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FURTHER PROPERTIES OF 2–INNER PRODUCT SPACES

ABBAS ALIPOUR AND MOHAMMAD HADI AKHBARI

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Abstract. This paper aims to establish some results on the structure of fixed point sets for mappings in 2-inner product spaces. To this end, we employ some well-known techniques of 2-inner product spaces.

1. Introduction and preliminaries

The concept of 2-metric spaces, linear 2-normed spaces, and 2-inner product spaces was introduced by Gähler [6]. After that, several authors like White [17], Lewandowska [11, 12], Freese [5], and Diminnie [3], worked on possible applications of Metric Geometry, Functional Analysis, and Topology in these settings. Some other related results are also concerned in [2, 8, 10, 14].

Let $\mathscr{X}$ be a linear space of dimension greater than one over the field $K = \mathbb{R}$ of real numbers or the field $K = \mathbb{C}$ of complex numbers and let $x, y, z \in \mathscr{X}$. Suppose that $\langle \cdot,\cdot|\cdot\rangle$ is a K-valued function defined on $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$ satisfying the subsequent conditions:

(I1) $\langle x,x|z\rangle \geqslant 0$ and $\langle x,x|z\rangle = 0$, if and only if x and z are linearly dependent;

(I2) $\langle x,x|z\rangle = \langle z,z|x\rangle$;

(I3) $\langle y,x|z\rangle = \overline{\langle x,y|z\rangle}$;

(I4) $\langle \alpha x,y|z\rangle = \alpha \langle x,y|z\rangle$ for any scalar $\alpha \in K$;

(I5) $\langle x+x',y|z\rangle = \langle x,y|z\rangle + \langle x',y|z\rangle$.

$\langle \cdot,\cdot|\cdot\rangle$ is called a 2-inner product on $\mathscr{X}$ and $(\mathscr{X}, \langle \cdot,\cdot|\cdot\rangle)$ is called a 2-inner product space (or 2-pre-Hilbert space). Some basic properties of 2-inner product $\langle \cdot,\cdot|\cdot\rangle$ can be immediately obtained as follows:

(P1) $\langle 0,y|z\rangle = \langle x,0|z\rangle = \langle x,y|0\rangle = 0$;

(P2) $\langle x,\alpha y|z\rangle = \overline{\alpha} \langle x,y|z\rangle$ for any $\alpha \in K$;

(P3) $\langle x,y|\alpha z\rangle = |\alpha|^2 \langle x,y|z\rangle$, for all $x,y,z \in \mathscr{X}$ and $\alpha \in K$.

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By the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x,y|z\rangle|^2 \leqslant \langle x,x|z\rangle \langle y,y|z\rangle .$$

The most common example for a linear 2-inner product $\langle \cdot,\cdot|\cdot\rangle$ is defined on $\mathscr{X}$ by

$$\langle x,y|z\rangle := \det \begin{bmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{bmatrix}$$

for all $x,y,z \in \mathscr{X}$. In [3], it is shown that, in any given 2-inner product space $(\mathscr{X},\langle \cdot,\cdot|\cdot\rangle)$, we can define a function

$$\|x,z\| = \sqrt{\langle x,x|z\rangle} \tag{1.1}$$

for all $x,z \in \mathscr{X}$. It is easy to see that this function satisfies the following conditions:

(N1) $\|x,y\| = 0$, if and only if x and y are linearly dependent;

(N2) $\|x,y\| = \|y,x\|$;

(N3) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$ for any real number α;

(N4) $\|x,y+z\| \leqslant \|x,y\| + \|x,z\|$.

Any function $\|\cdot,\cdot\|$ defined on $\mathscr{X} \times \mathscr{X}$ and satisfying the above conditions is called a 2-norm on $\mathscr{X}$ and $(\mathscr{X},\|\cdot,\cdot\|)$ is called linear 2-normed space.

