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HERMITE—HADAMARD–TYPE INEQUALITIES FOR *DIFFERENTIABLE MULTIPLICATIVE m–PREINVEXITY AND (s,m)–PREINVEXITY VIA THE MULTIPLICATIVE TEMPERED FRACTIONAL INTEGRALS

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Abstract. In virtue of the conception of the multiplicative tempered fractional integrals, put forward by Fu et al. in the published article [AIMS Math., 6 (7): 7456–7478, 2021], we present a fractional integral identity for *differentiable functions. Based upon it, we develop several inequalities of Hermite–Hadamard type in association with *differentiable multiplicative m-preinvexity and (s,m)-preinvexity.

1. Introduction

Throughout this paper let $\mathcal{K} \subseteq \mathbb{R}$ be a real interval, $\mathcal{K}^\circ$ be the interior of $\mathcal{K}$ and $\mathbb{R}^+ = (0,\infty)$.

Let $f : \mathcal{K} \to \mathbb{R}$ be a convex function defined on the real-valued interval $\mathcal{K}$ and $a,b \in \mathcal{K}$ along with $a < b$. The following inequalities

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2}$$

are well known in the literature as Hermite–Hadamard's inequalities.

Such type of classical inequality, under various generalized convexity conditions, has been deeply investigated in the sense of Riemann integrals by many researchers. For example, one can refer to Kórus [25] for s-convex functions, to Delavar and Sen [14] for h-convex functions, to Eken et al. [19] for p-convex functions, to Latif et al. [26] for harmonically-convex and harmonically quasi-convex functions, to Andrić and Pečarić [6] for $(h,g;m)$-convex functions, to Nikodem and Rajba [32] for (k,h)-convex set-valued functions, to Du and Zhou [18] for interval-valued co-ordinated convex functions and so on. For more recent results involving with this topical subject, we recommend the minded readers to consult the published articles [7, 22, 27, 40] and the bibliographies quoted in them.

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Recently, the generalizations and extensions for the conceptions of convexity have been considered by some scholars. Among them, one of the most important generalizations of convexity is the conception of preinvex sets and preinvex functions. Let us review them in the following ways:

DEFINITION 1.1. [41] The set $\Delta \subseteq \mathbb{R}^n$, with respect to the mapping $\zeta : \Delta \times \Delta \to \mathbb{R}^n$, is said to be invex, if it satisfies the following

$$x + t\zeta(y,x) \in \mathcal{K}, \quad \forall\, x,y \in \Delta, \quad t \in [0,1].$$

If we consider taking the mapping $\zeta(y,x) = y - x$, then the invex set becomes a convex set, but there are also some invex sets which are not convex.

DEFINITION 1.2. [16] A set $\Delta \subseteq \mathbb{R}^n$, with respect to the mapping $\eta : \Delta \times \Delta \times (0,1] \to \mathbb{R}^n$, is named as m-invex, if $mx + t\eta(y,x,m) \in \Delta$ holds true for all $x,y \in \Delta$ and $t \in [0,1]$ as well as certain fixed $m \in (0,1]$.

DEFINITION 1.3. [42] It is assumed that $\Delta \subseteq \mathbb{R}^n$ is an m-invex set with respect to the mapping $\eta : \Delta \times \Delta \times (0,1] \to \mathbb{R}^n$. For any $x,y \in \Delta$ and $m \in (0,1]$, the η_m-path $P_{\tau_1\tau_2}(y,x,m)$ linking the points $\tau_1 = mx$ and $\tau_2 = mx + \eta(y,x,m)$ is defined by the following expression

$$P_{\tau_1\tau_2}(y,x,m) = \Big\{\theta \,|\, \theta = mx + t\eta(y,x,m), t \in [0,1]\Big\}.$$

DEFINITION 1.4. [16] It is assumed that $\Delta \subseteq \mathbb{R}^n$ is an open m-invex set with respect to the mapping $\eta : \Delta \times \Delta \times (0,1] \to \mathbb{R}^n$. For certain fixed $s,m \in (0,1]$, the function $f : \Delta \to \mathbb{R}$ is said to be generalized (s,m)-preinvex if the following inequality

$$f\big(mx + t\eta(y,x,m)\big) \leqslant m(1-t)^s f(x) + t^s f(y)$$

holds for all $x,y \in \Delta$ and $t \in [0,1]$.

Noor provided the conception of the multiplicative preinvexity, also called log-preinvexity, in the following way:

DEFINITION 1.5. [33] The strictly positive function f defined on the invex set Δ, with respect to the mapping $\zeta : \Delta \times \Delta \to \mathbb{R}^n$, is said to be multiplicatively preinvex, if it satisfies the following inequality

$$f(x + t\zeta(y,x)) \leqslant (f(x))^{1-t}(f(y))^t, \quad \forall\, x,y \in \Delta, \quad t \in [0,1].$$

In 2022, Cao et al. introduced the following conception of the (λ,η)-incomplete gamma functions.

DEFINITION 1.6. [13] It is assumed that the mapping $\eta : \mathcal{K}\times\mathcal{K}\times(0,1]\to\mathbb{R}$, where $\mathcal{K}\subseteq\mathbb{R}$ be an open m-invex subset with some fixed $m\in(0,1]$. For any real numbers $\alpha,\lambda>0$ along with $x\geqslant 0$, the (λ,η)-incomplete gamma function is defined by in the following ways:

$$\gamma_{\lambda\eta(b,a,m)}(\alpha,x)=\int_0^x t^{\alpha-1}e^{-\lambda\eta(b,a,m)t}\mathrm{d}t.$$

If we consider taking the mapping $\eta(b,a,m)=1$, then the definition of the (λ,η)-incomplete gamma functions reduces to the definition of λ-incomplete gamma functions. In particular, if we consider choosing $\lambda=1$, then it transfers to the incomplete gamma functions.

The relations between the (λ,η)-incomplete gamma functions and λ-incomplete gamma functions are the following ones (see [13])

(1) $\gamma_{\lambda\eta(b,a,m)}(\alpha,1)=\int_0^1 u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}\mathrm{d}u=\frac{1}{\eta^{\alpha}(b,a,m)}\gamma_{\lambda}\Big(\alpha,\eta(b,a,m)\Big).$

(2) $\int_0^1 \gamma_{\lambda\eta(b,a,m)}(\alpha,x)\mathrm{d}u=\frac{\gamma_{\lambda}(\alpha,\eta(b,a,m))}{\eta^{\alpha}(b,a,m)}-\frac{\gamma_{\lambda}(\alpha+1,\eta(b,a,m))}{\eta^{\alpha+1}(b,a,m)}.$

Abdeljawad and Grossman introduced a family of fractional integrals, called the multiplicative Riemann–Liouville (RL) fractional integrals, in the following way:

DEFINITION 1.7. [1] The multiplicative left-sided RL-fractional integrals ${}_a\mathcal{I}_*^{\alpha}f(x)$ of order $\alpha\in\mathbb{C},\mathrm{Re}(\alpha)>0$ is defined by

$${}_a\mathcal{I}_*^{\alpha}f(x)=\exp\left\{\left(\mathcal{I}_{a+}^{\alpha}(\ln\circ f)\right)(x)\right\},$$

and the multiplicative right-sided one ${}_*\mathcal{I}_b^{\alpha}f(x)$ is defined by

$${}_*\mathcal{I}_b^{\alpha}f(x)=\exp\left\{\left(\mathcal{I}_{b-}^{\alpha}(\ln\circ f)\right)(x)\right\},$$

where the symbols $\mathcal{I}_{a+}^{\alpha}f(x)$ and $\mathcal{I}_{b-}^{\alpha}f(x)$ denote respectively the left- and right-sided RL-fractional integrals, which are defined, correspondingly, by the following ones

$$\mathcal{I}_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t,\quad x>a,$$

and

$$\mathcal{I}_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t,\quad x<b.$$

A generalization of the multiplicative RL-fractional integrals appeared in [21], called as multiplicative tempered fractional integrals, is defined in the following ways:

DEFINITION 1.8. [21] The multiplicative left-sided tempered fractional integrals ${}_a\mathcal{I}_*^{\alpha,\lambda} f(x)$ of order $\alpha \in \mathbb{C}$, together with $\mathrm{Re}(\alpha) > 0$, is defined by

$$ {}_a\mathcal{I}_*^{\alpha,\lambda} f(x) = \exp\left\{ \left(\mathcal{I}_{a+}^{\alpha,\lambda} (\ln \circ f) \right) (x) \right\}, \ \ \lambda \geqslant 0, $$

and the multiplicative right-sided one ${}_*\mathcal{I}_b^{\alpha,\lambda} f(x)$ is defined by

$$ {}_*\mathcal{I}_b^{\alpha,\lambda} f(x) = \exp\left\{ \left(\mathcal{I}_{b-}^{\alpha,\lambda} (\ln \circ f) \right) (x) \right\}, \ \ \lambda \geqslant 0, $$

where the symbols $\mathcal{I}_{a+}^{\alpha,\lambda} f(x)$ and $\mathcal{I}_{b-}^{\alpha,\lambda} f(x)$ denote respectively the left- and right-sided tempered fractional integrals, which are defined, correspondingly, by the following ones (see [36])

$$ \mathcal{I}_{a+}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t) \mathrm{d}t, \quad x > a, $$

and

$$ \mathcal{I}_{b-}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t) \mathrm{d}t, \quad x < b. $$

For recent results in connection with multiplicative integral inequalities, the readers can refer to [4, 9, 29, 31, 34] and the references therein.

