)$, respectively. The have obtained that

$$|\mathscr{H}_{3,1}(f)| \leqslant \begin{cases} \frac{4}{135}, & f \in \mathscr{K}, \\ \frac{1}{9}, & f \in \mathscr{S}^*\left(\frac{1}{2}\right). \end{cases}$$

In 2022, the third Hankel determinant of starlike functions was proved to $\frac{4}{9}$, see [18]. In most cases, the problem of finding bounds for the inverse function is much more difficult than finding bounds for the function itself. Thus, there are relatively little sharp bounds of Hankel determinant on the inverse functions, see [3, 15, 32].

In [30], Kanika Sharma, Naveen Kumar Jain and V. Ravichandran introduced an subclass of starlike functions $\mathscr{S}^*_{car}$ defined by

$$\mathscr{S}^*_{car} := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathbb{D} \right\}.$$

For function in this class, it means that $\frac{zf'(z)}{f(z)}$ lying in the region bounded by the cardioid given by the equation

$$(9x^2+9y^2-18x+5)^2-16(9x^2+9y^2-6x+1)=0. \tag{12}$$

Later, its properties were intensively studied in [1, 9, 22].

Motivated by the above works, we introduce a subclass of bounded turning functions $\mathscr{R}_{car}$ defined by

$$\mathscr{R}_{car} := \left\{ f \in \mathscr{S} : \quad f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathbb{D} \right\}. \tag{13}$$

The goal of this paper is to compute the sharp bounds of coefficient results, Fekete-Szegö type problems, and Hankel determinants of second and third order for the inverse functions of this class.

## 2. A set of lemmas

To prove our main results, we need the following Lemmas.

LEMMA 1. (see [20]) *Let $p \in \mathscr{P}$ be given by (2). Then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{14}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\delta, \tag{15}$$

$$\begin{aligned} 8c_4 = {} & c_1^4 + (4 - c_1^2)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4(4 - c_1^2)(1 - |x|^2) \\ & \times \left[c_1(x-1)\delta + \overline{x}\delta^2 - (1 - |\delta|^2)\rho\right]. \end{aligned} \tag{16}$$

*for some $x, \delta, \rho \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$.*

LEMMA 2. (see [14]) *Let* $\mu \in \mathbb{C}$. *If* $p \in \mathscr{P}$ *is represented as* (2), *then*

$$|c_{n+k} - \mu c_n c_k| \leqslant 2\max(1, |2\mu - 1|), \tag{17}$$

$$|c_n| \leqslant 2, \quad n \geqslant 1. \tag{18}$$

## 3. Coefficient inequalities for the class $\mathscr{R}_{car}$

We start by calculating the first two initial coefficients bounds for $f^{-1} \in \mathscr{R}_{car}$.

THEOREM 1. *Let* $f \in \mathscr{R}_{car}$ *be the form of* (1). *Then*

$$|B_2| \leqslant \frac{2}{3} \quad and \quad |B_3| \leqslant \frac{2}{3}. \tag{19}$$

*These bounds are sharp with the extremal functions given by*

$$f(z) = \int_0^z \left(1 + \frac{4}{3}t + \frac{2}{3}t^2\right) dt = z + \frac{2}{3}z^2 + \frac{2}{9}z^3, \tag{20}$$

*whose inverse function can be written as*

$$f^{-1}(w) = w - \frac{2}{3}w^2 + \frac{2}{3}w^3 - \frac{20}{27}w^4 + \cdots, \quad |w| < \frac{1}{4}. \tag{21}$$

*Proof.* From the definition of the class $\mathscr{R}_{car}$ along with subordination principal, there exist a Schwarz function $\omega$ such that

$$f'(z) = 1 + \frac{4}{3}\omega(z) + \frac{2}{3}(\omega(z))^2 := \chi(z), \quad z \in \mathbb{D}.$$

Let

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{22}$$

Clearly, we have $p \in \mathscr{P}$ and

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}.$$

By simplifications and using the series expansion of $\omega$, we get

$$\begin{aligned} \chi(z) = 1 &+ \frac{2}{3}c_1 z + \left(\frac{2}{3}c_2 - \frac{1}{6}c_1^2\right) z^2 + \left(\frac{2}{3}c_3 - \frac{1}{3}c_1 c_2\right) z^3 \\ &+ \left(\frac{2}{3}c_4 - \frac{1}{6}c_2^2 + \frac{1}{24}c_1^4 - \frac{1}{3}c_1 c_3\right) z^4 + \cdots. \end{aligned} \tag{23}$$