Whenever a 2-inner product space $(\mathscr{X},\langle \cdot,\cdot|\cdot\rangle)$ is given, we consider it as a linear 2-normed space $(\mathscr{X},\|\cdot,\cdot\|)$ with the 2-norm represented by (1.1).

An operator A is said to be bounded, if there exists $M > 0$ such that

$$\|Ax,y\| \leqslant M\,\|x,y\|,$$

for every $x,y \in \mathscr{X}$ (we write $A \in \mathscr{B}(\mathscr{X})$).

Let $(\mathscr{X},\langle \cdot,\cdot|\cdot\rangle)$ be a 2-inner product space, and $z \in \mathscr{X}$. A sequence $\{x_n,z\}$ in $\mathscr{X}$ is a z-Cauchy sequence if

$$\forall \varepsilon > 0\ \exists N > 0,\ \text{s.t}\ \forall\, m,n \geqslant N\ 0 < \|x_m - x_n, z\| < \varepsilon.$$

Meanwhile, $\mathscr{X}$ is called z-Hilbert if every z-Cauchy sequence is converges in the semi normed $(\mathscr{X},\|\cdot,z\|)$.

2. Main properties

Let C be a nonempty closed convex subset of a 2-inner product space. A mapping $A : C \to C$ is named non spreading if

$$2\|Ax - Ay, z\|^2 \leqslant \|Ax - y, z\|^2 + \|Ay - x, z\|^2$$

for all $x,y \in C$.

We say $A : C \to C$ is an asymptotic non-spreading mapping if there exists two functions $\alpha : C \to [0,2)$ and $\beta : C \to [0,k]$, $k < 2$, such that

(a) $2\|Ax-Ay,z\|^2 \leqslant \alpha(x)\|Ax-y,z\|^2+\beta(x)\|Ay-x,z\|^2$, for all $x,y,z\in C$.

(b) $0<\alpha(x)+\beta(x)\leqslant 2$, for all $x\in C$.

It is required to remark that

(a') If $\alpha(x)=\beta(x)=1$, for all $x\in C$, then A is a non-spreading mapping.

(b') If $\alpha(x)=\frac{4}{3}$ and $\beta(x)=\frac{2}{3}$ for all $x\in C$, then A is a AJ-2 mapping.

Let $\mathscr{X}$ be a real 2-inner product space and C be a nonempty subset of $\mathscr{X}$. A mapping $A:C\to\mathscr{X}$ is named symmetric generalized hybrid if there exist $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that

$$\alpha\|Ax-Ay,z\|^2+\beta\left(\|x-Ay,z\|^2+\|Ax-y,z\|^2\right)+\gamma\|x-y,z\|^2 \\ +\delta\left(\|x-Ax,z\|^2+\|y-Ay,z\|^2\right)\leqslant 0$$

for all $x,y,z\in\mathscr{X}$. Such mapping A is also called $(\alpha,\beta,\gamma,\delta)$- symmetric generalized hybrid.

THEOREM 2.1. *Let C be a nonempty closed convex subset of a 2-inner product space $\mathscr{X}\times\mathscr{X}$. Let α,β be the same as in the above. Then $A:C\to C$ is an asymptotic non-spreading mapping if*

$$\begin{aligned}&\|Ax-Ay,z\|^2\\ &\leqslant \frac{\alpha(x)-\beta(x)}{2-\beta(x)}\|Ax-x,z\|^2\\ &\quad+\frac{\alpha(x)\|x-y,z\|}{2-\beta(x)}\frac{2\langle Ax-x,\alpha(x)(x-y)+\beta(x)(Ay-x)|z\rangle}{2-\beta(x)}.\end{aligned}$$