Fractional calculus, as one of the fastest developing parts in mathematical analysis, has played a crucial cornerstone in approximation theory, especially the fractional integral operators have become a prevalent tool in dealing with integral inequalities. Here, we list a bunch of different kinds of inequalities established by different fractional integral operators. For example, in the sense of RL- and Hadamard fractional integrals, Bounoua and Yin [10] built the Simpson-type integral inequalities. Further, the approximate schemes of the results obtained in their work were studied as well. In the frame of the tempered fractional integrals, Mohammed et al. [30] researched a few inequalities of the Hermite–Hadamard type refer to the λ-incomplete gamma functions. Furthermore, Akhtar and Awan [3] generalized the tempered fractional integrals to the δ-tempered fractional integrals, by means of it, and obtained Hermite–Hadamard's inequalities for harmonically convex functions. By virtue of the fractional integral operators with exponential kernels, Ahmad et al. [2] established Hermite–Hadamard-, Hermite–Hadamard–Fejér-, Dragomir–Agarwal- and Pachpatte-type inequalities, which generalized classical inequalities involving convex functions. By using AB-fractional integral operators, Set et al. [38] presented inequalities of Hermite–Hadamard type involving with twice differentiable convex functions. As a form of generalization of AB-fractional integral operators, the authors in Ref. [12] gave Hermite–Hadamard-type inequalities under the setting of ABK-fractional integrals. And they gave some simulation results via different parameters values. Taking advantage of local fractional integral operators, Meftah et al. [28] derived the fractal Maclaurin-type integral inequalities. In terms of generalized fractional integrals, Du et al. [17] deduced Bullen-type fractional inequalities and considered its applications. Additionally, the authors in the paper [39], with the help of the RL-fractional integrals, set up the Newton-type inequalities for differentiable convex functions. They also pointed out the results established are the extension of already existing results in the literature. For more interesting outcomes with relation to the fractional integrals by different approaches, we recommend the minded readers to glance over the published articles [15, 20, 24, 37] for reference.

Inspired by the above-mentioned outcomes, especially these developed in [11] and [21], the present article focus on investigating some inequalities of Hermite–Hadamard type through the multiplicative tempered fractional integrals. For this purpose, we propose an identity for *differentiable functions. And using it as an auxiliary result, we obtain certain estimates of the upper bounds involving the multiplicative tempered fractional integral inequalities.

2. Multiplicative calculus

In 2008, Bashirov et al. [8] proposed a family of multiplicative integral operators, called the *integral operators, which is denoted by $\int_a^b (f(x))^{\mathrm{d}x}$. And the classical Riemann integrals is denoted by $\int_a^b f(x)\mathrm{d}x$. Let us retrospect that the function f is multiplicatively integrable defined on the real-valued interval $[a,b]$, if f is positive and Riemann integrable on the real-valued interval $[a,b]$. The *integral operator, that is to say, is given by in the following way:

$$\int_a^b (f(x))^{\mathrm{d}x} = \exp\left\{\int_a^b \ln(f(x))\mathrm{d}x\right\}.$$

PROPOSITION 2.1. [8] *It is assumed that the positive function f belongs to *integrable on the real-valued interval $[a,b]$. Then, we have the following properties:*

$$(i)\ \int_a^b ((f(x))^p)^{\mathrm{d}x} = \left(\int_a^b (f(x))^{\mathrm{d}x}\right)^p,\ p\in\mathbb{R},$$

$$(ii)\ \int_a^b (f(x)g(x))^{\mathrm{d}x} = \int_a^b (f(x))^{\mathrm{d}x}.\int_a^b (g(x))^{\mathrm{d}x},$$

$$(iii)\ \int_a^b \left(\frac{f(x)}{g(x)}\right)^{\mathrm{d}x} = \frac{\int_a^b (f(x))^{\mathrm{d}x}}{\int_a^b (g(x))^{\mathrm{d}x}},$$

$$(iv)\ \int_a^b (f(x))^{\mathrm{d}x} = \int_a^c (f(x))^{\mathrm{d}x}.\int_c^b (f(x))^{\mathrm{d}x}, \quad a\leqslant c\leqslant b,$$

$$(v)\ \int_a^a (f(x))^{\mathrm{d}x} = 1 \quad \textit{and} \quad \int_a^b (f(x))^{\mathrm{d}x} = \left(\int_b^a (f(x))^{\mathrm{d}x}\right)^{-1}.$$

Bashirov et al. also proposed the multiplicative derivative of the functions.

DEFINITION 2.1. [8] Given that the function $f:\mathbb{R}\to\mathbb{R}^{+}$. The multiplicative derivative of the function f is given by

$$\frac{\mathrm{d}^{*}f(t)}{\mathrm{d}t}=f^{*}(t)=\lim_{h\to 0}\left(\frac{f(t+h)}{f(t)}\right)^{\frac{1}{h}}.$$

The relation between the f^{*} and the ordinary derivative f' is the following

$$f^{*}(t)=\exp\left\{[\ln f(t)]'\right\}=\exp\left\{\frac{f'(t)}{f(t)}\right\}.$$

PROPOSITION 2.2. [8] *Given that the functions f and g are both multiplicative derivative, and h is differentiable. If c is a positive constant, then functions cf, $f+g$, fg, $\frac{f}{g}$, f^{h} and $f\circ h$ are all multiplicative derivative, and we have the following properties*

(*i*) $(cf)^{*}(x)=f^{*}(x)$,

(*ii*) $(f+g)^{*}(x)=f^{*}(x)^{\frac{f(x)}{f(x)+g(x)}}\cdot g^{*}(x)^{\frac{g(x)}{f(x)+g(x)}}$,

(*iii*) $(fg)^{*}(x)=f^{*}(x)g^{*}(x)$,

(*iv*) $\left(\frac{f}{g}\right)^{*}(x)=\frac{f^{*}(x)}{g^{*}(x)}$,

(*v*) $\left(f^{h}\right)^{*}(x)=f^{*}(x)^{h(x)}\cdot f(x)^{h'(x)}$,

(*vi*) $(f\circ h)^{*}(x)=f^{*}(h(x))^{h'(x)}$.

The formulas of the multiplicative integration by parts are the following ones.

THEOREM 2.1. [8] *Let $f:[a,b]\to\mathbb{R}$ be multiplicative differentiable, and let $g:[a,b]\to\mathbb{R}$ be differentiable. Then, the function f^{g} is multiplicative integrable. And we have that*

$$\int_{a}^{b}\left(f^{*}(x)^{g(x)}\right)^{\mathrm{d}x}=\frac{f(b)^{g(b)}}{f(a)^{g(a)}}\cdot\frac{1}{\int_{a}^{b}\left(f(x)^{g'(x)}\right)^{\mathrm{d}x}}.$$

LEMMA 2.1. [5] *Let $f:[a,b]\to\mathbb{R}$ be multiplicative differentiable, and let $g:[a,b]\to\mathbb{R}$ and $h:J\subset\mathbb{R}\to[a,b]$ be two differentiable functions. Then, we have that*

$$\int_{a}^{b}\left(f^{*}(h(x))^{g(x)h'(x)}\right)^{\mathrm{d}x}=\frac{f(h(b))^{g(b)}}{f(h(a))^{g(a)}}\cdot\frac{1}{\int_{a}^{b}\left(f(h(x))^{g'(x)}\right)^{\mathrm{d}x}}.$$

3. Main results

Before establishing our primary results, we first introduce the following definitions.

