PARAMETERIZED MORE ACCURATE
HARDY–HILBERT–TYPE INEQUALITIES AND APPLICATIONS

YONG HONG, YANRU ZHONG* AND BICHENG YANG

(*Communicated by M. Krnić*)

Abstract. By means of the weight coefficients, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate Hardy-Hilbert-type inequality with the general homogeneous kernel and the discrete intermediate variables is given. The equivalent form and a few equivalent statements of the best possible constant factor related to some parameters are obtained. As applications, the operator expressions, a few particular cases and some examples are considered.

1. Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geqslant 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have a more accurate Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ as follows (cf. [3], Theorem 323):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1}$$

Since $\frac{1}{m+n} < \frac{1}{m+n-1}$ $(m, n \in \mathbf{N} = \{1, 2, \cdots\})$, inequality (1) reduces to the following well known Hardy-Hilbert's inequality with the sane best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{2}$$

Stting $\mu_i, v_i > 0$ $(i, j \in \mathbf{N})$ and

$$U_m := \sum_{i=1}^{m} \mu_i, V_n := \sum_{j=1}^{n} v_j \ (m, n \in \mathbf{N}), \tag{3}$$

we still obtain the following Hardy-Hilbert-type inequality (cf. [1], Theorem 320):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{v_{nj}^{q-1}}\right)^{\frac{1}{q}}. \tag{4}$$

For $\mu_i = v_i = 1$ $(i, j \in \mathbf{N})$, inequality (4) reduces to (2).

If $f(x), g(y) \geqslant 0$, $0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(y)dy < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^q(y)dy\right)^{\frac{1}{q}}, \tag{5}$$

with the best constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], Theorem 316). In 1998, by introducing an independent parameter $\lambda > 0$, Yang [23, 24] gave an extension of (2) (for $p = q = 2$) with the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ as follows:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy$$

$$< B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_0^{\infty}x^{1-\lambda}f^2(x)dx\int_0^{\infty}y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}}, \tag{6}$$

where, $B(u,v) := \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt$ $(u, v > 0)$ is the beta function.

Inequalities (1), (2) and (4)–(6) with their extensions are important in analysis and its applications (cf. [1, 2, 4, 5, 12, 15, 19–21, 25, 26, 30]).

The following half-discrete Hilbert-type inequality was provided (cf. [3], Theorem 351): if $K(x)$ $(x > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^{\infty} K(t)t^{s-1}dt < \infty$, then for $a_n \geqslant 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$,

$$\int_0^{\infty}x^{p-2}\left(\sum_{n=1}^{\infty}K(nx)a_n\right)^p dx < \phi^p(\frac{1}{q})\sum_{n=1}^{\infty}a_n^p. \tag{7}$$

Some new extensions of (7) with their applications were provided by [6, 16–18, 27, 28].

In 2016, by the use of the technique of real analysis, Hong et al. [7] considered some equivalent statements of the extensions of (2) with the best possible constant factor related to a few parameters. The other similar works about the extensions of (5) and (6) were given by [8–11, 22].

In this paper, according to the way of [7], by means of the weight functions, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate Hardy-Hilbert-type inequality with the general homogeneous kernel and the discrete intermediate variables is given, which is a more accurate extension of (4). The equivalent form and the equivalent statements of the best possible constant factor related to some parameters are considered. As applications, the operator expressions, a few particular cases and some examples are obtained.

2. Some lemmas

In what follows, we suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha, \beta \leqslant 1$, $\lambda \in R_+ = (0, \infty)$, λ_2, $\lambda - \lambda_1 \leqslant \frac{1}{\alpha}$, λ_1, $\lambda - \lambda_2 \leqslant \frac{1}{\beta}$, both $\{\mu_m\}_{m=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ are positive decreasing sequences, with $\widehat{\mu} \in [0, \frac{\mu_1}{2}]$ and $\widehat{\nu} \in [0, \frac{\nu_1}{2}]$, $k_\lambda(x, y)$ is a positive homogeneous function of degree $-\lambda$, satisfying for any $u, x, y > 0$,

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y).$$

Also, $k_\lambda(x, y)$ is a strictly decreasing and strictly convex function with respect to $x, y > 0$, such that

$$(-1)^i \frac{\partial}{\partial x^i} k_\lambda(x, y) > 0, \quad (-1)^i \frac{\partial}{\partial y^i} k_\lambda(x, y) > 0 \quad (i = 1, 2),$$

and for $\gamma = \lambda_1, \ \lambda - \lambda_2,$

$$k_\lambda(\gamma) := \int_0^\infty k_\lambda(u, 1) u^{\gamma - 1} du \in \mathbf{R}_+. \tag{8}$$