Proof. We have that for $x,y,z\in C$

$$\begin{aligned}2\|Ax-Ay,z\|^2 &\leqslant \alpha(x)\|Ax-y,z\|^2+\beta(x)\|Ay-x,z\|^2\\ &=\alpha(x)\|Ax-x,z\|^2+2\alpha(x)\langle Ax-x,x-y|z\rangle\\ &\quad+\alpha(x)\|x-y,z\|^2+\beta(x)\|Ay-Ax,z\|^2\\ &\quad+2\beta(x)\langle Ay-Ax,Ax-x|z\rangle+\beta(x)\|Ax-x,z\|^2\\ &=(\alpha(x)+\beta(x))\|Ax-x,z\|^2+\beta(x)\|Ay-Ax,z\|^2\\ &\quad+\alpha(x)\|x-y,z\|^2+2\alpha(x)\langle Ax-x,x-y|z\rangle\\ &\quad+2\beta(x)\langle Ay-x+x-Ax,Ax-x|z\rangle\\ &=(\alpha(x)-\beta(x))\|Ax-x,z\|^2+\beta(x)\|Ay-Ax,z\|^2\\ &\quad+\alpha(x)\|x-y,z\|^2+\langle Ax-x,2\alpha(x)(x-y)\\ &\qquad+2\beta(x)Ay-x|z\rangle,\end{aligned}$$

and this indicates the desired result. □

THEOREM 2.2. *Let $\mathscr{X}\times\mathscr{X}$ be a real 2-inner product space, let C be a nonempty closed convex subset of $\mathscr{X}$ and let A be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions*

(i) $\alpha+2\beta+\gamma\geqslant 0$

(ii) $\alpha+\beta+\delta>0$

(iii) $\delta\geqslant 0$

hold. Then A has a fixed point if and only if there exists $y\in C$ such that $\{A^n y:\ n\in\{0,1,\dots\}\}$ is bounded. In particular, a fixed point of A is unique in the case of $\alpha+2\beta+\gamma>0$ on the condition.

Proof. Assume that A has a fixed point y. Then $\{A^n y:\ n\in\{0,1,\dots\}\}=\{y\}$ and hence $\{A^n y:\ n\in\{0,1,\dots\}\}$ is bounded. Conversely, suppose that there exists $y\in C$ such that $\{A^n y:\ n\in\{0,1,\dots\}\}$ is bounded. Since A is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping of C into itself, we have that

$$\alpha\left\|Ax-A^{n+1}y,z\right\|^2+\beta\left(\left\|x-A^{n+1}y,z\right\|^2+\|Ax-A^n y,z\|^2\right)$$
$$+\gamma\|x-A^n y,z\|^2+\delta\left(\|x-Ax,z\|^2+\left\|A^n y-A^{n+1}y,z\right\|^2\right)\leqslant 0$$

for all $n\in\mathbb{N}\cup\{0\}$ and $x\in C$. Since $\{A^n y\}$ is bounded, we can apply Banach limit μ to both sides of the inequality. Since $\mu_n\|Ax-A^n y,z\|^2=\mu_n\left\|Ax-A^{n+1}y,z\right\|^2$ and $\mu_n\left\|x-A^{n+1}y,z\right\|^2=\mu_n\|x-A^n y,z\|^2$, we have that

$$(\alpha+\beta)\,\mu_n\|Ax-A^n y,z\|^2+(\beta+\gamma)\,\mu_n\|x-A^n y,z\|^2$$
$$+\delta\left(\|x-Ax,z\|^2+\mu_n\left\|A^n y-A^{n+1}y,z\right\|^2\right)\leqslant 0.$$

Also, since

$$\mu_n\|Ax-A^n y,z\|^2=\|Ax-x,z\|^2+2\mu_n\,(Ax-x,x-A^n,z)+\mu_n\|x-A^n x,z\|^2$$

we have that

$$(\alpha+\beta+\delta)\,\|Ax-x,z\|^2+2\,(\alpha+\beta)\,\mu_n\,(Ax-x,x-A^n|z)$$
$$+(\alpha+2\beta+\gamma)\,\mu_n\|x-A^n y,z\|^2+\delta\mu_n\left\|A^n x-A^{n+1}x,z\right\|^2\leqslant 0.$$