DEFINITION 3.1. It is assumed that $\mathcal{K} \subseteq \mathbb{R}$ is an m-invex set with respect to the mapping $\eta : \mathcal{K} \times \mathcal{K} \times (0,1] \to \mathbb{R}$. A function $f\colon\ \mathcal{K} \to \mathbb{R}^+$, with respect to the mapping η, is said to be multiplicatively m-preinvex or m-log-preinvex, if it satisfies the following inequality

$$f(mx + t\eta(y,x,m)) \leqslant [f(x)]^{m(1-t)}[f(y)]^t, \tag{3.1}$$

for all x, y $\in \mathcal{K}$ and $t \in [0,1]$ and some fixed $m \in (0,1]$.

DEFINITION 3.2. It is assumed that $\mathcal{K} \subseteq \mathbb{R}$ is an m-invex set with respect to the mapping $\eta : \mathcal{K} \times \mathcal{K} \times (0,1] \to \mathbb{R}$. A function $f\colon\ \mathcal{K} \to \mathbb{R}^+$, with respect to the mapping η, is said to be multiplicatively (s,m)-preinvex or (s,m)-log-preinvex, if it satisfies the following inequality

$$f(mx + t\eta(y,x,m)) \leqslant [f(x)]^{m(1-t)^s}[f(y)]^{t^s}, \tag{3.2}$$

for all $x, y \in \mathcal{K}$ and $t \in [0,1]$ together with some fixed $s, m \in (0,1]$.

REMARK 3.1. The following conclusions can be drawn by considering some special cases:

(i) If we consider taking $m = 1$ in Definition 3.1, then we have the conception of the multiplicative preinvexity. Furthermore, if the mapping $\eta(y,x,m)$ reduces to $\eta(y,x,m) = y - mx$ with $m = 1$, then we have the conception of multiplicative convexity.

(ii) If we consider taking the mapping $\eta(y,x,m) = y - mx$ with $m = 1$ in Definition 3.2, then we have the conception of multiplicative s-convexity.

Next, we discuss some properties for multiplicatively m-preinvex functions and multiplicatively (s,m)-preinvex functions.

PROPOSITION 3.1. *Let $f, g : \mathcal{K} \to [1,\infty)$. If f and g are both multiplicatively m-preinvex functions, then fg is a multiplicatively m-preinvex function.*

PROPOSITION 3.2. *If $f_i : \mathcal{K} \to [1,\infty]$ are multiplicatively m-preinvex functions with respect to the same mapping $\eta : \mathbb{R} \times \mathbb{R} \times (0,1] \to \mathbb{R}$ for the same fixed $m \in (0,1]$, then the function $f = \prod a_i f_i, a_i \geqslant 0, (i = 1,2\cdots n)$ is also a multiplicatively m-preinvex function with respect to the same mapping η.*

PROPOSITION 3.3. *Let $f : \mathcal{K} \to [1,\infty)$. Every multiplicatively m-preinvex function is a multiplicatively (s,m)-preinvex function.*

Proof. The proof is clear from the following inequalities

$$t \leqslant t^s \text{ and } 1-t \leqslant (1-t)^s,$$

for all $t \in [0,1]$ with some fixed $s \in (0,1]$.

Therefore,

$$f(mx+t\eta(y,x,m)) \leqslant [f(x)]^{m(1-t)}[f(y)]^t \leqslant [f(x)]^{m(1-t)^s}[f(y)]^{t^s},$$

for all $x,y \in \mathcal{K}, t \in [0,1]$ and fixed $s \in (0,1]$. Thus, the desired result is obtained.

PROPOSITION 3.4. *Let $f : \mathbb{R}^+ \to [1,\infty)$ is a multiplicatively m-preinvex function with respect to the mapping $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \times (0,1] \to \mathbb{R}^+$ for some fixed $m \in (0,1]$. Assumed that the function f is monotone decreasing, and the mapping η is monotone increasing regarding m for fixed $x,y \in \mathbb{R}^+$ along with $m_1 \leqslant m_2$, $m_1,m_2 \in (0,1]$. If f is a multiplicatively m_1-preinvex function on $\mathbb{R}^+$ with respect to the mapping η, then f is a multiplicatively m_2-preinvex function on $\mathbb{R}^+$ with respect to the same mapping η.*

Proof. Since f is a multiplicatively m-preinvex function, for all $x,y \in \mathbb{R}^+$, we have that

$$f(m_1x+t\eta(y,x,m_1)) \leqslant [f(x)]^{m_1(1-t)}[f(y)]^t.$$

Combining the conditions f is monotone decreasing, and the mapping η is monotone increasing regarding m for fixed $x,y \in \mathbb{R}^+$ and $m_1 \leqslant m_2$, it follows that

$$f(m_2x+t\eta(y,x,m_2)) \leqslant f(m_1x+t\eta(y,x,m_1)),$$

and

$$[f(x)]^{m_1(1-t)}[f(y)]^t \leqslant [f(x)]^{m_2(1-t)}[f(y)]^t.$$

Following the above two inequalities, we have that

$$f(m_2x+t\eta(y,x,m_2)) \leqslant [f(x)]^{m_2(1-t)}[f(y)]^t.$$

Hence, f is also a multiplicatively m_2-preinvex function on $\mathbb{R}^+$ with respect to the mapping η, which completes the proof.

Our main results depend on the following lemma.

LEMMA 3.1. *Let $\mathcal{K} \subseteq \mathbb{R}$ be an open m-invex subset with respect to the mapping $\eta : \mathcal{K} \times \mathcal{K} \times (0,1] \to \mathbb{R}^+$ for certain fixed $m \in (0,1]$. And let ma, $ma+\eta(b,a,m)$ lie on the η_m-path $P_{\tau_1\tau_2}(b,a,m)$, for $a,b \in \mathcal{K}$ together with $a < b$. It is supposed that $f : \mathcal{K} \to \mathbb{R}^+$ is a *differentiable function on $\mathcal{K}^\circ$. If f^* is integrable on interval*

$[ma, ma+\eta(b,a,m)]$, then for $\alpha > 0$ and $\lambda \geqslant 0$, the following equality with regard to the multiplicative tempered fractional integrals holds:

$$
\begin{aligned}
{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m) = &\int_0^1 \left[f^*\big(ma+t\eta(b,a,m)\big)^{\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)} \right]^{\mathrm{d}t} \\
&\times \int_0^1 \left[f^*\big(ma+(1-t)\eta(b,a,m)\big)^{-\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)} \right]^{\mathrm{d}t},
\end{aligned} \tag{3.3}
$$

where

$$
\begin{aligned}
&{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m) \\
&:= \frac{\sqrt{f(ma)f(ma+\eta(b,a,m))}}{\left[{}_*\mathcal{I}^{\alpha,\lambda}_{[ma+\eta(b,a,m)]} f(ma) \cdot {}_{ma}\mathcal{I}^{\alpha,\lambda}_{*} f(ma+\eta(b,a,m)) \right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,\eta(b,a,m))}}},
\end{aligned} \tag{3.4}
$$

and

$$
\omega = \frac{\eta^\alpha(b,a,m)}{2\gamma_\lambda\big(\alpha,\eta(b,a,m)\big)}. \tag{3.5}
$$

Proof. For the convenience of expression, let us define the quantities

$$
\rho_1 = \int_0^1 \left[\Big(f^*\big(ma+t\eta(b,a,m)\big)\Big)^{\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)} \right]^{\mathrm{d}t},
$$

and

$$
\rho_2 = \int_0^1 \left[\Big(f^*\big(ma+(1-t)\eta(b,a,m)\big)\Big)^{-\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)} \right]^{\mathrm{d}t}.
$$

Applying the multiplicative integration by parts, we have that

$$
\begin{aligned}
\rho_1 = &\frac{\big[f\big(ma+\eta(b,a,m)\big)\big]^{\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,1)}}{[f(ma)]^0} \\
&\times \frac{1}{\int_0^1 \left[\Big(f\big(ma+t\eta(b,a,m)\big)\Big)^{\omega t^{\alpha-1}e^{-\lambda\eta(b,a,m)t}} \right]^{\mathrm{d}t}},
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_2 = &\frac{[f(ma)]^{\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,1)}}{\big[f\big(ma+\eta(b,a,m)\big)\big]^0} \\
&\times \frac{1}{\int_0^1 \left[\Big(f\big(ma+(1-t)\eta(b,a,m)\big)\Big)^{\omega t^{\alpha-1}e^{-\lambda\eta(b,a,m)t}} \right]^{\mathrm{d}t}}.
\end{aligned}
$$