Using the expressions (3), we still assume that $a_m, b_n \geqslant 0$, such that

$$0 < \sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p[1 - \alpha(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q})] - 1} \frac{a_m^p}{\mu_{m+1}^{p-1}} < \infty$$

and

$$0 < \sum_{n=1}^{\infty} (V_n - \widehat{\nu})^{q[1 - \beta(\frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p})] - 1} \frac{b_n^q}{\nu_{n+1}^{q-1}} < \infty.$$

We set $\mu(t) := \mu_m$, $t \in (m-1, m]$, $\nu(t) := \nu_n$, $t \in (n-1, n]$ $(m, n \in \mathbf{N})$, and

$$U(x) := \int_0^x \mu(t) dt, \quad V(y) := \int_0^y \nu(t) dt \quad (x, y \geqslant 0).$$

It follows that $U(m) = U_m$, $V(n) = V_n$, $U(\frac{1}{2}) = \frac{\mu_1}{2}$, $V(\frac{1}{2}) = \frac{\nu_1}{2}$ and

$$U'(x) = : \mu(x) = \mu_m, \ x \in (m-1, m],$$
$$V'(y) = : \nu(y) = \nu_n, \ y \in (n-1, n] \quad (m, n \in \mathbf{N}).$$

LEMMA 1. *For fixed $m \in \mathbf{N}$, the following continuous function*

$$f_m(y) := k_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{\nu})^\beta)(V(y) - \widehat{\nu})^{\lambda_2 \beta - 1}$$

is strictly decreasing and strictly convex with respect to $y \in (n - \frac{1}{2}, n + \frac{1}{2})$ $(n \in \mathbf{N})$.

Proof. For $y \in (n - \frac{1}{2}, n)$, $0 < \beta \leqslant 1$, $\lambda_2 \beta - 1 \leqslant 0$, we find

$$
\begin{aligned}
f'_m(y) = [&\beta k'_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{\beta-1} \\
&+ (\lambda_2 \beta - 1)k_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{-1}] \\
&\times (V(y) - \widehat{v})^{\lambda_2\beta-1} v_n < 0.
\end{aligned}
$$

Replacing v_n by v_{n+1} in the above expression, we have $f'_m(y) < 0$ $(y \in (n, n + \frac{1}{2}))$. In view of $f_m(y)$ is continuous in $(n - \frac{1}{2}, n + \frac{1}{2})$, it follows that $f_m(y)$ is a strictly decreasing function with respect to $(n - \frac{1}{2}, n + \frac{1}{2})$ $(n \in \mathbf{N})$.

For $y \in (n - \frac{1}{2}, n)$, we find

$$
\begin{aligned}
f''_m(y) = [&\beta^2 k''_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{2\beta-2} \\
&+ \beta(\beta - 1)k'_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{\beta-2} \\
&+ (\lambda_2 \beta - 1)\beta k'_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{\beta-2} \\
&+ (1 - \lambda_2 \beta)k_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{-2}] \\
&\times (V(y) - \widehat{v})^{\lambda_2\beta-1} v_n^2 + [\beta k'_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{\beta-1} \\
&+ (\lambda_2 \beta - 1)k_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{v})^\beta)(V(y) - \widehat{v})^{-1}] \\
&\times (V(y) - \widehat{v})^{\lambda_2\beta-2} v_n^2 > 0.
\end{aligned}
$$

Replacing v_n by v_{n+1} in the above expression, it follows that $f''_m(y) > 0$ $(y \in (n, n + \frac{1}{2}))$.

For $n \in \mathbf{N}$, since $v_n \geqslant v_{n+1} > 0$, we find that $f'_m(n-0) \geqslant f'_m(n+0)$, in the above expressions. In view of $f''_m(y) > 0$ $(y \in (n - \frac{1}{2}, n + \frac{1}{2}))$, it follows that $f'_m(y)$ is a strictly increasing function in $(n - \frac{1}{2}, n + \frac{1}{2})$, and then $f_m(y)$ is a strictly convex function with respect to $y \in (n - \frac{1}{2}, n + \frac{1}{2})$ $(n \in \mathbf{N})$.

The lemma is proved. $\square$

DEFINITION 1. The following weight coefficients are defined: for $m, n \in \mathbf{N}$,

$$
\omega(\lambda_2, m) := \sum_{n=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V(n) - \widehat{v})^\beta) \frac{(V(n) - \widehat{v})^{\beta\lambda_2-1} v_{n+1}}{(U_m - \widehat{\mu})^{\alpha(\lambda_2-\lambda)}}. \tag{9}
$$

$$
\varpi(\lambda_1, n) := \sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V(n) - \widehat{v})^\beta) \frac{(U_m - \widehat{\mu})^{\alpha\lambda_1-1} \mu_{m+1}}{(V(n) - \widehat{v})^{\beta(\lambda_1-\lambda)}}. \tag{10}
$$

LEMMA 2. *The following inequalities are valid:*

$$
\omega(\lambda_2, m) < \frac{1}{\beta} k_\lambda(\lambda - \lambda_2) \ (m \in \mathbf{N}), \tag{11}
$$

$$
\varpi(\lambda_1, n) < \frac{1}{\alpha} k_\lambda(\lambda_1) \ (n \in \mathbf{N}). \tag{12}
$$

