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THE RELATIONSHIP BETWEEN r–CONVEXITY AND SCHUR–CONVEXITY AND ITS APPLICATION

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Abstract. In this paper, the relationship between r-convexity and Schur-convexity is investigated first. Moreover, the r-convexity and Schur-convexity of a class of functions are studied. As applications, some new inequalities on Minkowski's inequality are established.

1. Introduction

Convex functions play an important role in the inequalities theory. Many famous inequalities can be proved by constructing some special convex functions [1]. Schur-convex function is a powerful tool for the study of symmetrical inequalities theory [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The combination of convexity and Schur-convexity has rich applications [7], because there is a strong connection between the two convexities. Schur, Marshall et al. [7, p, 97] proved that:

THEOREM A. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ be symmetric and convex, and $f : \Omega \to \mathbb{R}$ be symmetric and convex. Then $f(\boldsymbol{x})$ is Schur-convex on Ω.*

Recently, the concept of convex functions has been generalized in various ways. For example, see [2, 5, 6]. However, the relationship between generalized convexity and Schur-convexity has not been found. r-convex function is a generalized convex function [3, 4], because 1-convex functions are ordinary convex functions. The first aim of this paper is to establish the relationship between r-convexity and Schur-convexity, then Theorem A is generalized.

Let $f : I \subset \mathbb{R} \to (0, +\infty)$ be a function, $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $k = 1, \cdots, n$, $r \in \mathbb{R}$, $r \neq 0$, and let

$$W_{k,r}(f,\boldsymbol{x}) = \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^{k} f^r(x_{i_j}) \right)^{\frac{1}{r}}.$$

The second aim of this paper is to study the r-convexity and Schur-convexity of $W_{k,r}(f,\boldsymbol{x})$. As applications, we establish some new inequalities on Minkowski's inequality.

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2. Definitions and Lemmas

Let $\boldsymbol{x} = (x_1, \cdots, x_n)$, $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$, $\lambda \in [0,1]$. The weighted arithmetic mean of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$A(\boldsymbol{x}, \boldsymbol{y}, \lambda) = \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} = (\lambda x_1 + (1-\lambda)y_1, \cdots, \lambda x_n + (1-\lambda)y_n).$$

For $x, y > 0$, $\lambda \in [0,1]$, the weighted power mean of order r of x and y is defined by [4]

$$M_r(x,y,\lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{\frac{1}{r}}, & r \neq 0; \\ x^\lambda y^{1-\lambda}, & r = 0. \end{cases}$$

Next we introduce the notations of r-convex function and Schur-convex function.

DEFINITION 2.1. ([4]) Let $\Omega \subseteq \mathbb{R}^n$ be a set, $r \in \mathbb{R}$, and let $f : \Omega \to (0, +\infty)$ be a function.

(i) Ω is called a convex set if $A(\boldsymbol{x}, \boldsymbol{y}, \lambda) \in \Omega$ for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $\lambda \in [0,1]$.

(ii) Let Ω be a convex set, f is called a r-convex (r-concave) function, if

$$f(A(\boldsymbol{x}, \boldsymbol{y}, \lambda)) \leqslant (\geqslant) M_r(f(\boldsymbol{x}), f(\boldsymbol{y}), \lambda), \ \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \Omega, \ \lambda \in [0,1].$$

Clearly, 1-convex functions are ordinary convex functions (need to be positive).

DEFINITION 2.2. ([7]) (i) $\boldsymbol{x} = (x_1, \cdots, x_n)$ is said to be majorized by $\boldsymbol{y} = (y_1 \cdots, y_n)$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if

$$\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]} \quad \text{for} \quad k = 1, 2, \cdots, n-1 \quad \text{and} \quad \sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$$

where $x_{[i]}$ denotes the ith largest component in $\boldsymbol{x}$.

(ii) $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ $(n \geqslant 2)$ is said to be a Schur-convex (Schur-concave) function on Ω if

$$f(\boldsymbol{x}) \leqslant (\geqslant) f(\boldsymbol{y})$$

holds for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ with $\boldsymbol{x} \prec \boldsymbol{y}$.