Thus, we derive that

$$\begin{aligned}
\rho_1 \times \rho_2 = & \left[[f(ma+\eta(b,a,m))]^{\frac{1}{2}} \cdot [f(ma)]^{\frac{1}{2}} \right] \\
& : \left[\int_0^1 \left[\left(f\left(ma+t\eta(b,a,m)\right)\right)^{\omega t^{\alpha-1} e^{-\lambda\eta(b,a,m)t}} \right]^{\mathrm{d}t} \right. \\
& \left. \times \int_0^1 \left[\left(f\left(ma+(1-t)\eta(b,a,m)\right)\right)^{\omega t^{\alpha-1} e^{-\lambda\eta(b,a,m)t}} \right]^{\mathrm{d}t} \right] \\
= & \frac{\left[f\left(ma+\eta(b,a,m)\right)\right]^{\frac{1}{2}}}{\exp\left\{ \int_0^1 \omega \cdot t^{\alpha-1} e^{-\lambda\eta(b,a,m)t} \cdot \ln f\left(ma+t\eta(b,a,m)\right)\mathrm{d}t \right\}} \\
& \times \frac{[f(ma)]^{\frac{1}{2}}}{\exp\left\{ \int_0^1 \omega \cdot t^{\alpha-1} e^{-\lambda\eta(b,a,m)t} \cdot \ln f\left(ma+(1-t)\eta(b,a,m)\right)\mathrm{d}t \right\}} \\
= & \frac{\left[f\left(ma+\eta(b,a,m)\right)\right]^{\frac{1}{2}} \cdot [f(ma)]^{\frac{1}{2}}}{\exp\{I_1+I_2\}}.
\end{aligned} \tag{3.6}$$

Taking advantage of the change of variable $u = ma + t\eta(b,a,m)$, we have that

$$\begin{aligned}
I_1 = & \int_0^1 \omega \cdot t^{\alpha-1} e^{-\lambda\eta(b,a,m)t} \cdot \ln f\left(ma+t\eta(b,a,m)\right)\mathrm{d}t \\
= & \,\omega \int_{ma}^{ma+\eta(b,a,m)} \left(\frac{u-ma}{\eta(b,a,m)} \right)^{\alpha-1} e^{-\lambda(u-ma)} \\
& \times \ln f(u) \cdot \frac{1}{\eta(b,a,m)} \mathrm{d}u \\
= & \frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,\eta(b,a,m))} \mathcal{I}^{\alpha,\lambda}_{[ma+\eta(b,a,m)]^-} \ln f(ma).
\end{aligned} \tag{3.7}$$

Analogously, we can deduce that

$$\begin{aligned}
I_2 = & \int_0^1 \omega \cdot t^{\alpha-1} e^{-\lambda\eta(b,a,m)t} \cdot \ln\left(f\left(ma+(1-t)\eta(b,a,m)\right)\right)\mathrm{d}t \\
= & \frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,\eta(b,a,m))} \mathcal{I}^{\alpha,\lambda}_{[ma]^+} \ln f\left(ma+\eta(b,a,m)\right).
\end{aligned} \tag{3.8}$$

Substituting equations (3.7) and (3.8) into (3.6), we can get the required identity. This ends the proof.

COROLLARY 3.1. *In Lemma 3.1, if we consider taking $\lambda = 0$, then we have the following identity in association with multiplicative RL-fractional integrals*

$$\begin{aligned}
&\frac{\sqrt{f(ma)f\big(ma+\eta(b,a,m)\big)}}{\Big[{}_*\mathcal{I}^{\alpha}_{[ma+\eta(b,a,m)]}f(ma)\cdot{}_{ma}\mathcal{I}^{\alpha}_{*}f\big(ma+\eta(b,a,m)\big)\Big]^{\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a,m)}}} \\
&\quad=\int_0^1\Big[\big(f^*\big(ma+t\eta(b,a,m)\big)\big)^{\frac{t^{\alpha}}{2}\eta(b,a,m)}\Big]^{\mathrm{d}t} \\
&\qquad\times\int_0^1\Big[\big(f^*\big(ma+(1-t)\eta(b,a,m)\big)\big)^{-\frac{t^{\alpha}}{2}\eta(b,a,m)}\Big]^{\mathrm{d}t}.
\end{aligned} \tag{3.9}$$

COROLLARY 3.2. *In Lemma 3.1, if we consider taking $\lambda = 0$ and $\alpha = 1$, then we have the following identity in association with multiplicative Riemann integrals*

$$\begin{aligned}
&\frac{\sqrt{f(ma)f\big(ma+\eta(b,a,m)\big)}}{\Big(\int_{ma}^{ma+\eta(b,a,m)}(f(u))^{\mathrm{d}u}\Big)^{\frac{1}{\eta(b,a,m)}}} \\
&\quad=\int_0^1\Big[\big(f^*\big(ma+t\eta(b,a,m)\big)\big)^{\frac{t}{2}\eta(b,a,m)}\Big]^{\mathrm{d}t} \\
&\qquad\times\int_0^1\Big[\big(f^*\big(ma+(1-t)\eta(b,a,m)\big)\big)^{-\frac{t}{2}\eta(b,a,m)}\Big]^{\mathrm{d}t}.
\end{aligned} \tag{3.10}$$

We are now in a position to establish the following multiplicative fractional integral inequalities for multiplicatively m-preinvex functions.

THEOREM 3.1. *Let $\mathcal{K} \subseteq \mathbb{R}$ be an open m-invex subset with respect to the mapping $\eta : \mathcal{K}\times\mathcal{K}\times(0,1] \to \mathbb{R}^+$ for certain fixed $m \in (0,1]$. And let ma, $ma+\eta(b,a,m)$ lie on the η_m-path $P_{\tau_1\tau_2}(b,a,m)$, for $a,b \in \mathcal{K}$ together with $a < b$. It is supposed that $f : \mathcal{K} \to \mathbb{R}^+$ is an increasing *differentiable function on $\mathcal{K}^\circ$. And if the function f^* is multiplicatively m-preinvex on $\mathcal{K}$, then the following inequality holds:*

$$\big|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\big| \leqslant \Big[(f^*(a))^m\cdot f^*(b)\Big]^{\frac{1}{2}\varphi}, \tag{3.11}$$

where

$$\varphi = \eta(b,a,m) - \frac{\gamma_\lambda\big(\alpha+1,\eta(b,a,m)\big)}{\gamma_\lambda\big(\alpha,\eta(b,a,m)\big)}. \tag{3.12}$$

Proof. Making use of Lemma 3.1, and the increasing property of f, we deduce that

$$\begin{aligned}
&\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \\
&= \left| \int_0^1 \left[\left(f^*\left(ma+t\eta(b,a,m)\right)\right)^{\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)}\right]^{\mathrm{d}t} \right. \\
&\quad \left. \times \int_0^1 \left[\left(f^*\left(ma+(1-t)\eta(b,a,m)\right)\right)^{-\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)}\right]^{\mathrm{d}t} \right| \\
&\leqslant \exp\left\{ \int_0^1 \left|\ln\left(f^*\left(ma+t\eta(b,a,m)\right)^{\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)}\right)\right| \mathrm{d}t \right\} \\
&\quad \times \exp\left\{ \int_0^1 \left|\ln\left(f^*\left(ma+(1-t)\eta(b,a,m)\right)^{-\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)}\right)\right| \mathrm{d}t \right\} \\
&\leqslant \exp\left\{ \int_0^1 \left|\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right| \cdot \ln f^*\left(ma+t\eta(b,a,m)\right)\mathrm{d}t \right\} \\
&\quad \times \exp\left\{ \int_0^1 \left|-\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right| \cdot \ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\mathrm{d}t \right\} \\
&= \exp\left\{ \int_0^1 \left|\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right| \right. \\
&\quad \left. \times \left[\ln f^*\left(ma+t\eta(b,a,m)\right) + \ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right]\mathrm{d}t \right\}.
\end{aligned} \tag{3.13}$$

Since f^* is a multiplicatively m-preinvex function, we get that

$$\ln f^*\left(ma+t\eta(b,a,m)\right) \leqslant m(1-t)\ln f^*(a) + t\ln f^*(b), \tag{3.14}$$

and

$$\ln f^*\left(ma+(1-t)\eta(b,a,m)\right) \leqslant mt\ln f^*(a) + (1-t)\ln f^*(b). \tag{3.15}$$

Combining (3.14) and (3.15) with (3.13), we obtain that

$$\begin{aligned}
&\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \\
&\leqslant \exp\left\{ \eta(b,a,m)\omega\left[m\ln f^*(a) + \ln f^*(b)\right] \right. \\
&\quad \left. \times \left[\frac{\gamma_\lambda\left(\alpha,\eta(b,a,m)\right)}{\eta^\alpha(b,a,m)} - \frac{\gamma_\lambda\left(\alpha+1,\eta(b,a,m)\right)}{\eta^{\alpha+1}(b,a,m)}\right] \right\} \\
&= \exp\left\{ \left[\frac{\eta(b,a,m)}{2} - \frac{\gamma_\lambda\left(\alpha+1,\eta(b,a,m)\right)}{2\gamma_\lambda\left(\alpha,\eta(b,a,m)\right)}\right] \cdot \left[m\ln f^*(a) + \ln f^*(b)\right] \right\} \\
&= \left[(f^*(a))^m \cdot f^*(b)\right]^{\frac{\eta(b,a,m)}{2} - \frac{\gamma_\lambda(\alpha+1,\eta(b,a,m))}{2\gamma_\lambda(\alpha,\eta(b,a,m))}} \\
&= \left[(f^*(a))^m \cdot f^*(b)\right]^{\frac{1}{2}\varphi},
\end{aligned}$$

which is the required result. The proof is completed.