Proof. According to Lemma 1, by Hermite-Hadamard's inequality (cf. [13]), we find

$$
\begin{aligned}
\omega(\lambda_2,m) &\leqslant \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{v})^\beta) \frac{(V_n-\widehat{v})^{\beta\lambda_2-1}V'(y)}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}} dy \\
&\leqslant \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda((U_m-\widehat{\mu})^\alpha,(V(y)-\widehat{v})^\beta) \frac{(V(y)-\widehat{v})^{\beta\lambda_2-1}V'(y)}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}} dy \\
&= \int_{\frac{1}{2}}^{\infty} k_\lambda((U_m-\widehat{\mu})^\alpha,(V(y)-\widehat{v})^\beta) \frac{(V(y)-\widehat{v})^{\beta\lambda_2-1}}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}} d(V(y)-\widehat{v}).
\end{aligned}
$$

Setting $t = \frac{(U_m-\widehat{\mu})^\alpha}{(V(y)-\widehat{v})^\beta}$, since $\widehat{v} \in [0,\frac{v_1}{2}]$, we find

$$
\begin{aligned}
\omega(\lambda_2,m) &< \frac{1}{\beta} \int_{\frac{(U_m-\widehat{\mu})^\alpha}{V(\infty)-\widehat{v})^\beta}}^{\frac{(U_m-\widehat{\mu})^\alpha}{V(\frac{1}{2})-\widehat{v})^\beta}} k_\lambda(t,1)t^{(\lambda-\lambda_2)-1} dt \\
&\leqslant \frac{1}{\beta} \int_0^{\infty} k_\lambda(t,1)t^{(\lambda-\lambda_2)-1} dt = \frac{1}{\beta} k_\lambda(\lambda-\lambda_2).
\end{aligned}
$$

Hence, we have (11).

In view of Lemma 1 and in the same way, for fixed $n \in \mathbf{N}$ and $0 < \alpha \leqslant 1$, $\alpha\lambda_1 - 1 \leqslant 0$, we can conclude that the following continuous function

$$
g_n(x) := k_\lambda((U(x)-\widehat{\mu})^\alpha,(V_n-\widehat{v})^\beta)(U(x)-\widehat{\mu})^{\alpha\lambda_1-1}
$$

is also a strictly decreasing and strictly convex function with respect to $x \in (m-\frac{1}{2},m+\frac{1}{2})$ $(m \in \mathbf{N})$. Setting, $u = \frac{(U(x)-\widehat{\mu})^\alpha}{(V_n-\widehat{v})^\beta}$, sine $\widehat{\mu} \in [0,\frac{\mu_1}{2}]$, we find

$$
\begin{aligned}
\varpi(\lambda_1,n) &\leqslant \sum_{m=1}^{\infty} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{v})^\beta) \frac{(U_m-\widehat{\mu})^{\alpha\lambda_1-1}U'(x)}{(V_n-\widehat{v})^{\beta(\lambda_1-\lambda)}} dx \\
&< \sum_{m=1}^{\infty} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} k_\lambda((U(x)-\widehat{\mu})^\alpha,(V_n-\widehat{v})^\beta) \frac{(U(x)-\widehat{\mu})^{\alpha\lambda_1-1}U'(x)}{(V_n-\widehat{v})^{\beta(\lambda_1-\lambda)}} dx \\
&= \int_{\frac{1}{2}}^{\infty} k_\lambda((U(x)-\widehat{\mu})^\alpha,(V_n-\widehat{v})^\beta) \frac{(U(x)-\widehat{\mu})^{\alpha\lambda_1-1}}{(V_n-\widehat{v})^{\beta(\lambda_1-\lambda)}} d(U(x)-\widehat{\mu}) \\
&= \frac{1}{\alpha} \int_{\frac{(U(\frac{1}{2})-\widehat{\mu})^\alpha}{(V_n-\widehat{v})^\beta}}^{\frac{(U\infty x)-\widehat{\mu})^\alpha}{(V_n-\widehat{v})^\beta}} k_\lambda(u,1)u^{\lambda_1-1} du \leqslant \frac{1}{\alpha} \int_0^{\infty} k_\lambda(u,1)u^{\lambda_1-1} du = \frac{1}{\alpha} k_\lambda(\lambda_1).
\end{aligned}
$$

Hence, we have (12).