LEMMA 2.1. [7, p. 85. A.5] *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric convex set, and let $f : \Omega \to \mathbb{R}$ be a symmetric function. If $f(x_1, x_2, x_3, \cdots, x_n)$ is Schur-convex (Schur-concave) for any fixed $x_3, \cdots, x_n$, then $f(\boldsymbol{x})$ is Schur-convex (Schur-concave) on Ω.*

LEMMA 2.2. [1, P. 11] *Let $x_i, y_i > 0$, $i = 1, 2, \cdots$, then*

$$\left(\prod_{i=1}^n (x_i + y_i)\right)^{\frac{1}{n}} \geqslant \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n y_i\right)^{\frac{1}{n}}.$$

3. Main results

In this section, we will study the relationship between r-convexity and Schur-convexity. Moreover, the r-convexity and Schur-convexity of $W_k(f,\boldsymbol{x})$ are investigated.

THEOREM 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ be a symmetric convex set. If $f : \Omega \to (0,+\infty)$ is symmetric r-convex (r-concave), then $f(\boldsymbol{x})$ is Schur-convex (Schur-concave) on Ω.*

Proof. We only need to prove that the result holds for $n = 2$ by Lemma 2.1. So we let $\boldsymbol{x} = (x_1,x_2)$, $\boldsymbol{y} = (y_1,y_2) \in \Omega$. If $(x_1,x_2) \prec (y_1,y_2)$, suppose that $y_1 \leqslant y_2$, then $y_1 \leqslant x_1 \leqslant y_2$. So there is $\lambda \in [0,1]$ such that

$$x_1 = \lambda y_1 + (1-\lambda)y_2.$$

Note that $x_1 + x_2 = y_1 + y_2$, it follows that

$$\begin{aligned}(x_1,x_2) &= (\lambda y_1 + (1-\lambda)y_2, y_1 + y_2 - \lambda y_1 - (1-\lambda)y_2)\\ &= (\lambda y_1 + (1-\lambda)y_2, \lambda y_2 + (1-\lambda)y_1)\\ &= A((y_1,y_2),(y_2,y_1),\lambda).\end{aligned}$$

Moreover, if $f(\boldsymbol{x})$ is symmetric r-convex, then we have

$$\begin{aligned}f(x_1,x_2) &= f(A((y_1,y_2),(y_2,y_1),\lambda))\\ &\leqslant M_r(f(y_1,y_2), f(y_2,y_1),\lambda)\\ &= M_r(f(y_1,y_2), f(y_1,y_2),\lambda)\\ &= f(y_1,y_2).\end{aligned}$$

So $f(\boldsymbol{x})$ is Schur-convex on Ω. Similarly, r-concavity corresponds to Schur-concavity. □

By Lemma 2.1 and Theorem 3.1 we have

THEOREM 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ be a symmetric convex set, and let $f : \Omega \to (0,+\infty)$ be a symmetric function. If $f(x_1,x_2,x_3,\cdots,x_n)$ is r-convex (r-concave) for any fixed $x_3,\cdots,x_n$, then $f(\boldsymbol{x})$ is Schur-convex (Schur-concave) on Ω.*

THEOREM 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ be a symmetric convex set, and let $f : \Omega \to (0,+\infty)$ be a symmetric function. If $f(x_1,s-x_1,x_3,\cdots,x_n)$ is r-convex (r-concave) for any fixed $s,x_3,\cdots,x_n$, then $f(\boldsymbol{x})$ is Schur-convex (Schur-concave) on Ω.*

Proof. We only need to prove that the result holds for $n = 2$ by Theorem 3.2.

For any $(x_1,x_2),(y_1,y_2) \in \Omega$, $(x_1,x_2) \prec (y_1,y_2)$, then there is $\lambda \in [0,1]$ such that

$$x_1 = A(y_1,y_2,\lambda).$$

Let $x_1+x_2=s$, $f(x_1,s-x_1)=g(x_1)$, then $g(x_1)$ is r-convex. It follows that

$$\begin{aligned}f(x_1,x_2)&=f(x_1,s-x_1)=g(x_1)=g(A(y_1,y_2,\lambda))\\&\leqslant M_r(g(y_1),g(y_2),\lambda)=M_r(f(y_1,y_2),f(y_2,y_1),\lambda)=f(y_1,y_2).\end{aligned}$$