For displaying the result of Theorem 3.1 more visually, we here offer an example to illustrate the correctness of Theorem 3.1.

EXAMPLE 3.1. Let the multiplicatively preinvex function $\frac{f'(x)}{f(x)}:\ (0,\infty)\to(0,\infty)$ be defined by $\frac{f'(x)}{f(x)}=\frac{1}{x}$ with respect to the mapping $\eta(y,x,m)=y-mx$ with $m=1$. We can get $f^*(x)=e^{\frac{1}{x}}$ is a generalized multiplicatively 1-preinvex function. It is easy to check that $f(x)=x$. Thus, all assumptions in Theorem 3.1 are satisfied. If we take $a=1$, $b=2$, $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{2}$, then the left-hand side term of (3.11) is

$$\begin{aligned}
\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| &= \left|\frac{\sqrt{f(1)f(2)}}{\left[e^{\mathcal{I}_{1+}^{\frac{1}{2},\frac{1}{2}}\ln f(2)}\cdot e^{\mathcal{I}_{2-}^{\frac{1}{2},\frac{1}{2}}\ln f(1)}\right]^{\frac{\Gamma\left(\frac{1}{2}\right)}{2\gamma_{\frac{1}{2}}\left(\frac{1}{2},1\right)}}}\right| \\
&= \left|\frac{\sqrt{2}}{\left[e^{\int_1^2\ln u\cdot(2-u)^{-\frac{1}{2}}e^{-\frac{1}{2}(2-u)}\mathrm{d}u+\int_1^2\ln u\cdot(u-1)^{-\frac{1}{2}}e^{-\frac{1}{2}(u-1)}\mathrm{d}u}\right]^{\frac{1}{2\gamma_{\frac{1}{2}}\left(\frac{1}{2},1\right)}}}\right| \\
&\approx 0.9702,
\end{aligned}$$

and the right-hand side term of (3.11) is

$$\left[(f^*(a))^m\cdot f^*(b)\right]^{\frac{1}{2}\varphi}=\left(e^{\frac{3}{2}}\right)^{\frac{1}{2}\left[1-\frac{\gamma_{\frac{1}{2}}\left(\frac{3}{2},1\right)}{\gamma_{\frac{1}{2}}\left(\frac{1}{2},1\right)}\right]}\approx 1.7017.$$

It is clear that $0.9702<1.7017$, which demonstrates the correctness of the result described in Theorem 3.1.

REMARK 3.2. *Case* 1: It is assumed that the parameter α is not a fixed constant in Example 3.1. For instance, if we consider putting the parameter $\alpha\in(0,1]$, in accordance with Theorem 3.1, then we get the result for the parameter α as below

$$-\left[f^*(1)\cdot f^*(2)\right]^{\frac{1}{2}\varphi}\leqslant\frac{\sqrt{2}}{\left[e^{\mathcal{I}_{1+}^{\alpha,\frac{1}{2}}\ln f(2)}\cdot e^{\mathcal{I}_{2-}^{\alpha,\frac{1}{2}}\ln f(1)}\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\frac{1}{2}}(\alpha,1)}}}\leqslant\left[f^*(1)\cdot f^*(2)\right]^{\frac{1}{2}\varphi}, \tag{3.16}$$

where

$$\varphi=1-\frac{\gamma_{\frac{1}{2}}(\alpha+1,1)}{\gamma_{\frac{1}{2}}(\alpha,1)}.$$

Three functions given by the inequalities (3.16) pertaining to the left-, middle- and right-sides are plotted in Figure 3.1 for the parameter $\alpha \in (0,1]$. From Figure 3.1, we can intuitively observe that the value on the left is less than the value on the middle, and the value on the middle is less than the value on the right, which is consistent with the theoretical result given in Theorem 3.1.

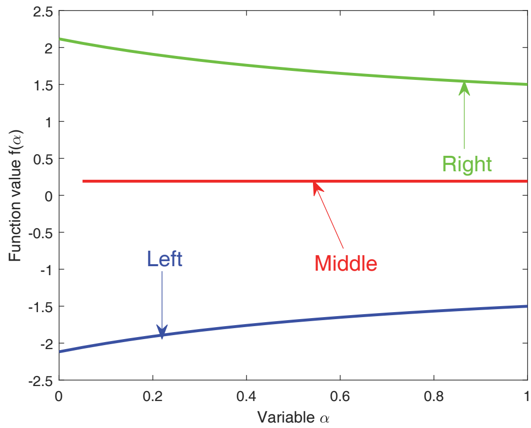

Figure 3.1: *Graphical representation for Example* 3.1 *for the variable* $\alpha \in (0,1]$ *with* $\lambda = \frac{1}{2}$

Case 2: It is assumed that the parameter λ is not a fixed constant in Example 3.1. For instance, if we consider putting the parameter $\lambda \in [0,1]$ with $\alpha = \frac{1}{2}$ in Theorem 3.1, then we get the result for the parameter λ as below

$$-\left[f^*(1)\cdot f^*(2)\right]^{\frac{1}{2}\varphi} \leqslant \frac{\sqrt{2}}{\left[e^{\mathcal{I}_{1+}^{\frac{1}{2},\lambda}\ln f(2)} \cdot e^{\mathcal{I}_{2-}^{\frac{1}{2},\lambda}\ln f(1)}\right]^{\frac{\Gamma(\frac{1}{2})}{2\gamma_\lambda\left(\frac{1}{2},1\right)}}} \leqslant \left[f^*(1)\cdot f^*(2)\right]^{\frac{1}{2}\varphi}, \tag{3.17}$$

where

$$\varphi = 1 - \frac{\gamma_\lambda\left(\frac{3}{2},1\right)}{\gamma_\lambda\left(\frac{1}{2},1\right)}.$$

The visualization results of three functions given by the inequalities (3.17) pertaining to the left-, middle- and right-sides are plotted in Figure 3.2. From the visual perspective of graphics, it vividly describes the result exhibited in the inequalities (3.17). The result displayed in the Figure 3.2 is consistent with the theoretical result given in Theorem 3.1.

Case 3: It is assumed that the parameters α and λ are not two fixed constants in Example 3.1. For instance, if we consider putting the parameters $\alpha \in (0,1]$ and $\lambda \in [0,1]$ in Theorem 3.1, then we get the result for the parameters α and λ as below

$$-\left[f^*(1)\cdot f^*(2)\right]^{\frac{1}{2}\varphi} \leqslant \frac{\sqrt{2}}{\left[e^{\mathcal{I}_{1+}^{\alpha,\lambda}\ln f(2)} \cdot e^{\mathcal{I}_{2-}^{\alpha,\lambda}\ln f(1)}\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,1)}}} \leqslant \left[f^*(1)\cdot f^*(2)\right]^{\frac{1}{2}\varphi}, \tag{3.18}$$

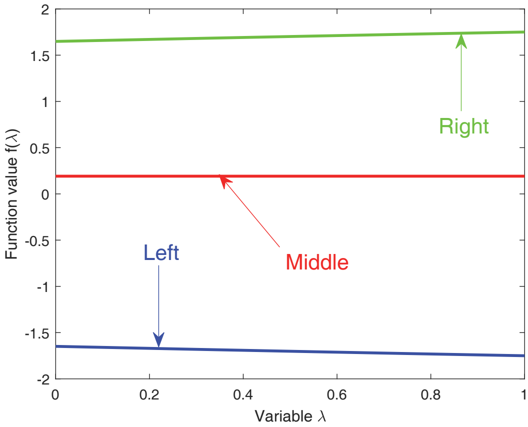

Figure 3.2: *Graphical representation for Example 3.1 for the variable* $\lambda \in [0,1]$ *with* $\alpha = \frac{1}{2}$

where

$$\varphi = 1 - \frac{\gamma_\lambda(\alpha+1,1)}{\gamma_\lambda(\alpha,1)}.$$

The visualization results of three functions given by the inequalities (3.18) pertaining to the left-, middle- and right-sides are plotted in Figure 3.3. From the visual perspective of graphics, it vividly describes the result exhibited in the inequalities (3.18). The result displayed in the Figure 3.3 is consistent with the theoretical result given in Theorem 3.1.