The lemma is proved. $\square$

LEMMA 3. *The following inequality is valid:*

$$I := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{\lambda}((U_m - \widehat{\mu})^{\alpha}, (V_n - \widehat{v})^{\beta}) a_m b_n$$

$$< \frac{k_{\lambda}^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p} \alpha^{1/q}} k_{\lambda}^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p[1-\alpha(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} (V_n - \widehat{v})^{q[1-\beta(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} \frac{b_n^q}{v_{n+1}^{q-1}} \right\}^{\frac{1}{q}}. \tag{13}$$

Proof. By Hölder's inequality with weight (cf. [13]), we obtain

$$I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{\lambda}((U_m - \widehat{\mu})^{\alpha}, (V_n - \widehat{v})^{\beta}) \left[\frac{(V_n - \widehat{v})^{(\beta\lambda_2-1)/p} v_{n+1}^{1/p} a_m}{(U_m - \widehat{\mu})^{(\alpha\lambda_1-1)/q} \mu_{m+1}^{1/q}} \right]$$

$$\times \left[\frac{(U_m - \widehat{\mu})^{(\alpha\lambda_1-1)/q} \mu_{m+1}^{1/q} b_n}{(V_n - \widehat{v})^{(\beta\lambda_2-1)/p} v_{n+1}^{1/p}} \right]$$

$$\leqslant \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{\lambda}((U_m - \widehat{\mu})^{\alpha}, (V_n - \widehat{v})^{\beta}) \frac{(V_n - \widehat{v})^{(\beta\lambda_2-1)} v_{n+1} a_m^p}{(U_m - \widehat{\mu})^{(\alpha\lambda_1-1)(p-1)} \mu_{m+1}^{p-1}} \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{\lambda}((U_m - \widehat{\mu})^{\alpha}, (V_n - \widehat{v})^{\beta}) \frac{(U_m - \widehat{\mu})^{(\alpha\lambda_1-1)} \mu_{m+1} b_n^q}{(V_n - \widehat{v})^{(\beta\lambda_2-1)(q-1)} v_{n+1}^{q-1}} \right]^{\frac{1}{q}}$$

$$= \left\{ \sum_{m=1}^{\infty} \omega(\lambda_2, m)(U_m - \widehat{\mu})^{p[1-\alpha(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} \varpi(\lambda_1, n)(V_n - \widehat{v})^{q[1-\beta(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} \frac{b_n^q}{v_{n+1}^{q-1}} \right\}^{\frac{1}{q}}.$$

Then, using (11) and (12), we obtain (13).

The lemma is proved. □

REMARK 1. By (13) with the assumption, for $\lambda_1 + \lambda_2 = \lambda$, we find

$$0 < \sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\lambda_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} < \infty,$$

$$0 < \sum_{n=1}^{\infty} (V_n - \widehat{v})^{q(1-\beta\lambda_2)-1} \frac{b_n^q}{v_{n+1}^{q-1}} < \infty,$$

and the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{v})^\beta)a_m b_n$$

$$< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p(1-\alpha\lambda_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}$$

$$\times\left[\sum_{n=1}^{\infty}(V_n-\widehat{v})^{q(1-\beta\lambda_2)-1}\frac{b_n^q}{v_{n+1}^{q-1}}\right]^{\frac{1}{q}} \tag{14}$$

In particular, for $\widehat{\mu}=\widehat{v}=0$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m^\alpha,V_n^\beta)a_m b_n$$

$$< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}U_m^{p(1-\alpha\lambda_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}V_n^{q(1-\beta\lambda_2)-1}\frac{b_n^q}{v_{n+1}^{q-1}}\right]^{\frac{1}{q}} \tag{15}$$

Hence, inequality (14) is a more accurate extension of (15).

For $\lambda=\alpha=\beta=1$, $k_1(x,y)=\frac{1}{x+y}$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, inequality (14) reduces to the following more accurate extension of (4) (replacing μ_{m+1} (v_{n+1}) by μ_m (v_n)): for $\widehat{\kappa}\in[0,\frac{\mu_1+v_1}{2}]$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n-\widehat{\kappa}} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{v_{n+1}^{q-1}}\right)^{\frac{1}{q}}. \tag{16}$$

In particular, for $\mu_m=v_n=\widehat{\kappa}=1$, inequality (16) reduces to (1).

LEMMA 4. *If* $U(\infty)=V(\infty)=\infty$, *then the constant factor* $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ *in* (14) *is the best possible.*

Proof. For any $\varepsilon>0$, we set

$$\widetilde{a}_m:=U_m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}\mu_{m+1},\widetilde{b}_n:=V_n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}v_{n+1}\ (m,n\in\mathbf{N}).$$

If there exists a constant M $(\leqslant\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}})$, such that (14) is valid when we replace $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ by M. Then in particular, for $\widehat{\mu}=\widehat{v}=0$, we have

$$\widetilde{I}:=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m^\alpha,V_n^\beta)\widetilde{a}_m\widetilde{b}_n$$

$$< M\left[\sum_{m=1}^{\infty}U_m^{p(1-\alpha\lambda_1)-1}\frac{\widetilde{a}_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}V_n^{q(1-\beta\lambda_2)-1}\frac{\widetilde{b}_n^q}{v_{n+1}^{q-1}}\right]^{\frac{1}{q}}.$$