From (i) and (iii) we have

$$(\alpha+\beta+\delta)\,\|Ax-x,z\|^2+2\,(\alpha+\beta)\,\mu_n\,(Ax-x,x-A^n,z)\leqslant 0. \tag{2.1}$$

Since there exists $p\in\mathscr{X}$ such that

$$\mu_n\,(w,A^n y,z)=(w,p,z)$$

for all $w \in \mathscr{X}$. We have from (2.1) that

$$(\alpha+\beta+\delta)\,\|Ax-x,z\|^2+2\,(\alpha+\beta)\,\mu_n\,(Ax-x,x-p,z)\leqslant 0. \tag{2.2}$$

Since C is closed and convex, we have that

$$p\in\overline{co}\,\{A^n x:\ n\in\mathbb{N}\}\subset C.$$

Placing $x=p$ we receive from (2.2) that

$$(\alpha+\beta+\delta)\,\|Ap-p,z\|^2\leqslant 0. \tag{2.3}$$

We have from (ii) that $\|Ap-p,b\|^2\leqslant 0$. This means that p is a fixed point in A.

New assume that $\alpha+2\beta+\gamma>0$. Let p_1 and p_2 be fixed points of A. Then we have that

$$\begin{aligned}\alpha\|Ap_1-Ap_2,z\|^2&+\beta\left(\|p_1-Ap_2,z\|^2+\|Ap_1-p_2,z\|^2\right)\\&+\gamma\|p_1-p_2,z\|^2+\delta\left(\|p_1-Ap_1,z\|^2+\|p_2-Ap_2,z\|^2\right)\leqslant 0\end{aligned}$$

and hence $(\alpha+2\beta+\gamma)\,\|p_1-p_2,z\|^2\leqslant 0$. We have from $\alpha+2\beta+\gamma>0$ that $p_1=p_2$. Consequently, a fixed point of A is unique. This completes the proof. □

COROLLARY 2.3. *Let $\mathscr{X}\times\mathscr{X}$ be a real 2-inner product space, let C be a non-empty closed convex subset of $\mathscr{X}$ and let A be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions*

1. $\alpha+2\beta+\gamma\geqslant 0$
2. $\alpha+\beta+\delta>0$
3. $\delta\geqslant 0$

hold. Then A has a fixed. In particular, a fixed point of A is unique in the case of $\alpha+2\beta+\gamma>0$ on the condition.

The next theorem is the generalization of the Banach contraction principle in the 2-inner product space, involving four rational square terms in the inequality.

THEOREM 2.4. *Let $A:\mathscr{X}\to\mathscr{X}$ be a self-mapping satisfying the following condition*

$$\begin{aligned}\|Ax-Ay,z\|^2\leqslant a_1&\frac{\|y-Ay,z\|^2\left(1+\|x-Ax,z\|^2\right)}{1+\|x-y,z\|^2}\\&+a_2\frac{\|x-Ax,z\|^2\left(1+\|y-Ay,z\|^2\right)}{1+\|x-y,z\|^2}\end{aligned}$$

$$
\begin{aligned}
&+a_3 \frac{\|x-Ay,z\|^2\left(1+\|y-Ax,z\|^2\right)}{1+\|x-y,z\|^2} \\
&+a_4 \frac{\|y-Ax,z\|^2\left(1+\|x-Ay,z\|^2\right)}{1+\|x-y,z\|^2} \\
&+a_5\|x-y,z\|^2
\end{aligned}
$$

for all $x,y,z \in \mathscr{X}$ *and* $x \neq y$*, where* a_1,a_2,a_3,a_4,a_5 *are non negative reals with* $a_1+a_2+a_3+4a_4+a_5<1$*. Therefore,* A *has a unique fixed point in* $\mathscr{X}$*.*

Proof. For some $x_0 \in \mathscr{X}$, we define a sequence $\{x_n\}$ of iterates of A as follows

$$
x_1 = Ax_0,z, \;\; x_2 = Ax_1,z, \;\; x_3 = Ax_2,z,\ldots, \;\; x_{n+1} = Ax_n,z
$$

for $n \in \{0,1,\ldots\}$.