Using (1), it is seen that

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{24}$$

By comparing (23) and (24) we get

$$a_2 = \frac{1}{3}c_1, \tag{25}$$

$$a_3 = \frac{1}{3}\left(\frac{2}{3}c_2 - \frac{1}{6}c_1^2\right), \tag{26}$$

$$a_4 = \frac{1}{4}\left(\frac{2}{3}c_3 - \frac{1}{3}c_1c_2\right), \tag{27}$$

$$a_5 = \frac{1}{5}\left(\frac{2}{3}c_4 - \frac{1}{6}c_2^2 + \frac{1}{24}c_1^4 - \frac{1}{3}c_1c_3\right). \tag{28}$$

Substituting (25), (26), (27) and (28) into (5), (6) , (7) and (8), we get

$$B_2 = -\frac{1}{3}c_1, \tag{29}$$

$$B_3 = -\frac{2}{9}\left(c_2 - \frac{5}{4}c_1^2\right), \tag{30}$$

$$B_4 = -\frac{1}{6}\left(c_3 - \frac{49}{18}c_1c_2 - \frac{5}{3}c_1^3\right), \tag{31}$$

$$B_5 = -\frac{2}{15}\left(c_4 - 3c_1c_3 - \frac{49}{36}c_2^2 + \frac{205}{36}c_1^2c_2 - \frac{983}{432}c_1^4\right). \tag{32}$$

For the bounds of $B_2$ and $B_3$, it follows directly from Lemma 2. This completes the proof of Theorem 1. $\square$

Now we examine the Fekete-Szegö type result for the inverse function of $f \in \mathscr{R}_{car}$.

THEOREM 2. *Let $\gamma \in \mathbb{C}$. If $f \in \mathscr{R}_{car}$ is in the form of* (1) *, then*

$$\left|B_3 - \gamma B_2^2\right| \leqslant \max\left\{\frac{4}{9}, \frac{2}{9}\left|3 - 2\gamma\right|\right\}.$$

*This inequality is sharp and can be obtained from the extremal function defined by*

$$f(z) = \int_0^z \left(1 + \frac{4}{3}t^2 + \frac{2}{3}t^4\right) dt = z + \frac{4}{9}z^3 + \frac{2}{15}z^5. \tag{33}$$

*The inverse of $f$ is given by*

$$f^{-1}(w) = w - \frac{4}{9}w^3 + \frac{62}{135}w^5 + \cdots, \quad |w| < \frac{1}{4}. \tag{34}$$

*Proof.* From (29) and (30), we easily obtain that

$$\left|B_3 - \gamma B_2^2\right| = \frac{2}{9}\left|c_2 - \left(\frac{5}{4} - \frac{1}{2}\gamma\right)c_1^2\right|.$$

An application of (17) leads to

$$\left|B_3-\gamma B_2^2\right| \leqslant \max\left\{\frac{4}{9}, \frac{2}{9}\left|3-2\gamma\right|\right\}.$$

This completes the proof. $\square$

Putting $\gamma=1$, we get the below inequality.

COROLLARY 1. *If $f\in\mathscr{R}_{car}$ is in the form of (1), then*

$$\left|B_3-B_2^2\right| \leqslant \frac{4}{9}.$$

In the following we will discuss the second order Hankel determinant for the inverse function of $f\in\mathscr{R}_{car}$.

THEOREM 3. *If $f\in\mathscr{R}_{car}$ is in the form of (1), then*

$$\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right| \leqslant \frac{16}{81}.$$

*The inequality is sharp and can be obtained by the function defined by*

$$f(z)=z+\frac{4}{9}z^3+\frac{2}{15}z^5, \quad z\in\mathbb{D}. \tag{35}$$

*The inverse function is given by*

$$f^{-1}(w)=w-\frac{4}{9}w^3+\frac{62}{135}w^5+\cdots, \quad |w|<\frac{1}{4}. \tag{36}$$

*Proof.* The determinant $\mathscr{H}_{2,2}\left(f^{-1}\right)$ can be described as

$$\mathscr{H}_{2,2}\left(f^{-1}\right)=B_2B_4-B_3^2.$$

From (29), (30) and (31), we have

$$\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right|=\frac{1}{324}\left|5c_1^4-9c_1^2c_2+18c_1c_3-16c_2^2\right|.$$