So $f(\boldsymbol{x})$ is Schur-convex on Ω. Similarly, r-concavity corresponds to Schur-concavity. □

THEOREM 3.4. *Let $f: I\subset\mathbb{R}\to(0,+\infty)$ be a function, $\boldsymbol{x}=(x_1,\cdots,x_n)\in I^n$. Then we have*

(i) *If f is convex and $r\geqslant 1$, then $W_{1,r}(f,\boldsymbol{x})$ is convex and Schur-convex on I^n.*

(ii) *If f is 0-convex and $r\geqslant 1$, then $W_{k,r}(f,\boldsymbol{x})$ $(2\leqslant k\leqslant n)$ is convex and Schur-convex on I^n.*

(iii) *If f is concave and $r\leqslant\frac{1}{k}$, $r\neq 0$, then $W_{k,r}(f,\boldsymbol{x})$ $(1\leqslant k\leqslant n)$ is $\frac{1}{k}$-concave and Schur-concave on I^n.*

Proof. Clearly $W_{k,r}(f,\boldsymbol{x})$ $(1\leqslant k\leqslant n)$ is symmetric. By Theorem 3.1, we only need to prove the convexity of $W_{k,r}(f,\boldsymbol{x})$.

(i) For any $\boldsymbol{x}=(x_1,\cdots,x_n)$, $\boldsymbol{y}=(y_1,\cdots,y_n)\in I^n$, $\lambda\in[0,1]$, using the convexity of f and Minkowski's inequality we have

$$\begin{aligned}W_{1,r}(f,A(\boldsymbol{x},\boldsymbol{y},\lambda))&=\left[\sum_{i=1}^{n}f^r(\lambda x_i+(1-\lambda)y_i)\right]^{\frac{1}{r}}\\&\leqslant\left[\sum_{i=1}^{n}(\lambda f(x_i)+(1-\lambda)f(y_i))^r\right]^{\frac{1}{r}}\\&\leqslant\left[\sum_{i=1}^{n}(\lambda f(x_i))^r\right]^{\frac{1}{r}}+\left[\sum_{i=1}^{n}((1-\lambda)f(y_i))^r\right]^{\frac{1}{r}}\\&=A(W_{1,r}(f,\boldsymbol{x}),W_{1,r}(f,\boldsymbol{y}),\lambda).\end{aligned}$$

Hence $W_{1,r}(f,\boldsymbol{x})$ is convex.

(ii) For any $\boldsymbol{x}=(x_1,\cdots,x_n)$, $\boldsymbol{y}=(y_1,\cdots,y_n)\in I^n$, $\lambda\in[0,1]$, by the 0-convexity of f and AG inequality and Minkowski's inequality we have

$$\begin{aligned}&W_{k,r}(f,A(\boldsymbol{x},\boldsymbol{y},\lambda))\\&=\left\{\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left[\prod_{i=1}^{k}\left(f(\lambda x_{i_j}+(1-\lambda)y_{i_j})\right)\right]^r\right\}^{\frac{1}{r}}\\&\leqslant\left\{\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left[\left(\prod_{i=1}^{k}f(x_{i_j})\right)^{\lambda}\left(\prod_{i=1}^{k}f(y_{i_j})\right)^{1-\lambda}\right]^r\right\}^{\frac{1}{r}}\end{aligned}$$

$$\leqslant \left\{ \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left[\lambda \prod_{i=1}^{k} f(x_{i_j}) + (1-\lambda) \prod_{i=1}^{k} f(y_{i_j}) \right]^r \right\}^{\frac{1}{r}}$$

$$\leqslant \left\{ \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left[\lambda \prod_{i=1}^{k} f(x_{i_j}) \right]^r \right\}^{\frac{1}{r}} + \left\{ \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left[(1-\lambda) \prod_{i=1}^{k} f(y_{i_j}) \right]^r \right\}^{\frac{1}{r}}$$

$$= A(W_{k,r}(f,\boldsymbol{x}), W_{k,r}(f,\boldsymbol{y}), \lambda).$$

So $W_{k,r}(f,\boldsymbol{x})$ is convex.