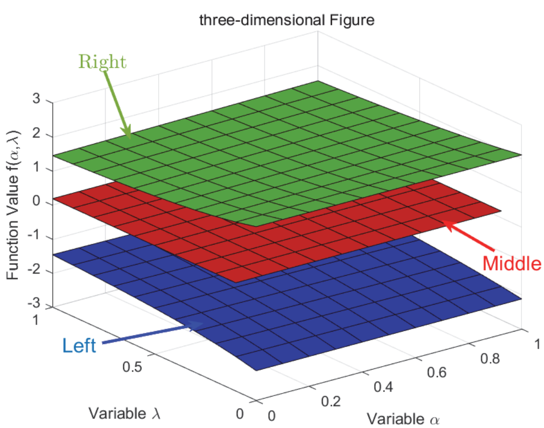

Figure 3.3: *Graphical representation of Example 3.1 for Three-dimensional*

THEOREM 3.2. *Let $\mathcal{K} \subseteq \mathbb{R}$ be an open m-invex subset with respect to the mapping $\eta : \mathcal{K} \times \mathcal{K} \times (0,1] \to \mathbb{R}^+$ for certain fixed $m \in (0,1]$. And let ma, $ma+\eta(b,a,m)$ lie on the η_m-path $P_{\tau_1 \tau_2}(b,a,m)$, for $a,b \in \mathcal{K}$ together with $a<b$. It is supposed that $f : \mathcal{K} \to \mathbb{R}^+$ is an increasing *differentiable function on $\mathcal{K}^\circ$. And if, for $q>1$ with $p^{-1}+q^{-1}=1$, the function $(\ln f^*)^q$ is m-preinvex on $\mathcal{K}$, then the following inequality holds:*

$$\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \leqslant \exp\left\{2\eta(b,a,m)\cdot\omega\cdot\tau^{\frac{1}{p}}\left(\frac{m(\ln f^*(a))^q+(\ln f^*(b))^q}{2}\right)^{\frac{1}{q}}\right\}, \tag{3.19}$$

where ω is defined in Lemma 3.1 and

$$\tau = \int_0^1 \left|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right|^p \mathrm{d}t. \tag{3.20}$$

Proof. Making use of Lemma 3.1 and Hölder's inequality, we deduce that

$$\begin{aligned}
&\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \\
&= \left| \int_0^1 \left[\left(f^*\left(ma+t\eta(b,a,m)\right)\right)^{\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)}\right]^{\mathrm{d}t} \right.\\
&\quad \left. \times \int_0^1 \left[\left(f^*\left(ma+(1-t)\eta(b,a,m)\right)\right)^{-\eta(b,a,m)\omega\gamma_{\lambda\eta(b,a,m)}(\alpha,t)}\right]^{\mathrm{d}t} \right| \\
&\leqslant \exp\left\{\eta(b,a,m)\omega\int_0^1 \left|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right| \right.\\
&\quad \left. \times\left[\left|\ln f^*\left(ma+t\eta(b,a,m)\right)\right|+\left|\ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right|\right]\mathrm{d}t\right\} \\
&= \exp\left\{\eta(b,a,m)\omega\int_0^1 \left|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right|\cdot\left|\ln f^*(ma+t\eta(b,a,m))\right|\mathrm{d}t\right\} \\
&\quad \times\exp\left\{\eta(b,a,m)\omega\int_0^1 \left|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right|\cdot\left|\ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right|\mathrm{d}t\right\} \\
&\leqslant \exp\left\{\eta(b,a,m)\omega\left(\int_0^1 \left|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\right.\\
&\quad \left.\times\left(\int_0^1 \left|\ln f^*\left(ma+t\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\} \\
&\quad \times\exp\left\{\eta(b,a,m)\omega\left(\int_0^1 \left|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\right.\\
&\quad \left.\times\left(\int_0^1 \left|\ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.21}$$

Since $(\ln f^*)^q$ is an m-preinvex function, and using the increasing property of f, we get that

$$\begin{aligned}\int_0^1 \left|\ln f^*\left(ma+t\eta(b,a,m)\right)\right|^q \mathrm{d}t &\leqslant \int_0^1 \left[m(1-t)(\ln f^*(a))^q + t(\ln f^*(b))^q\right]\mathrm{d}t \\ &= \frac{m(\ln f^*(a))^q+(\ln f^*(b))^q}{2},\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\int_0^1 \left|\ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right|^q \mathrm{d}t &\leqslant \int_0^1 \left[mt(\ln f^*(a))^q + (1-t)(\ln f^*(b))^q\right]\mathrm{d}t \\ &= \frac{m(\ln f^*(a))^q+(\ln f^*(b))^q}{2}.\end{aligned} \tag{3.23}$$

Combining (3.22) and (3.23) with (3.21), we can get the required result. This finishes the proof. □

REMARK 3.3. Considering Theorem 3.2, we have the following conclusions:
(i) If we choose $\lambda = 0$, then we have that

$$\begin{aligned}&\left|\frac{\sqrt{f(ma)f\left(ma+\eta(b,a,m)\right)}}{\left[{}_*\mathcal{I}^{\alpha}_{\left[ma+\eta(b,a,m)\right]} f(ma)\cdot {}_{ma}\mathcal{I}^{\alpha}_{*} f\left(ma+\eta(b,a,m)\right)\right]^{\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a,m)}}}\right| \\ &\quad\leqslant \exp\left\{\eta(b,a,m)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{m(\ln f^*(a))^q+(\ln f^*(b))^q}{2}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

(ii) If we choose $\lambda = 0$ and $\alpha = 1$, then we have that

$$\begin{aligned}&\left|\frac{\sqrt{f(ma)f\left(ma+\eta(b,a,m)\right)}}{\int_{ma}^{ma+\eta(b,a,m)}\left(f(u)^{\frac{1}{\eta(b,a,m)}}\right)^{\mathrm{d}u}}\right| \\ &\quad\leqslant \exp\left\{\eta(b,a,m)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{m(\ln f^*(a))^q+(\ln f^*(b))^q}{2}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

THEOREM 3.3. *Let $\mathcal{K}\subseteq\mathbb{R}$ be an open m-invex subset with respect to the mapping $\eta:\mathcal{K}\times\mathcal{K}\times(0,1]\to\mathbb{R}^+$ for certain fixed $m\in(0,1]$. And let ma, $ma+\eta(b,a,m)$ lie on the η_m-path $P_{\tau_1\tau_2}(b,a,m)$, for $a,b\in\mathcal{K}$ together with $a<b$. It is supposed that*

$f:\mathcal{K}\to\mathbb{R}^+$ is an increasing *differentiable function on $\mathcal{K}^\circ$. And if, for $q>1$, the function $(\ln f^*)^q$ is m-preinvex on $\mathcal{K}$, then the following inequality holds:

$$
\begin{aligned}
&\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \\
&\leqslant \exp\left\{\eta(b,a,m)\cdot\omega\cdot J_1^{1-\frac{1}{q}}\cdot\left(m(\ln f^*(a))^q\cdot\kappa_1+(\ln f^*(b))^q\cdot\kappa_2\right)^{\frac{1}{q}}\right\} \\
&\quad\times\exp\left\{\eta(b,a,m)\cdot\omega\cdot J_1^{1-\frac{1}{q}}\cdot\left((\ln f^*(b))^q\cdot\kappa_1+m(\ln f^*(a))^q\cdot\kappa_2\right)^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{3.24}
$$

where ω is defined in Lemma 3.1 and

$$
J_1=\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+1,\eta(b,a,m))}{\eta^{\alpha+1}(b,a,m)},
$$

$$
\kappa_1=\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{2\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+1,\eta(b,a,m))}{\eta^{\alpha+1}(b,a,m)}+\frac{\gamma_\lambda(\alpha+2,\eta(b,a,m))}{2\eta^{\alpha+2}(b,a,m)},
$$

together with

$$
\kappa_2=\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{2\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+2,\eta(b,a,m))}{2\eta^{\alpha+2}(b,a,m)}.
$$