Now, we demonstrate that $\{x_n,z\}$ is a z-Cauchy sequence in $\mathscr{X} \times \mathscr{X}$. For this, consider

$$
\|x_{n+1}-x_n,z\|^2 = \|Ax_n - Ax_{n-1},z\|^2.
$$

Then by utilizing the assumption, we have

$$
\begin{aligned}
\|x_{n+1}-x_n,z\|^2 \leqslant a_1 &\frac{\|x_{n-1}-Ax_{n-1},z\|^2\left(1+\|x_n-Ax_n,z\|^2\right)}{1+\|x_n-x_{n-1},z\|^2} \\
&+a_2 \frac{\|x_n-Ax_n,z\|^2\left(1+\|x_{n-1}-Ax_{n-1},z\|^2\right)}{1+\|x_n-x_{n-1},z\|^2} \\
&+a_3 \frac{\|x_n-Ax_{n-1},z\|^2\left(1+\|x_{n-1}-Ax_n,z\|^2\right)}{1+\|x_n-x_{n-1},z\|^2} \\
&+a_4 \frac{\|x_{n-1}-Ax_n,z\|^2\left(1+\|x_n-Ax_{n-1},z\|^2\right)}{1+\|x_n-x_{n-1},z\|^2} \\
&+a_5\|x_n-x_{n-1},z\|^2,
\end{aligned}
$$

which indicates that

$$
\begin{aligned}
&(1-a_2-2a_4)\,\|x_{n+1}-x_n,z\|^2 \\
&\quad +(1-a_1-a_2)\,\|x_{n+1}-x_n,z\|^2\|x_n-x_{n-1},z\|^2 \\
&\leqslant \left((a_1+2a_4+a_5)+a_5\|x_n-x_{n-1},z\|^2\right)\|x_n-x_{n-1},z\|^2.
\end{aligned}
$$

Resulting in

$$\|x_{n+1}-x_n,z\|^2 \leqslant p(n)\,\|x_n-x_{n-1},b\|^2$$

where

$$p(n)=\frac{(a_1+2a_4+a_5)+a_5\|x_n-x_{n-1},z\|^2}{(1-a_2-2a_4)+(1-a_1-a_2)\,\|x_n-x_{n-1},z\|^2}$$

for $n\in\{0,1,\ldots\}$. Obviously $p(n)<1$, for all n as $a_1+a_2+a_3+4a_4+a_5<1$. Repeating the same argument, we find some $S<1$, such that

$$\|x_{n+1}-x_n,z\|^2 \leqslant \lambda^n\|x_1-x_0,z\|^2$$

where $\lambda=S^2$. Letting $n\to\infty$, we obtain $\|x_{n+1}-x_n,z\|\to 0$. It follows that $\{x_n,z\}$ is a z-Cauchy sequence in $\mathscr{X}$. So by the completeness of $\mathscr{X}$ there exists a point $\mu\in\mathscr{X}$ such that $x_n\to\mu$ as $n\to\infty$. Also $\{x_{n+1},z\}=\{Ax_n,z\}$ is sub sequence of $\{x_n,z\}$ converges to the same limit μ. Since A is continuous, we get

$$A(\mu)=A\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}x_{n+1}=\mu.$$

Thus μ is a fixed point of $\mathscr{X}$. Now, we establish the uniqueness of μ. If A has another fixed point γ and $\mu\neq\gamma$, then