(iii) For any $\boldsymbol{x} = (x_1,\cdots,x_n)$, $\boldsymbol{y} = (y_1,\cdots,y_n) \in I^n$, $\lambda \in [0,1]$, by the concavity of f and Lemma 2.2 and Minkowski's inequality we have

$$W_{k,r}(f, A(\boldsymbol{x},\boldsymbol{y},\lambda))$$

$$\geqslant \left\{ \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left[\prod_{i=1}^{k} \left(\lambda f(x_{i_j}) + (1-\lambda) f(y_{i_j}) \right) \right]^r \right\}^{\frac{1}{r}}$$

$$\geqslant \left\{ \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left[\left(\prod_{i=1}^{k} \lambda f(x_{i_j}) \right)^{\frac{1}{k}} + \left(\prod_{i=1}^{k} (1-\lambda) f(y_{i_j}) \right)^{\frac{1}{k}} \right]^{kr} \right\}^{\frac{1}{kr}\cdot k}$$

$$\geqslant \left\{ \left[\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left(\prod_{j=1}^{k} \lambda f(x_{i_j}) \right)^{\frac{1}{k}\cdot kr} \right]^{\frac{1}{kr}} + \left[\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \left(\prod_{j=1}^{k} (1-\lambda) f(y_{i_j}) \right)^{\frac{1}{k}\cdot kr} \right]^{\frac{1}{kr}} \right\}^{k}$$

$$= \left\{ \lambda \left[\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \prod_{j=1}^{k} f^r(x_{i_j}) \right]^{\frac{1}{r}\cdot\frac{1}{k}} + (1-\lambda) \left[\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \prod_{j=1}^{k} f^r(y_{i_j}) \right]^{\frac{1}{r}\cdot\frac{1}{k}} \right\}^{k}$$

$$= M_{\frac{1}{k}}(W_{k,r}(f,\boldsymbol{x}), W_{k,r}(f,\boldsymbol{y}), \lambda).$$

So $W_{k,r}(f,\boldsymbol{x})$ is $\frac{1}{k}$-concave. $\square$

4. Some new inequalities on Minkowski's inequality

In this section, we will use Theorem 3.4 to establish some new inequalities on Minkowski's inequality.

Let $\boldsymbol{x} = (x_1,\cdots,x_n) \in \mathbb{R}^n$, $x_i > 0$, $1 \leqslant i \leqslant n$, the arithmetic mean and geometric mean of $x_1,\cdots,x_n$ are respectively defined by

$$A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n(\boldsymbol{x}) = \left(\prod_{i=1}^{n} x_i \right)^{1/n}.$$

THEOREM 4.1. *Let* $\boldsymbol{x}=(x_1,\cdots,x_n)$, $\boldsymbol{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$, $x_i,y_i>0$, $1\leqslant i\leqslant n$. (i) *If* $s\geqslant r\geqslant 1$, *then*

$$2^{1-\frac{s}{r}}n^{\frac{1}{r}}A_n(\boldsymbol{x}+\boldsymbol{y})^{\frac{s}{r}}\leqslant 2^{1-\frac{s}{r}}\left(\sum_{i=1}^n(x_i+y_i)^s\right)^{\frac{1}{r}}\leqslant\left(\sum_{i=1}^n x_i^s\right)^{\frac{1}{r}}+\left(\sum_{i=1}^n y_i^s\right)^{\frac{1}{r}}. \tag{4.1}$$

(ii) *For any* $2\leqslant k\leqslant n$, *if* $s<0$, $r\geqslant 1$, *then*

$$2^{1-\frac{ks}{r}}\binom{n}{k}^{\frac{1}{r}}A_n(\boldsymbol{x}+\boldsymbol{y})^{\frac{ks}{r}}\leqslant 2^{1-\frac{ks}{r}}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^k(x_{i_j}+y_{i_j})^s\right)^{\frac{1}{r}}$$
$$\leqslant\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^k x_{i_j}^s\right)^{\frac{1}{r}}+\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^k y_{i_j}^s\right)^{\frac{1}{r}}. \tag{4.2}$$

(iii) *For any* $1\leqslant k\leqslant n$, *if* $0<s\leqslant r\leqslant\frac{1}{k}$ *or* $r\leqslant s<0$, *then*

$$2^{k-\frac{ks}{r}}\binom{n}{k}^{\frac{1}{r}}A_n(\boldsymbol{x}+\boldsymbol{y})^{\frac{ks}{r}}\geqslant 2^{k-\frac{ks}{r}}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^k(x_{i_j}+y_{i_j})^s\right)^{\frac{1}{r}}$$
$$\geqslant\left\{\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^k x_{i_j}^s\right)^{\frac{1}{kr}}+\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^k y_{i_j}^s\right)^{\frac{1}{kr}}\right\}^k. \tag{4.3}$$