Proof. Continuing from inequality (3.21) in the proof of Theorem 3.2, and using the power-mean inequality, we have that

$$
\begin{aligned}
&\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \\
&\leqslant \exp\left\{\eta(b,a,m)\omega\left(\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\mathrm{d}t\right)^{1-\frac{1}{q}}\right. \\
&\quad\left.\times\left(\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\cdot\left|\ln f^*(ma+t\eta(b,a,m))\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\} \\
&\quad\times\exp\left\{\eta(b,a,m)\omega\left(\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\mathrm{d}t\right)^{1-\frac{1}{q}}\right. \\
&\quad\left.\times\left(\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\cdot\left|\ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{3.25}
$$

For the convenience of expression, let us define the quantities

$$
J_1=\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\mathrm{d}t,
$$

$$
J_2=\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\left|\ln f^*\left(ma+t\eta(b,a,m)\right)\right|^q\mathrm{d}t,
$$

and

$$
J_3=\int_0^1|\gamma_{\lambda\eta(b,a,m)}(\alpha,t)|\left|\ln f^*\left(ma+(1-t)\eta(b,a,m)\right)\right|^q\mathrm{d}t.
$$

According to the Definition 1.6, we derive that

$$J_1=\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+1,\eta(b,a,m))}{\eta^{\alpha+1}(b,a,m)}. \tag{3.26}$$

Utilizing the m-preinvexity of $(\ln f^*)^q$, and using the increasing property of f, we obtain that

$$\begin{aligned}
J_2&=\int_0^1\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\left|\ln f^*\left(ma+t\eta(b,a,m)\right)\right|^q\mathrm{d}t\\
&\leqslant\int_0^1\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\left[m(1-t)\left(\ln f^*(a)\right)^q+t\left(\ln f^*(b)\right)^q\right]\mathrm{d}t\\
&=\int_0^1\int_0^t u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}[m(1-t)(\ln f^*(a)^q+t(\ln f^*(b))^q]\mathrm{d}u\mathrm{d}t\\
&=m(\ln f^*(a))^q\cdot\int_0^1\int_0^t u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}(1-t)\mathrm{d}u\mathrm{d}t+(\ln f^*(b))^q\\
&\quad\times\int_0^1\int_0^t u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}t\mathrm{d}u\mathrm{d}t\\
&=m(\ln f^*(a))^q\left[\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{2\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+1,\eta(b,a,m))}{\eta^{\alpha+1}(b,a,m)}+\frac{\gamma_\lambda(\alpha+2,\eta(b,a,m))}{2\eta^{\alpha+2}(b,a,m)}\right]\\
&\quad+(\ln f^*(b))^q\left[\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{2\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+2,\eta(b,a,m))}{2\eta^{\alpha+2}(b,a,m)}\right].
\end{aligned} \tag{3.27}$$

Analogously, we can deduce that

$$\begin{aligned}
J_3&=\int_0^1\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\left|\ln f^*(ma+(1-t)\eta(b,a,m))\right|^q\mathrm{d}t\\
&\leqslant\int_0^1\gamma_{\lambda\eta(b,a,m)}(\alpha,t)\left[mt(\ln f^*(a))^q+(1-t)(\ln f^*(b))^q\right]\mathrm{d}t\\
&=\int_0^1\int_0^t u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}\left[mt(\ln f^*(a))^q+(1-t)(\ln f^*(b))^q\right]\mathrm{d}u\mathrm{d}t\\
&=m(\ln f^*(a))^q\cdot\int_0^1\int_0^t u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}t\mathrm{d}u\mathrm{d}t+(\ln f^*(b))^q\\
&\quad\times\int_0^1\int_0^t u^{\alpha-1}e^{-\lambda\eta(b,a,m)u}(1-t)\mathrm{d}u\mathrm{d}t\\
&=(\ln f^*(b))^q\left[\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{2\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+1,\eta(b,a,m))}{\eta^{\alpha+1}(b,a,m)}+\frac{\gamma_\lambda(\alpha+2,\eta(b,a,m))}{2\eta^{\alpha+2}(b,a)}\right]\\
&\quad+m(\ln f^*(a))^q\left[\frac{\gamma_\lambda(\alpha,\eta(b,a,m))}{2\eta^\alpha(b,a,m)}-\frac{\gamma_\lambda(\alpha+2,\eta(b,a,m))}{2\eta^{\alpha+2}(b,a,m)}\right].
\end{aligned} \tag{3.28}$$

Combining (3.26), (3.27) and (3.28) with (3.25), we can get the required result. The proof is completed. □

Next, we establish multiplicative fractional integral inequalities for multiplicatively (s,m)-preinvex.

THEOREM 3.4. *Let $\mathcal{K} \subseteq \mathbb{R}$ be an open m-invex subset with respect to the mapping $\eta : \mathcal{K}\times\mathcal{K}\times(0,1] \to \mathbb{R}^+$ for certain fixed $m \in (0,1]$. And let ma, $ma+\eta(b,a,m)$ lie on the η_m-path $P_{\tau_1\tau_2}(b,a,m)$, for $a,b \in \mathcal{K}$ together with $a<b$. It is supposed that $f : \mathcal{K} \to \mathbb{R}^+$ is an increasing *differentiable function on $\mathcal{K}^\circ$. And if the function f^* is multiplicatively (s,m)-preinvex on $\mathcal{K}$, then the following inequality holds:*

$$\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \leqslant \left[(f^*(a))^m \cdot f^*(b)\right]^{2^{-s}\varphi}, \tag{3.29}$$

where φ is defined in Theorem 3.1.

Proof. The desired result can be obtained by applying the strategy used in the proof of Theorem 3.1, combing with the (s,m)-multiplicative preinvexity of f^*, and the inequality $t^s+(1-t)^s \leqslant 2^{1-s}$ for all $t \in [0,1]$ with some fixed $s \in (0,1]$. Thus, the proof is omitted. □

REMARK 3.4. In Theorem 3.4, if we consider taking $s=1$, then can get the same result in Theorem 3.1.

THEOREM 3.5. *Let $\mathcal{K} \subseteq \mathbb{R}$ be an open m-invex subset with respect to the mapping $\eta : \mathcal{K}\times\mathcal{K}\times(0,1] \to \mathbb{R}^+$ for certain fixed $m \in (0,1]$. And let ma, $ma+\eta(b,a,m)$ lie on the η_m-path $P_{\tau_1\tau_2}(b,a,m)$, for $a,b \in \mathcal{K}$ together with $a<b$. It is supposed that $f : \mathcal{K} \to \mathbb{R}^+$ is an increasing *differentiable function on $\mathcal{K}^\circ$. And if, for $q>1$ with $p^{-1}+q^{-1}=1$, the function $(\ln f^*)^q$ is (s,m)-preinvex on $\mathcal{K}$, then the following inequality holds:*

$$\begin{aligned} &\left|{}_*\mathcal{J}_f(\alpha,\lambda;a,b,m)\right| \\ &\quad \leqslant \exp\left\{2\eta(b,a,m)\cdot\omega\cdot\tau^{\frac{1}{p}}\left(\frac{m(\ln f^*(a))^q+(\ln f^*(b))^q}{s+1}\right)^{\frac{1}{q}}\right\}, \end{aligned} \tag{3.30}$$

where ω is defined in Lemma 3.1 and τ is defined in Theorem 3.2.

Proof. On the same parallel lines as used in the proof of Theorem 3.2, and taking account of the (s,m)-preinvexity of $(\ln f^*)^q$, we get the desired result of Theorem 3.5. Thus, the proof is omitted. □

REMARK 3.5. In Theorem 3.5, if we consider taking $s=1$, then we can get the same result in Theorem 3.2.

4. Conclusions

In this study, we first present a fractional integral identity for * differentiable functions. By applying it and the multiplicative m-preinvexity and (s,m)-preinvexity, we deduce a series of multiplicative fractional integral inequalities. The obtained results here can be transferred to the multiplicative Riemann–Liouville fractional integral inequalities for $\lambda = 0$, and the multiplicative Riemann integral inequalities for $\alpha = 1$ together with $\lambda = 0$. With the aid of ideas developed in this paper, interested researchers can consider using different multiplicative fractional integrals, such as multiplicative fractional integrals having exponential kernels [35], generalized multiplicative fractional integrals [23] and others. And then the inequalities for generalized multiplicative convex functions can be established similarly, which is an interesting and new research subject.