Let $f_\theta(z):=e^{-i\theta}f(e^{i\theta}z), \theta\in\mathbb{R}$. It is observed that

$$\mathscr{H}_{2,2}\left(f^{-1}\right)=e^{4i\theta}\mathscr{H}_{2,2}\left(f^{-1}\right).$$

Thus $\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right|$ is rotation invariant for $f\in\mathscr{R}_{car}$, we may assume $c_1=c\in[0,2]$. Using (14) and (15) to express $c_2$ and $c_3$ in terms of $c_1=c$, we obtain

$$\begin{aligned}\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right|=\frac{1}{324}\bigg|&c^4-\frac{9}{2}c^2x^2\left(4-c^2\right)-\frac{7}{2}c^2x\left(4-c^2\right)\\&+9c\left(4-c^2\right)\left(1-|x|^2\right)\delta-4x^2\left(4-c^2\right)^2\bigg|.\end{aligned}$$

After implementing the triangle inequality and replacing $|\delta| \leqslant 1,\ |x| = t,$ we achieve

$$\begin{aligned}\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right| \leqslant \frac{1}{324}\Bigg[& c^4 + \frac{9}{2}c^2t^2\left(4-c^2\right) + \frac{7}{2}c^2t\left(4-c^2\right) \\ & + 9c\left(4-c^2\right)\left(1-t^2\right) + 4t^2\left(4-c^2\right)^2\Bigg] =: \Lambda(c,t).\end{aligned}$$

Differentiating about the parameter $t,$ we have

$$\frac{\partial \Lambda}{\partial t} = \frac{1}{324}\left[\left(c^2 - 18c + 32\right)\left(4-c^2\right)t + \frac{7}{2}c^2\left(4-c^2\right)\right].$$

It is a straightforward task to illustrate that $\frac{\partial \Lambda}{\partial t} \geqslant 0$ on $t \in [0,1]$ and hence $\Lambda(c,t) \leqslant \Lambda(c,1)$. Setting $t = 1$ gives

$$\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right| \leqslant \frac{1}{324}\left[c^4 + 8c^2\left(4-c^2\right) + 4\left(4-c^2\right)^2\right] =: \eta(c).$$

Also $\eta'(c) \leqslant 0$ shows that $\eta(c)$ is decreasing on $c \in [0,2]$. Hence, the maximum value of $\eta$ is $64/324$ achieved on $c = 0$. Therefore, we obtain

$$\left|\mathscr{H}_{2,2}\left(f^{-1}\right)\right| \leqslant \frac{64}{324} = \frac{16}{81}.$$

The proof is thus completed. □

## 4. Third Hankel determinant for the class $\mathscr{R}_{car}$

THEOREM 4. *If $f \in \mathscr{R}_{car}$ is described by* (1)*, then*

$$\left|\mathscr{H}_{3,1}\left(f^{-1}\right)\right| \leqslant \frac{424}{3645}.$$

*This inequality is sharp with the extremal function given by*

$$f(z) = z + \frac{4}{9}z^3 + \frac{2}{15}z^5, \quad z \in \mathbb{D}, \tag{37}$$

*whose inverse function can be expressed as*

$$f^{-1}(w) = w - \frac{4}{9}w^3 + \frac{62}{135}w^5 + \cdots, \quad |w| < \frac{1}{4}. \tag{38}$$

*Proof.* The determinant $\mathscr{H}_{3,1}\left(f^{-1}\right)$ can be expressed as

$$\mathscr{H}_{3,1}\left(f^{-1}\right) = 2B_2B_3B_4 - B_3^3 - B_4^2 + B_3B_5 - B_2^2B_5.$$

As $\left|\mathscr{H}_{3,1}(f^{-1})\right|$ is also rotation invariant for $f \in \mathscr{R}_{car}$, we still assume $c_1 = c \in [0,2]$. By the virtue of (29) , (30), (31) and (32) along with $c_1 = c,$ we get

$$\mathscr{H}_{3,1}\left(f^{-1}\right) = \frac{1}{58320}\left(199c^6 - 912c^4c_2 + 288c^3c_3 + 1119c^2c_2^2 - 1296c^2c_4 \right.$$
$$\left. + 2196cc_2c_3 - 1712c_2^3 + 1728c_2c_4 - 1620c_3^2\right). \tag{39}$$