We obtain

$$\widetilde{I} < M \left[\sum_{m=1}^{\infty} U_m^{p(1-\alpha\lambda_1)-1} \frac{U_m^{p\alpha(\lambda_1-\frac{\varepsilon}{p})-p} \mu_{m+1}^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} V_n^{q(1-\beta\lambda_2)-1} \frac{V_n^{q\beta(\lambda_2-\frac{\varepsilon}{q})-q} v_{n+1}^q}{v_{n+1}^{q-1}} \right]^{\frac{1}{q}}$$

$$\leqslant M \left(\mu_1^{-\alpha\varepsilon-1} \mu_2 + \sum_{m=2}^{\infty} U_m^{-\alpha\varepsilon-1} \mu_m \right)^{\frac{1}{p}} \left(v_1^{-\beta\varepsilon-1} v_2 + \sum_{n=2}^{\infty} V_n^{-\beta\varepsilon-1} v_n \right)^{\frac{1}{q}}$$

$$= M \left[\mu_1^{-\alpha\varepsilon-1} \mu_2 + \sum_{m=2}^{\infty} \int_{m-1}^m U_m^{-\alpha\varepsilon-1} U'(x) dx \right]^{\frac{1}{p}}$$

$$\times \left[(v_1^{-\beta\varepsilon-1} v_2 + \sum_{n=2}^{\infty} \int_{n-1}^n V_n^{-\beta\varepsilon-1} V'(y) dy \right]^{\frac{1}{q}}$$

$$< M \left[\mu_1^{-\alpha\varepsilon-1} \mu_2 + \sum_{m=2}^{\infty} \int_{m-1}^m U^{-\alpha\varepsilon-1}(x) dU(x) \right]^{\frac{1}{p}}$$

$$\times \left[v_1^{-\beta\varepsilon-1} v_2 + \sum_{n=2}^{\infty} \int_{n-1}^n V^{-\beta\varepsilon-1}(y) dV(y) \right]^{\frac{1}{q}}$$

$$= M \left(\mu_1^{-\alpha\varepsilon-1} \mu_2 + \int_1^{\infty} U^{-\alpha\varepsilon-1}(x) dU(x) \right)^{\frac{1}{p}}$$

$$\times \left(v_1^{-\beta\varepsilon-1} v_2 + \int_1^{\infty} V^{-\beta\varepsilon-1}(y) dV(y) \right)^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon} M \left(\varepsilon\mu_1^{-\alpha\varepsilon-1} \mu_2 + \frac{1}{\alpha} \mu_1^{-\alpha\varepsilon} \right)^{\frac{1}{p}} \left(\varepsilon v_1^{-\beta\varepsilon-1} v_2 + \frac{1}{\beta} v_1^{-\beta\varepsilon} \right)^{\frac{1}{q}}.$$

By the decreasingness property of series and Fubini theorem (cf. [14]), in view of $U(\infty) = V(\infty) = \infty$, we find

$$\widetilde{I} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_\lambda(U_m^\alpha, V_n^\beta) \frac{U_m^{\alpha\lambda_1-1} \mu_{m+1}}{U_m^{\alpha\varepsilon/p}} \frac{V_n^{\beta\lambda_2-1} v_{n+1}}{V_n^{\beta\varepsilon/q}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_n^{n+1} \int_m^{m+1} k_\lambda(U_m^\alpha, V_n^\beta) \frac{U_m^{\alpha\lambda_1-1}}{U_m^{\alpha\varepsilon/p}} \frac{V_n^{\beta\lambda_2-1}}{V_n^{\beta\varepsilon/q}} U'(x) V'(y) dx dy$$

$$\geqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_n^{n+1} \left[\int_m^{m+1} k_\lambda(U^\alpha(x), V^\beta(y)) \frac{U^{\alpha\lambda_1-1}(x)}{U^{\alpha\varepsilon/p}(x)} U'(x) dx \right] \frac{V^{\beta\lambda_2-1}(y)}{V^{\beta\varepsilon/q}(y)} V'(y) dy$$

$$= \int_1^{\infty} \left[\int_1^{\infty} k_\lambda(U^\alpha(x), V^\beta(y)) \frac{U^{\alpha\lambda_1-1}(x)}{U^{\alpha\varepsilon/p}(x)} U'(x) dx \right] \frac{V^{\beta\lambda_2-1}(y)}{V^{\beta\varepsilon/q}(y)} V'(y) dy.$$