$$\begin{aligned}
\left\|p-p',z\right\|^2 &= \left\|Ap-Ap',z\right\|^2\\
&\leqslant a_1\frac{\|p'-Ap',z\|^2\left(1+\|p-Ap,z\|^2\right)}{1+\|p-p',b\|^2}\\
&\quad+a_2\frac{\|p-Ap,z\|^2\left(1+\|p'-Ap',z\|^2\right)}{1+\|p-p',z\|^2}\\
&\quad+a_3\frac{\|p-Ap',z\|^2\left(1+\|p'-Ap,z\|^2\right)}{1+\|p-p',z\|^2}\\
&\quad+a_4\frac{\|p'-Ap,z\|^2\left(1+\|p-Ap',z\|^2\right)}{1+\|p-p',z\|^2}\\
&\quad+a_5\left\|p-p',z\right\|^2
\end{aligned}$$

which, in turn, indicates that

$$\left\|p-p',z\right\|^2\leqslant(a_3+a_4+a_5)\left\|p-p',z\right\|^2.$$

This provides a contradiction, for $a_3+a_4+a_5<1$. Accordingly, p is a unique fixed point of A in $\mathscr{X}$. $\square$

3. Frame in 2-Hilbert space

This section introduces frame notions in a 2-Hilbert space $\mathscr{H}$. Some results concerning these notions are surveyed. We refer the reader to [16] for other fresh generalizations of the frame.

Many authors have obtained significant results on the classical definition of the 2-inner product (e.g., see [1, 4, 5]). However, the most important fault in the classical definition of a 2-inner product space was that they could not define 2-Hilbert spaces. Gordji [7] presented the best-generalized definition of the 2-inner product space, which includes all the previous definitions and the geometric approach to the concept of a more appropriate. In this section, we recall some fundamental definitions of generalized 2-inner product spaces that will be used in the sequel.

DEFINITION 3.1. A complex vector space $\mathscr{X}$ is called a 2-inner product space if there exists a complex-valued function $\langle(\cdot,\cdot),(\cdot,\cdot)\rangle$ on $\mathscr{X}^2\times\mathscr{X}^2$ such that, for all $x,y,z,w\in\mathscr{X}$ and $\alpha\in\mathbb{C}$ satisfying the following conditions:

(a) $\langle(x,y),(z,w)\rangle=\overline{\langle(z,w),(x,y)\rangle}$;

(b) If x and y are linearly independent in X, then $\langle(x,y),(x,y)\rangle>0$;

(c) $\langle(x,y),(z,w)\rangle=-\langle(y,x),(w,z)\rangle$;

(d) $\langle(\alpha x+x',y),(z,w)\rangle=\alpha\langle(x,y),(z,w)\rangle+\langle(x',y),(z,w)\rangle$.

In any given 2-inner product space $(\mathscr{X},\langle(\cdot,\cdot),(\cdot,\cdot)\rangle)$, we can define a function $\|\cdot,\cdot\|$ on $\mathscr{X}^2\times\mathscr{X}^2$ by

$$\|x,y\|=\langle(x,y),(x,y)\rangle^{\frac{1}{2}}.$$

Let $\mathscr{X}$ be a linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on $\mathscr{X}\times\mathscr{X}$ satisfying the following conditions:

(a) $\|x,y\|=0$ if and only if x and y are linearly dependent;

(b) $\|x,y\|=\|y,x\|$;

(c) $\|\alpha x,y\|=|\alpha|\,\|x,y\|$, where α is real;

(d) $\|x,y+z\|\leqslant\|x,y\|+\|x,z\|$.

The function $\|\cdot,\cdot\|$ is called a 2-norm on $\mathscr{X}$ and $(\mathscr{X},\|\cdot,\cdot\|)$ a linear 2-normed space. Note that in the definition of 2-norm, if the following condition replaces the condition (d):

(e) $\|x,y\|=\|x,y-x\|$

then the function $\|\cdot,\cdot\|$ is called a semi 2-norm on $\mathscr{X}$ and $(\mathscr{X},\|\cdot,\cdot\|)$ a semi 2-normed space. Some basic properties of 2-inner product $\langle(\cdot,\cdot),(\cdot,\cdot)\rangle$ can be immediately obtained as follows:

(I) $\langle (x,y),(x,y)\rangle \geqslant 0,$

(II) $\langle (x,y),(z,\beta w+w')\rangle = \overline{\beta}\langle (x,y),(z,w)\rangle + \langle (x,y),(z,w')\rangle,$