Let $b = 4 - c^2$. Using (14) , (15) and (16) along by straightforward algebraic computations, we have

$$\begin{aligned}\mathscr{H}_{3,1}\left(f^{-1}\right) = \frac{1}{58320}\Big\{ & 432b^2x^3 - 214b^3x^3 - 216c^2bx^2 - 54c^4bx^3 + 18c^4bx^2 \\ & - 30c^4bx + \frac{27}{4}c^2b^2x^4 - \frac{387}{2}c^2b^2x^3 - 405b^2\left(1-|x|^2\right)^2\delta^2 \\ & + 72c^3b\left(1-|x|^2\right)\delta + 216c^3bx\left(1-|x|^2\right)\delta + 216c^2b\overline{x}\left(1-|x|^2\right)\delta^2 \\ & - 216c^2b\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho - 27cb^2x^2\left(1-|x|^2\right)\delta \\ & - 432b^2|x|^2\left(1-|x|^2\right)\delta^2 + 171cb^2x\left(1-|x|^2\right)\delta \\ & + 432b^2x\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho\Big\}.\end{aligned}$$

It can be seen that

$$\mathscr{H}_{3,1}\left(f^{-1}\right) = \frac{1}{58320}\left[v_1(c,x) + v_2(c,x)\delta + v_3(c,x)\delta^2 + \Psi(c,x,\delta)\rho\right],$$

where $x, \delta, \rho \in \overline{\mathbb{D}}$, and

$$\begin{aligned} v_1(c,x) &= \left(4-c^2\right)\Big[\left(4-c^2\right)\left(-424x^3 + \frac{41}{2}c^2x^3 + \frac{27}{4}c^2x^4 + \frac{423}{4}c^2x^2\right) \\ &\qquad - 216c^2x^2 - 54c^4x^3 + 18c^4x^2 - 30c^4x\Big], \\ v_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(171cx - 27cx^2\right) + 216c^3x + 72c^3\right], \\ v_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-27|x|^2 - 405\right) + 216c^2\overline{x}\right], \\ \Psi(c,x,\delta) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\left[-216c^2 + 432x\left(4-c^2\right)\right].\end{aligned}$$

With the use of $|x| = t,\ |\delta| = y$ along with $|\rho| \leqslant 1,$ we obtain

$$\begin{aligned}\left|\mathscr{H}_{3,1}\left(f^{-1}\right)\right| &\leqslant \frac{1}{58320}\left[|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2 + |\Psi(c,x,\delta)|\right]. \\ &\leqslant \frac{1}{58320}\Gamma(c,t,y),\end{aligned} \tag{40}$$

where

$$\Gamma(c,t,y) = \sigma_1(c,t) + \sigma_2(c,t)y + \sigma_3(c,t)y^2 + \sigma_4(c,t)\left(1-y^2\right), \tag{41}$$

with

$$\begin{aligned}
\sigma_1(c,t) &= \left(4-c^2\right)\Bigg[\left(4-c^2\right)\left(424t^3 + \frac{41}{2}c^2t^3 + \frac{27}{4}c^2t^4 + \frac{423}{4}c^2t^2\right) \\
&\qquad + 216c^2t^2 + 54c^4t^3 + 18c^4t^2 + 30c^4t\Bigg] \\
\sigma_2(c,t) &= \left(4-c^2\right)\left(1-t^2\right)\left[\left(4-c^2\right)\left(171ct + 27ct^2\right) + 216c^3t + 72c^3\right], \\
\sigma_3(c,t) &= \left(4-c^2\right)\left(1-t^2\right)\left[\left(4-c^2\right)\left(27t^2 + 405\right) + 216c^2t\right], \\
\sigma_4(c,t) &= \left(4-c^2\right)\left(1-t^2\right)\left[216c^2 + 432t\left(4-c^2\right)\right].
\end{aligned}$$