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REFERENCES

[1] T. ABDELJAWAD, M. GROSSMAN, *On geometric fractional calculus*, J. Semigroup Theory Appl., **2016** (2016), Article ID 2, 14 pages.
[2] B. AHMAD, A. ALSAEDI, M. KIRANE, B. T. TOREBEK, *Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals*, J. Comput. Appl. Math., **353** (2019), 120–129.
[3] N. AKHTAR, M. U. AWAN, *Some δ-tempered fractional Hermite–Hadamard inequalities involving harmonically convex functions and applications*, J. Math., **2021** (2021), Article ID 1948613, 14 pages.
[4] M. A. ALI, M. ABBAS, A. A. ZAFAR, *On some Hermite–Hadamard integral inequalities in multiplicative calculus*, J. Inequal. Spec. Funct., **10** (2019), 111–122.
[5] M. A. ALI, H. BUDAK, M. Z. SARIKAYA, Z. Y. ZHANG, *Ostrowski and Simpson type inequalities for multiplicative integrals*, Proyecciones, **40** (2021), 743–763.
[6] M. ANDRIĆ, J. PEČARIĆ, *On $(h,g;m)$-convexity and the Hermite–Hadamard inequality*, J. Convex Anal., **29** (2022), 257–268.
[7] M. U. AWAN, S. TALIB, M. A. NOOR, K. I. NOOR, *On γ-preinvex functions*, Filomat, **34** (2020), 4137–4159.
[8] A. E. BASHIROV, E. M. KURPINAR, A. ÖZYAPICI, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337** (2008), 36–48.
[9] H. BOULARES, B. MEFTAH, A. MOUMEN, R. SHAFQAT, H. SABER, T. ALRAQAD, E. E. ALI, *Fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions*, Symmetry, **15** (2023), 12 pages.
[10] M. D. BOUNOUA, C. T. YIN, *Some Simpson type fractional integral inequalities for functions of bounded variation*, J. Math. Inequal., **15** (2021), 1473–1486.
[11] H. BUDAK, K. ÖZÇELIK, *On Hermite–Hadamard type inequalities for multiplicative fractional integrals*, Miskolc Math. Notes, **21** (2020), 91–99.
[12] S. I. BUTT, S. YOUSAF, A. O. AKDEMIR, M. A. DOKUYUCU, *New Hadamard-type integral inequalities via a general form of fractional integral operators*, Chaos, Solitons and Fractals, **148** (2021), Article ID 111025, 14 pages.
[13] Y. CAO, J. F. CAO, P. Z. TAN, T. S. DU, *Some parameterized inequalities arising from the tempered fractional integrals involving the (μ,η)-incomplete gamma functions*, J. Math. Inequal., **16** (2022), 1091–1121.
[14] M. R. DELAVAR, M. DE LA SEN, *A mapping associated to h-convex version of the Hermite–Hadamard inequality with applications*, J. Math. Inequal., **14** (2020), 329–335.

[15] S. S. DRAGOMIR, *Hermite–Hadamard type inequalities for generalized Riemann–Liouville fractional integrals of h-convex functions*, Math. Methods Appl. Sci., **44** (2021), 2364–2380.

[16] T. S. DU, J. G. LIAO, Y. J. LI, *Properties and integral inequalities of Hadamard–Simpson type for the generalized (s,m)-preinvex functions*, J. Nonlinear Sci. Appl., **9** (2016), 3112–3126.

[17] T. S. DU, C. Y. LUO, Z. J. CAO, *On the Bullen-type inequalities via generalized fractional integrals and their applications*, Fractals, **29** (2021), Article ID 2150188, 20 pages.

[18] T. S. DU, T. C. ZHOU, *On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings*, Chaos, Solitons and Fractals, **156** (2022), Article ID 111846, 19 pages.

[19] Z. EKEN, S. KEMALI, G. TINAZTEPE, G. ADILOV, *The Hermite–Hadamard inequalities for p-convex functions*, Hacet. J. Math. Stat., **50** (2021), 1268–1279.

[20] A. EKINCI, M. E. ÖZDEMIR, *Some new integral inequalities via Riemann–Liouville integral operators*, Appl. Comput. Math., **18** (2019), 288–295.

[21] H. FU, Y. PENG, T. S. DU, *Some inequalities for multiplicative tempered fractional integrals involving the λ-incomplete gamma functions*, AIMS Mathematics, **6** (2021), 7456–7478.

[22] İ. İŞCAN, *Weighted Hermite–Hadamard–Mercer type inequalities for convex functions*, Numer. Methods Partial Differential Equations, **37** (2021), 118–130.

[23] A. KASHURI, S. K. SAHOO, M. ALJUAID, M. TARIQ, M. DE LA SEN, *Some new Hermite–Hadamard type inequalities pertaining to generalized multiplicative fractional integrals*, Symmetry, **15** (2023), Article ID 868, 14 pages.

[24] M. KUNT, D. KARAPINAR, S. TURHAN, İ. İSÇAN, *The left Riemann–Liouville fractional Hermite–Hadamard type inequalities for convex functions*, Math. Slovaca, **69** (2019), 773–784.

[25] P. KÓRUS, *An extension of the Hermite–Hadamard inequality for convex and s-convex functions*, Aequat. Math., **93** (2019), 527–534.

[26] M. A. LATIF, S. HUSSAIN, Y. M. CHU, *Generalized Hermite–Hadamard type inequalities for differentiable harmonically–convex and harmonically quasi-convex functions*, J. Math. Inequal., **15** (2021), 755–766.

[27] J. G. LIAO, S. H. WU, T. S. DU, *The Sugeno integral with respect to α-preinvex functions*, Fuzzy Sets and Systems, **379** (2020), 102–114.

[28] B. MEFTAH, A. SOUAHI, M. MERAD, *Some local fractional Maclaurin type inequalities for generalized convex functions and their applications*, Chaos, Solitons and Fractals, **162** (2022), Article ID 112504, 7 pages.

[29] B. MEFTAH, *Maclaurin type inequalities for multiplicatively convex functions*, Proc. Amer. Math. Soc., **151** (2023), 2115–2125.

[30] P. O. MOHAMMED, M. Z. SARIKAYA, D. BALEANU, *On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals*, Symmetry, **2020** (2020), Article ID 595, 17 pages.

[31] A. MOUMEN, H. BOULARES, B. MEFTAH, R. SHAFQAT, T. ALRAQAD, E. E. ALI, Z. KHALED, *Multiplicatively Simpson type inequalities via fractional integral*, Symmetry, **15** (2023), 13 pages.

[32] K. NIKODEM, T. RAJBA, *On Hermite–Hadamard inequalities for (k,h)-convex set-valued maps*, Math. Inequal. Appl., **25** (2022), 467–475.

[33] M. A. NOOR, K. I. NOOR, *New aspects of strongly log-preinvex functions*, Facta Universitatis. Ser. Math. Inform., **36** (2021), 783–800.

[34] S. ÖZCAN, *Some integral inequalities of Hermite–Hadamard type for multiplicatively preinvex functions*, AIMS Mathematics, **5** (2020), 1505–1518.

[35] Y. PENG, H. FU, T. S. DU, *Estimations of bounds on the multiplicative fractional integral inequalities having exponential kernels*, Commun. Math. Stat., 2022, online https://doi.org/10.1007/s40304-022-00285-8.

[36] F. SABZIKAR, M. M. MEERSCHAERT, J. H. CHEN, *Tempered fractional calculus*, J. Comput. Phys., **293** (2015), 14–28.

[37] M. Z. SARIKAYA, H. YILDIRIM, *On Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals*, Miskolc Math. Notes, **17** (2017), 1049–1059.

[38] E. SET, S. I. BUTT, A. O. AKDEMIR, A. KARAOĞLAN, T. ABDELJAWAD, *New integral inequalities for differentiable convex functions via Atangana–Baleanu fractional integral operators*, Chaos, Solitons and Fractals, **143** (2021), Article ID 110554, 14 pages.

[39] T. SITTHIWIRATTHAM, K. NONLAOPON, M. A. ALI, H. BUDAK, *Riemann–Liouville fractional Newton's type inequalities for differentiable convex functions*, Fractal Fract., **6**(2022), Article ID 175, 15 pages.

[40] H. M. SRIVASTAVA, S. MEHREZ, S. M. SITNIK, *Hermite–Hadamard-type integral inequalities for convex functions and their applications*, Mathematics, **10** (2022), Article ID 3127, 13 pages.

[41] X. M. YANG, X. Q. YANG, K. L. TEO, *Generalized invexity and generalized invariant monotonicity*, J. Optim. Theory Appl., **117** (2003), 607–625.

[42] Y. ZHANG, T. S. DU, H. WANG, *Some new k-fractional integral inequalities containing multiple parameters via generalized (s,m)-preinvexity*, Ital. J. Pure Appl. Math., **40** (2018), 510–527.

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