(III) $\langle (x,y),(x,y)\rangle = 0$ if and only if x and y are linearly,

for all $x,y,z,w,w' \in X$ and and $\beta \in \mathbb{C}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality:

$$|\langle (x,y),(z,w)\rangle|^2 \leqslant \langle (x,y),(x,y)\rangle \langle (z,w),(z,w)\rangle .$$

Let $(\mathscr{X},\|\cdot,\cdot\|)$ be a 2-normed space. Let $\mathscr{D}$ be a subspace of $\mathscr{X}$, $b \in \mathscr{D}$ be fixed, then a map $A : \mathscr{D}\times\langle b\rangle \to \mathscr{K}$ where $\langle b\rangle$ is the subspace of $\mathscr{X}$ generated by b, is called a b-operator on $\mathscr{D}\times\langle b\rangle$ whenever for every $x,y\in\mathscr{D}$ and $k\in\mathscr{K}$ holds:

1. $A(x+y,b) = A(x,b)+A(y,b)$.
2. $A(kx,b) = kA(x,b)$.

A b-operator $A : \mathscr{D}\times\langle b\rangle \to \mathscr{K}$ is said to be bounded if there exists a real number $\mathscr{M} > 0$ such that $|A(x,b)| \leqslant \mathscr{M}\|x,b\|$ for every $x\in\mathscr{D}$.

The norm of the b-operator is defined by

$$\|A\|_2 = \sup\{|A(x,b)| : \ \|x,b\| = 1\}.$$

Also we have $|A(x,b)| \leqslant \|A\|\,\|x,b\|$, where $A(x,b) = \langle x,y|b\rangle$ for every x in $\mathscr{X}$.

Harikrishnan et al. [9] proved the Riesz theorem in a 2-inner product for b-operators (see also [13]). In addition, Riyas in [15], presented the notion of adjoint for b-operators as follows:

DEFINITION 3.2. Let $\mathscr{X}$ be a 2-normed space. Restricting $\mathscr{X}$ to $\widetilde{\mathscr{X}}$, define $A^* : \widetilde{\mathscr{X}} \to \widetilde{\mathscr{X}}$ by $A^*(y) = z_y$. By construction,

$$\langle (A^*(y),b),(x,b)\rangle = \langle (y,b),(A(x),b)\rangle ,$$

or

$$\langle A^*(y),b\rangle_b = \langle y,A(x)\rangle_b .$$

We call A^* adjoint operator of A in $\mathscr{B}(\mathscr{X})$.

We refer the interested reader to [2] for the basic theory of 2-inner product spaces.

Let $\mathscr{V}$ be a finite-dimensional vector space equipped with a 2-inner product introduced in the previous section. A countable family of elements $\{f_i,h_i\}_{i\in I}$ in $\mathscr{V}\times\mathscr{V}$ is a frame for $\mathscr{V}\times\mathscr{V}$ if there exists constants $A,B>0$ such that

$$A\|f,g\|^2 \leqslant \sum_{i\in I}|\langle (f,g),(f_i,h_i)\rangle|^2 \leqslant B\|f,g\|^2,\ \ ((f,g)\in\mathscr{V}\times\mathscr{V}). \tag{3.1}$$

The numbers A,B are called frame bounds. The frame is normalized if $\|f_i,h_i\| = 1$ for each $i\in I$.