For finding the upper bound of $\left|\mathscr{H}_{3,1}\left(f^{-1}\right)\right|$, we have to maximize $\Gamma(c,t,y)$ in the closed cuboid $\Omega := [0,2]\times[0,1]\times[0,1]$. For this, we have to discuss the maximum values of $\Gamma(c,t,y)$ in the interior of $\Omega$ and the boundary $\partial\Omega$. By noting that $\Gamma(0,1,1) = 6784$, we know

$$\max_{(c,t,y)\in\Omega}\{\Gamma(c,x,y)\} \geqslant 6784. \tag{42}$$

Actually, we aim to prove that

$$\max_{(c,t,y)\in\Omega}\{\Gamma(c,t,y)\} = 6784. \tag{43}$$

On the face $c=2$,

$$\Gamma(2,t,y) \equiv 0, \quad t,y\in[0,1]. \tag{44}$$

On the face $t=1$,

$$\Gamma(c,t,y) = 31c^6 - 448c^4 - 400c^2 + 6784 =: g_1(c).$$

Differentiating $g_1$ with respect to $c$, we have

$$\frac{\partial g_1}{\partial c} = 186c^5 - 1792c^3 - 800c.$$

Since the only solution of the equation $\frac{\partial g_1}{\partial c} = 0$ lies in $[0,2]$ is $c=0$, we obtain that $g_1$ gets its maximum valve 6784 on $c=0$. Hence, we assume that $c\in[0,2)$ and $t\in[0,1)$ in the following.

Let $(c,t,y)\in[0,2)\times[0,1)\times(0,1)$. By differentiating partially (41) about $y$, we have

$$\begin{aligned}
\frac{\partial\Gamma}{\partial y} &= \left(4-c^2\right)\left(1-t^2\right)\Big\{54(t-1)\left[\left(4-c^2\right)(t-15) + 8c^2\right]y \\
&\qquad + 9c\left[t\left(4-c^2\right)(3t+19) + 8c^2(3t+1)\right]\Big\}.
\end{aligned}$$

Solving $\frac{\partial \Gamma}{\partial y}=0$ gives

$$y=\frac{c\left[t\left(4-c^2\right)(3t+19)+8c^2(3t+1)\right]}{6(1-t)\left[\left(4-c^2\right)(t-15)+8c^2\right]}=y_0.$$

If $y_0$ should belong to $(0,1)$, then it is possible only if

$$8c^3(3t+1)+ct\left(4-c^2\right)(3t+19)+6(1-t)\left(4-c^2\right)(15-t)<48c^2(1-t) \tag{45}$$

and

$$c^2>\frac{4(15-t)}{23-t}. \tag{46}$$

For the existence of such critical points, we must find solutions that meet both inequalities (45) and (46).

Let $\phi(t)=\frac{4(15-t)}{23-t}$. Then clearly $\phi'(t)<0$ in $(0,1)$. So $\phi(t)$ is decreasing over $(0,1)$. Hence $c^2>\frac{28}{11}$ and some basic calculations indicates that (45) can not be true for all $t\in\left[\frac{2}{5},1\right)$. Thus, $\Gamma(c,t,y)$ has no critical points in $[0,2)\times\left[\frac{2}{5},1\right)\times(0,1)$. Therefore, for any critical point $(c,t,y)$ of $\Gamma$ with $y\in(0,1)$, it must also satisfy $t\in(0,\frac{2}{5})$, which further leads to $c^2>\frac{292}{103}$.

Let $(c,t,y)$ be a critical point and $0<y<1$. Using $1-t^2\leqslant 1$, it is observed that

$$\sigma_1(c,t)\leqslant\sigma_1\left(c,\frac{2}{5}\right)=:\vartheta_1(c) \tag{47}$$

and

$$\sigma_j(c,t)\leqslant\frac{25}{21}\sigma_j\left(c,\frac{2}{5}\right)=:\vartheta_j(c),\quad j=2,3,4. \tag{48}$$

Then we have

$$\Gamma(c,t,y)\leqslant\vartheta_1(c)+\vartheta_2(c)y+\vartheta_3(c)y^2+\vartheta_4(c)\left(1-y^2\right)=:\Xi(c,y). \tag{49}$$

As $\vartheta_3(c)-\vartheta_4(c)\leqslant 0$ for $c^2>\frac{292}{103}$, it follows that

$$\frac{\partial\Xi}{\partial y}=\vartheta_2(c)+2\left[\vartheta_3(c)-\vartheta_4(c)\right]y\geqslant\vartheta_2(c)+2\left[\vartheta_3(c)-\vartheta_4(c)\right]\geqslant 0. \tag{50}$$

Hence, we get $\Xi(c,y)\leqslant\Xi(c,1)=:g_2(c)$. It is calculated that $g_2$ has a maximum value $2824.34$ at $c\approx 1.5954$. As the maximum value of $\Gamma$ only possibly obtained in the critical points or on the boundary $\partial\Omega$, we conclude that $\Gamma$ can not be achieved its maximum value in $[0,2)\times[0,1)\times(0,1)$.