Setting $u = \frac{U^\alpha(x)}{V^\beta(y)}$, we have

$$\widetilde{I} = \frac{1}{\alpha} \int_1^\infty \left(\int_{\frac{U^\alpha(1)}{V^\beta(y)}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right) \frac{V^{\beta\lambda_2 - 1}(y)}{V^{\beta\varepsilon/q}(y)} V'(y) dy$$

$$= \frac{1}{\alpha\beta} \int_{v_1^\beta}^\infty \left(\int_{\frac{\mu_1^\alpha}{t}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right) t^{-\varepsilon - 1} dt$$

$$= \frac{1}{\alpha\beta} \left[\int_{v_1^\beta}^\infty \left(\int_{\frac{\mu_1^\alpha}{t}}^{\frac{\mu_1^\alpha}{v_1^\beta}} k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right) t^{-\varepsilon - 1} dt \right.$$

$$\left. + \int_{v_1^\beta}^\infty \left(\int_{\frac{\mu_1^\alpha}{v_1^\beta}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right) t^{-\varepsilon - 1} dt \right]$$

$$= \frac{1}{\alpha\beta} \left[\int_0^{\frac{\mu_1^\alpha}{v_1^\beta}} \left(\int_{\frac{\mu_1^\alpha}{u}}^\infty t^{-\varepsilon - 1} dt \right) k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du + \frac{1}{\varepsilon v_1^{\beta\varepsilon}} \int_{\frac{\mu_1^\alpha}{v_1^\beta}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right]$$

$$= \frac{1}{\varepsilon\alpha\beta} \left[\frac{1}{\mu_1^{\alpha\varepsilon}} \int_0^{\frac{\mu_1^\alpha}{v_1^\beta}} k_\lambda(u,1) u^{\lambda_1 + \frac{\varepsilon}{q} - 1} du + \frac{1}{v_1^{\beta\varepsilon}} \int_{\frac{\mu_1^\alpha}{v_1^\beta}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right].$$

Then we obtain

$$\frac{1}{\alpha\beta} \left[\frac{1}{\mu_1^{\alpha\varepsilon}} \int_0^{\frac{\mu_1^\alpha}{v_1^\beta}} k_\lambda(u,1) u^{\lambda_1 + \frac{\varepsilon}{q} - 1} du + \frac{1}{v_1^{\beta\varepsilon}} \int_{\frac{\mu_1^\alpha}{v_1^\beta}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right]$$

$$\leqslant \varepsilon\widetilde{I} \leqslant M \left(\varepsilon\mu_1^{-\alpha\varepsilon - 1} \mu_2 + \frac{1}{\alpha} \mu_1^{-\alpha\varepsilon} \right)^{\frac{1}{p}} \left(\varepsilon v_1^{-\beta\varepsilon - 1} v_2 + \frac{1}{\beta} v_1^{-\beta\varepsilon} \right)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, by Fatou lemma (cf. [14]), we find

$$\frac{1}{\alpha\beta} k_\lambda(\lambda_1) = \frac{1}{\alpha\beta} \left[\lim_{\varepsilon\to 0^+} \frac{1}{\mu_1^{\alpha\varepsilon}} \int_0^{\frac{\mu_1^\alpha}{v_1^\beta}} \lim_{\varepsilon\to 0^+} k_\lambda(u,1) u^{\lambda_1 + \frac{\varepsilon}{q} - 1} du \right.$$

$$\left. + \lim_{\varepsilon\to 0^+} \frac{1}{v_1^{\beta\varepsilon}} \int_{\frac{\mu_1^\alpha}{v_1^\beta}}^\infty \lim_{\varepsilon\to 0^+} k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right]$$

$$\leqslant \frac{1}{\alpha\beta} \lim_{\varepsilon\to 0^+} \left[\frac{1}{\mu_1^{\alpha\varepsilon}} \int_0^{\frac{\mu_1^\alpha}{v_1^\beta}} k_\lambda(u,1) u^{\lambda_1 + \frac{\varepsilon}{q} - 1} du \right.$$

$$\left. + \frac{1}{v_1^{\beta\varepsilon}} \int_{\frac{\mu_1^\alpha}{v_1^\beta}}^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right] \leqslant M \left(\frac{1}{\alpha} \right)^{\frac{1}{p}} \left(\frac{1}{\beta} \right)^{\frac{1}{q}},$$

namely, $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1) \leqslant M$. Hence, $M = \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)$ is the best possible constant factor of (14).

The lemma is proved. $\square$

REMARK 2. Setting $\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\widehat{\lambda}_1 + \widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda,$$

$$\widehat{\lambda}_1 \leqslant \frac{1}{p\alpha} + \frac{1}{q\alpha} = \frac{1}{\alpha}, \quad \widehat{\lambda}_2 \leqslant \frac{1}{q\beta} + \frac{1}{p\beta} = \frac{1}{\beta},$$

and by Hölder's inequality (cf. [13]), we obtain

$$0 < k_\lambda(\lambda - \widehat{\lambda}_2) = k_\lambda(\widehat{\lambda}_1) = k_\lambda\left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}\right)$$

$$= \int_0^\infty k_\lambda(u,1)u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1}du = \int_0^\infty k_\lambda(u,1)(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du$$

$$\leqslant \left(\int_0^\infty k_\lambda(u,1)u^{\lambda-\lambda_2-1}du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du\right)^{\frac{1}{q}}$$

$$= (k_\lambda(\lambda - \lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}} < \infty. \tag{17}$$