In particular, if we let $r=s$ *in* (4.1), *and let* $k=1$, $r=s$ *in* (4.3), *respectively, then we have*

$$n^{\frac{1}{r}}A_n(\boldsymbol{x}+\boldsymbol{y})\leqslant\left(\sum_{i=1}^n(x_i+y_i)^r\right)^{\frac{1}{r}}\leqslant\left(\sum_{i=1}^n x_i^r\right)^{\frac{1}{r}}+\left(\sum_{i=1}^n y_i^r\right)^{\frac{1}{r}},\quad r\geqslant 1, \tag{4.4}$$

$$n^{\frac{1}{r}}A_n(\boldsymbol{x}+\boldsymbol{y})\geqslant\left(\sum_{i=1}^n(x_i+y_i)^r\right)^{\frac{1}{r}}\geqslant\left(\sum_{i=1}^n x_i^r\right)^{\frac{1}{r}}+\left(\sum_{i=1}^n y_i^r\right)^{\frac{1}{r}},\quad r<1,\ r\neq 0. \tag{4.5}$$

The right-hand inequalities in (4.4) *and* (4.5) *are Minkowski's inequality.*

If we let $k=n$ *in the right-hand inequalities* (4.2) *and* (4.3), *respectively, then we have*

$$2^{1-a}G_n(\boldsymbol{x}+\boldsymbol{y})^a\leqslant G_n(\boldsymbol{x})^a+G_n(\boldsymbol{y})^a,\quad a<0, \tag{4.6}$$
$$2^{1-a}G_n(\boldsymbol{x}+\boldsymbol{y})^a\geqslant G_n(\boldsymbol{x})^a+G_n(\boldsymbol{y})^a,\quad 0<a\leqslant 1. \tag{4.7}$$

Proof. First we prove the inequalities (4.3). Let $f(x)=x^{\frac{s}{r}}$. Note that $0<s\leqslant r\leqslant\frac{1}{k}$ or $r\leqslant s<0$, so $f(x)$ is concave on $(0,+\infty)$. It follows that $W_{k,r}(f,\boldsymbol{x})$ $(1\leqslant k\leqslant n)$

is $\frac{1}{k}$-concave by Theorem 3.4(iii). So for any $\lambda \in [0,1]$, we have

$$\left\{ \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left[\prod_{i=1}^{k} \left(\lambda x_{i_j} + (1-\lambda) y_{i_j} \right)^{\frac{s}{r}} \right]^r \right\}^{\frac{1}{r}}$$
$$\geqslant \left\{ \lambda \left[\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(\prod_{i=1}^{k} x_{i_j}^{\frac{s}{r}} \right)^r \right]^{\frac{1}{kr}} + (1-\lambda) \left[\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(\prod_{i=1}^{k} y_{i_j}^{\frac{s}{r}} \right)^r \right]^{\frac{1}{kr}} \right\}^k . \tag{4.8}$$

Let $\lambda = \frac{1}{2}$ in (4.8), then we obtain the right-hand inequality in (4.3).

Moreover, since $W_{k,r}(f, \boldsymbol{x})$ $(1 \leqslant k \leqslant n)$ is Schur-concave by Theorem 3.4(iii), and

$$(A_n(\boldsymbol{x}+\boldsymbol{y}), \cdots, A_n(\boldsymbol{x}+\boldsymbol{y})) \prec \boldsymbol{x}+\boldsymbol{y},$$

we can deduce that

$$\binom{n}{k}^{\frac{1}{r}} A_n(\boldsymbol{x}+\boldsymbol{y})^{\frac{ks}{r}} \geqslant \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^{k} (x_{i_j} + y_{i_j})^s \right)^{\frac{1}{r}} . \tag{4.9}$$

It follows that the left-hand inequality in (4.3) holds.

Similarly, we can obtain (4.1) and (4.2) using Theorem 3.4(i) and (ii) respectively. □

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