On the face $y=0$,

$$\begin{aligned}\Gamma(c,t,0)&=\sigma_1(c,t)+\sigma_4(c,t)\\&=\left(4-c^2\right)^2\left[\frac{27}{4}c^2t^4+\left(\frac{41}{2}c^2-8\right)t^3+\frac{423}{4}c^2t^2+432t\right]\\&\quad+\left(4-c^2\right)\left(54c^4t^3+18c^4t^2+30c^4t+216c^2\right)=:Q(c,t).\end{aligned}$$

For $c^2 > \frac{16}{41}$, we have $\frac{41}{2}c^2 - 8 \geqslant 0$. Then $Q(c,t) \leqslant Q(c,1) =: g_3(c)$. Since $g_3$ attains its maximum value $6561.52$ on $c \approx 0.6247$, we have $Q(c,t) < 6784$ for $(c,t) \in \left(\sqrt{\frac{16}{41}}, 2\right) \times (0,1)$. For $c^2 \leqslant \frac{16}{41}$, using $t \leqslant 1$ it is found that

$$\begin{aligned} Q(c,t) &\leqslant \left(4-c^2\right)^2 \left[\frac{225}{2}c^2 + 432t + \left(\frac{41}{2}c^2 - 8\right)t^3\right] \\ &\quad + \left(4-c^2\right)\left(102c^4 + 216c^2\right) =: K(c,t). \end{aligned}$$

It is seen that $\frac{\partial K}{\partial t} \geqslant 0$ for $t \in [0,1)$, thus we have $K(c,t) \leqslant K(c,1) =: g_4(c)$. As $g_4$ gains its maximum value $6784$ on $c = 0$, we conclude that $\Gamma$ has no point attains its value larger than $6784$ on the face $y = 0$.

On the face $y = 1$,

$$\begin{aligned} \Gamma(c,t,1) &= \sigma_1(c,t) + \sigma_2(c,t) + \sigma_3(c,t) \\ &= \left(4-c^2\right)^2 \left[\frac{27}{4}\left(c^2 - 4c - 4\right)t^4 + \left(\frac{41}{2}c^2 - 171c + 424\right)t^3 \right. \\ &\quad \left. + \left(\frac{423}{4}c^2 + 27c - 378\right)t^2 + 171ct + 405\right] \\ &\quad + \left(4-c^2\right)\left[72c^3 + \left(30c^2 + 216\right)c^2 t + 18\left(c^2 - 4c + 12\right)c^2 t^2 \right. \\ &\quad \left. + 54c^2\left(c^2 - 4\right)t^3\right]. \end{aligned}$$

For $t < \frac{2}{3}$, using $c^2 - 4c - 4 \leqslant 0$ and $c^2 - 4 \leqslant 0$, it is noted that

$$\begin{aligned} \Gamma(c,t,1) &\leqslant \left(4-c^2\right)^2 \left[\left(\frac{41}{2}c^2 - 171c + 424\right)t^3 + \left(\frac{423}{4}c^2 + 27c - 378\right)t^2 \right. \\ &\quad \left. + 171ct + 405\right] + \left(4-c^2\right)\left[72c^3 + \left(30c^2 + 216\right)c^2 t \right. \\ &\quad \left. + 18\left(c^2 - 4c + 12\right)c^2 t^2\right] := L(c,t). \end{aligned}$$

From $\frac{41}{2}c^2 - 171c + 424 \geqslant 0$ and $c^2 - 4c + 12 \geqslant 0$, using $t < \frac{2}{3}$ and $t^3 \leqslant \frac{2}{3}t^2$ we deduce that

$$L(c,t) \leqslant \left(4-c^2\right)^2 W(c,t) + 4\left(4-c^2\right)\left(7c^2 + 10c + 60\right)c^2,$$

where

$$W(c,t) = \frac{1}{12}\left(1433c^2 - 1044c - 1144\right)t^2 + 171ct + 405. \tag{51}$$