We can reduce (13) as follows:

$$I < \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^\infty (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}$$

$$\times \left[\sum_{n=1}^\infty (V_n - \widehat{v})^{q(1-\beta\widehat{\lambda}_2)-1}\frac{b_n^q}{v_{n+1}^{q-1}}\right]^{\frac{1}{q}}, \tag{18}$$

LEMMA 5. *If the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (13) is the best possible, then. $\lambda_1 + \lambda_2 = \lambda$.*

Proof. If the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (13) is the best possible, then by (18) and (14) (for $\lambda_1 = \widehat{\lambda}_1, \lambda_2 = \widehat{\lambda}_2$), we have the following inequality:

$$\frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant \frac{k_\lambda(\widehat{\lambda}_1)}{\beta^{1/p}\alpha^{1/q}},$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant k_\lambda(\widehat{\lambda}_1)$. We observe that inequality (17) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [13]) $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ *a.e.* in $\mathbf{R}_+$. Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_1-\lambda_2} = B/A$ *a.e.* in $\mathbf{R}_+$, and then $\lambda - \lambda_1 - \lambda_2 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. $\square$

3. Main results and some particular cases

THEOREM 1. *Inequality* (18) *(or* (13)*) is equivalent to the following inequality:*

$$
J := \left\{ \sum_{n=1}^{\infty} (V_n - \widehat{v})^{p\beta\widehat{\lambda}_2 - 1} v_{n+1} \left[\sum_{m=1}^{\infty} k_\lambda ((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{v})^\beta) a_m \right]^p \right\}^{\frac{1}{p}}
$$
$$
< \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p} \alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}}. \tag{19}
$$

If the constant factor in (18) *is the best possible, then, so is the constant factor in* (19).

Proof. Assuming that (19) is valid, by Hölder's inequality (cf. [13]), we find

$$
I = \sum_{n=1}^{\infty} \left[\frac{(V_n - \widehat{v})^{\frac{-1}{p} + \beta\widehat{\lambda}_2}}{v_{n+1}^{-1/p}} \sum_{m=1}^{\infty} k_\lambda ((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{v})^\beta) a_m \right]
$$
$$
\times \left[(V_n - \widehat{v})^{\frac{1}{p} - \beta\widehat{\lambda}_2} \frac{b_n}{v_{n+1}^{1/p}} \right]
$$
$$
\leqslant J \left[\sum_{n=1}^{\infty} (V_n - \widehat{v})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{v_{n+1}^{q-1}} \right]^{\frac{1}{q}}. \tag{20}
$$

Then by (19), we obtain (18).

On the other hand, assuming that (18) is valid, we set

$$
b_n := (V_n - \widehat{v})^{p\beta\widehat{\lambda}_2 - 1} v_{n+1} \left[\sum_{m=1}^{\infty} k_\lambda ((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{v})^\beta) a_m \right]^{p-1}, \quad n \in \mathbf{N}.
$$

If $J = 0$, then (19) is naturally valid; if $J = \infty$, then it is impossible that to make (19) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (18), it follows that

$$
\sum_{n=1}^{\infty} (V_n - \widehat{v})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{v_{n+1}^{q-1}}
$$
$$
= J^p = I < \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p} \alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}}
$$
$$
\times \left[\sum_{n=1}^{\infty} (V_n - \widehat{v})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{v_{n+1}^{q-1}} \right]^{\frac{1}{q}},
$$

$$J = \left[\sum_{n=1}^{\infty} (V_n - \widehat{v})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{v_{n+1}^{q-1}} \right]^{\frac{1}{p}}$$

$$< \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}},$$

namely, (19) follows, which is equivalent to (18).

If the constant factor in (18) is the best possible, then so is constant factor in (19) . Otherwise, by (20) (for $\lambda_1 + \lambda_2 = \lambda$), we would reach a contradiction that the constant factor in (14) is not the best possible.

The theorem is proved. $\square$

THEOREM 2. *If* $U(\infty) = V(\infty) = \infty$, *then the following statements (i), (ii), (iii), (iv) and (v) are equivalent:*

(i) Both $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda \left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} \right)$ are independent of p,q;
(ii) We have the following inequality:

$$k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant k_\lambda \left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} \right);$$

(iii) $\lambda_1 + \lambda_2 = \lambda$;

(iv) the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (18) (resp. (19)) is the best possible.