Cauchy-Schwarz inequality shaws that

$$\sum_{i=1}^{n} |\langle (f,g),(f_i,h_i)\rangle|^2 \leqslant \sum_{i=1}^{n} \|f,g\|^2 \|f_i,h_i\|^2, \ ((f,g) \in \mathscr{V} \times \mathscr{V}). \tag{3.2}$$

Let $\{f_i,b\}_{i=1}^{n}$ be a frame for $\mathscr{V} \times \langle b\rangle$ and define a linear mapping

$$T : \mathbb{C}^m \to \mathscr{V} \times \langle b\rangle, \ T\{c_i\}_{i=1}^{n} = \sum_{i=1}^{n} c_i (f_i,b)$$

T is usually called the pre-frame operator. Also, the adjoint operator is given

$$T^* : \mathscr{V} \times \langle b\rangle \to \mathbb{C}^m, \ (T^* f,b) = \{\langle (f,b),(f_i,b)\rangle\}_{i=1}^{n},$$

and is named the analysis operator. By composing T with its adjoint T^*, we obtain the frame operator

$$S : \mathscr{V} \times \langle b\rangle \to \mathscr{V} \times \langle b\rangle, \ S(f,b) = T(T^* f,b) = \sum_{i=1}^{n} \langle (f,b),(f_i,b)\rangle (f_i,b).$$

Note that in terms of the frame operator,

$$\langle (Sf,b)(f,b)\rangle = \sum_{i=1}^{n} |\langle (f,b),(f_i,b)\rangle|^2, \ ((f,b) \in \mathscr{V} \times \langle b\rangle).$$

DEFINITION 3.3. A frame $\{f_i,g_i\}_{i=1}^{n}$ is tight if

$$\sum_{i=1}^{n} |\langle (f,g),(f_i,h_i)\rangle|^2 = A\|f,g\|^2, \ ((f,g) \in \mathscr{V} \times \mathscr{V}).$$

PROPOSITION 3.4. *Let* $\{f_i,h_i\}_{i=1}^{n}$ *be a sequence in* $\mathscr{V} \times \mathscr{V}$. *Then* $\{f_i,h_i\}_{i=1}^{n}$ *is a frame for span* $\{f_i,h_i\}_{i=1}^{n}$.

Proof. We can assume that not all $\{f_i,h_i\}_{i=1}^{n}$ are zero. The upper frame condition is satisfied with

$$B = \sum_{i=1}^{n} \|f_i,h_i\|^2.$$

Now let

$$\mathscr{W} \times \mathscr{W} := \operatorname{span}\{f_i,h_i\}_{i=1}^{n},$$

and consider the continuous mapping

$$\phi : \mathscr{W} \times \mathscr{W} \to \mathbb{R}, \ \phi(f,g) = \sum_{i=1}^{n} |\langle (f,g),(f_i,h_i)\rangle|^2.$$

The unit ball is compact so

$$A =: \sum_{i=1}^{n} |\langle (w,k), (f_i,h_i)\rangle|^2$$
$$= \inf\left\{ \sum_{i=1}^{m} |\langle (f,g), (f_i,h_i)\rangle|^2 : \ (f,g) \in \mathscr{W} \times \mathscr{W}, \|f,g\| = 1 \right\}.$$

It is clear that $A > 0$. Now given $(f,g) \in \mathscr{W} \times \mathscr{W}$ and $(f,g) \neq 0$, we have

$$\sum_{i=1}^{n} |\langle (f,g), (f_i,h_i)\rangle|^2 = \sum_{i=1}^{n} \left| \left\langle \frac{(f,g)}{\|f,g\|}, (f_i,h_i) \right\rangle \right|^2 \|f,g\|^2 \geqslant A\|f,g\|^2. \quad \square$$

COROLLARY 3.5. *A family of elements* $\{f_i,h_i\}_{i\in I}$ *in* $\mathscr{V} \times \mathscr{V}$ *is a frame for* $\mathscr{V} \times \mathscr{V}$ *if and only if span* $\{f_i,h_i\}_{i\in I} = \mathscr{V} \times \mathscr{V}$.

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Abbas Alipour
Department of Mathematics
Kerman Branch, Islamic Azad University
Kerman, Iran
e-mail: abbasalipour1983@yahoo.com

Mohammad Hadi Akhbari
Department of Mathematics
Estahban Branch, Islamic Azad University
Estahban, Iran
e-mail: mhakhbari20@gmail.com