If $c > \frac{4}{3}$, we get $1433c^2 - 1044c - 1144 \geqslant 0$. It follows that $W(c,t) \leqslant W\left(c, \frac{2}{3}\right)$, which yields to

$$L(c,t) \leqslant \left(4-c^2\right)^2 W\left(c, \frac{2}{3}\right) + 4\left(4-c^2\right)\left(7c^2 + 10c + 60\right)c^2 =: g_5(c).$$

A basic calculation shows that $g_5$ has a maximum value about $4108.39$ for $c \in \left(\frac{4}{3}, 2\right)$. If $1 \leqslant c \leqslant \frac{4}{3}$, we have $1433c^2 - 1044c - 1144 \leqslant 12$. Then

$$W(c,t) \leqslant t^2 + 171ct + 405 \leqslant \frac{4}{9} + 114c + 405. \tag{52}$$

Hence, we get

$$L(c,t) \leqslant \left(4-c^2\right)^2 \left(\frac{4}{9} + 114c + 405\right) + 4\left(4-c^2\right)\left(7c^2+10c+60\right)c^2 =: g_6(c),$$

while $g_6$ achieves its maximum value 5599 for $c \in \left[1, \frac{4}{3}\right]$. If $c < 1$, we know $1433c^2 - 1044c - 1144 \leqslant -755$. It follows that

$$W(c,t) \leqslant -\frac{755}{12}t^2 + 171ct + 405 \leqslant 405 + \frac{87723}{755}c^2 \leqslant 405 + 117c^2, \tag{53}$$

where the second inequality is obtained by observing that the symmetric axis $x_0 = \frac{1026}{755}c \in \left[0, \frac{2}{3}\right)$. Hence, we obtain

$$L(c,t) \leqslant \left(4-c^2\right)^2 \left(405 + 117c^2\right) + 4\left(4-c^2\right)\left(7c^2+10c+60\right)c^2 := g_7(c).$$

A basic calculation indicates that $g_7$ has a maximum value 6480 on $c = 0$. Combining the above case, we deduce that $L(c,t) < 6784$ for all $(c,t) \in [0,2) \times \left[0, \frac{2}{3}\right)$. For $t \geqslant \frac{2}{3}$, it is not hard to be seen that $\frac{\partial L}{\partial t} \geqslant 0$ for all $c \in [0,2)$. Therefore, we know that $L(c,x) \leqslant L(c,1) := g_8(c)$. As $g_8$ attains its maximum value 6784 on $c = 0$, we conclude that $L(c,t) \leqslant 6784$ for all $[0,2) \times (0,1)$. From all of the preceding cases, we established that

$$\Gamma(c,t,y) \leqslant 6784, \quad [0,2] \times [0,1] \times [0.1].$$

Hence, from (40), we have

$$\left|\mathscr{H}_{3,1}\left(f^{-1}\right)\right| \leqslant \frac{1}{58320}\left[\Gamma(c,t,y)\right] \leqslant \frac{6784}{58320} = \frac{424}{3645}.$$

The proof is thus completed. □

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*Lei Shi*
*School of Mathematics and Statistics*
*Anyang Normal University*
*Anyang, 455002, Henan, China*
*e-mail:* shimath@163.com

*Muhammad Arif*
*Department of Mathematics*
*Abdul Wali Khan University*
*Mardan, Mardan 23200, Pakistan*
*e-mail:* marifmaths@awkum.edu.pk

*H. M. Srivastava*
*Department of Mathematics and Statistics*
*University of Victoria*
*Victoria, BC V8W 3R4, Canada*
*and*
*Department of Medical Research*
*China Medical University Hospital, China Medical University*
*Taichung, 40402, Taiwan*
*and*
*Department of Mathematics and Informatics*
*Azerbaijan University*
*Jeyhun Hajibeyli Street Taichung, Baku, AZ1007 40402, Azerbaijan*
*and*
*Section of Mathematics*
*International Telematic University Uninettuno*
*I-00186 Rome, Italy*
*e-mail:* harimsri@math.uvic.ca

*Muhammad Ihsan*
*Department of Mathematics*
*Abdul Wali Khan University*
*Mardan, Mardan 23200, Pakistan*
*e-mail:* mihsanmaths33@gmail.com