If the statement (iii) follows, namely, $\lambda_1 + \lambda_2 = \lambda$, then we have the following inequality equivalent to (14) with the best possible constant factor $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$:

$$\left\{ \sum_{n=1}^{\infty} (V_n - \widehat{v})^{p\beta\lambda_2-1} v_{n+1} \left[\sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^{\alpha}, (V_n - \widehat{v})^{\beta}) a_m \right]^p \right\}^{\frac{1}{p}}$$

$$< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} \left[\sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\lambda_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}}. \tag{21}$$

Proof. $(i) \Rightarrow (ii)$. Since both $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})$ are independent of p,q, we find

$$k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty}\lim_{q\to 1^+} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1),$$

and by Fatou lemma (cf. [14]), we have the following inequality:

$$k_\lambda \left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} \right) = \lim_{p\to\infty}\lim_{q\to 1^+} k_\lambda \left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} \right)$$

$$\geqslant k_\lambda(\lambda_1) = k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1).$$

$(ii) \Rightarrow (iii)$. If $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})$, then (17) keeps the form of equality. Based on the proof of Lemma 5, it follows that $\lambda_1 + \lambda_2 = \lambda$.

$(iii) \Rightarrow (i)$. If $\lambda_1 + \lambda_2 = \lambda$, then we have

$$k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}\right) = k_\lambda(\lambda_1).$$

Both $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})$ are independent of p, q.

Hence, it follows that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(iii) \Rightarrow (iv)$. By Lemm 4 and Theorem 1, we obtain the conclusions.

$(iv) \Rightarrow (iii)$. By Theorem 1 and Lemma 5, we obtain $\lambda_1 + \lambda_2 = \lambda$.

Therefore, the statements (i), (ii), (iii), (iv) and (v) are equivalent.

The theorem is proved. $\square$

REMARK 3. (i) For $\lambda = \alpha = \beta = 1, \lambda_1 = \frac{1}{q}, \lambda_1 = \frac{1}{p}$ in (14) and (21), we have the following equivalent inequalities with the best possible constant factor:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(U_m - \widehat{\mu}, V_n - \widehat{v})a_m b_n < k_1\left(\frac{1}{q}\right)\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{v_{n+1}^{q-1}}\right)^{\frac{1}{q}}, \quad (22)$$

$$\left[\sum_{n=1}^{\infty} v_{n+1}\left(\sum_{m=1}^{\infty} k_1(U_m - \widehat{\mu}, V_n - \widehat{v})a_m\right)^p\right]^{\frac{1}{p}} < k_1(\frac{1}{q})\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right)^{\frac{1}{p}}. \quad (23)$$

(ii) For $\lambda = \alpha = \beta = 1$, $\lambda_1 = \frac{1}{p}$, $\lambda_1 = \frac{1}{q}$ in (14) and (21), we have the following equivalent dual forms of (22) and (23) with the best possible constant factor:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(U_m - \widehat{\mu}, V_n - \widehat{v})a_m b_n$$

$$< k_1\left(\frac{1}{p}\right)\left[\sum_{m=1}^{\infty}(U_m - \widehat{\mu})^{p-2}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(V_n - \widehat{v})^{q-2}\frac{b_n^q}{v_{n+1}^{q-1}}\right]^{\frac{1}{q}}, \quad (24)$$

$$\left[\sum_{n=1}^{\infty}(V_n - \widehat{v})^{p-2}v_{n+1}\left(\sum_{m=1}^{\infty} k_1(U_m - \widehat{\mu}, V_n - \widehat{v})a_m\right)^p\right]^{\frac{1}{p}}$$

$$< k_1\left(\frac{1}{p}\right)\left[\sum_{m=1}^{\infty}(U_m - \widehat{\mu})^{p-2}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}. \quad (25)$$

(iii) For $p = q = 2$, both (22) and (24) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(U_m - \widehat{\mu}, V_n - \widehat{v})a_m b_n < k_1\left(\frac{1}{2}\right)\left(\sum_{m=1}^{\infty}\frac{a_m^2}{\mu_{m+1}}\sum_{n=1}^{\infty}\frac{b_n^2}{v_{n+1}}\right)^{\frac{1}{2}}, \quad (26)$$

and both (23) and (25) reduce to the equivalent form of (26) as follows:

$$\left[\sum_{n=1}^{\infty} v_{n+1} \left(\sum_{m=1}^{\infty} k_1(U_m - \widehat{\mu}, V_n - \widehat{v}) a_m \right)^2 \right]^{\frac{1}{2}} < k_1\left(\frac{1}{2}\right) \left(\sum_{m=1}^{\infty} \frac{a_m^2}{\mu_{m+1}} \right)^{\frac{1}{2}}. \quad (27)$$

4. Operator expressions and some particular inequalities

We set functions

$$\varphi(m) := (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{1}{\mu_{m+1}^{p-1}}, \qquad \psi(n) := (V_n - \widehat{v})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{1}{v_{n+1}^{q-1}},$$

wherefrom,

$$\psi^{1-p}(n) = (V_n - \widehat{v})^{p\beta\widehat{\lambda}_2-1} v_{n+1} \quad (m, n \in \mathbf{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{m=1}^{\infty}; ||a||_{p,\varphi} := \left(\sum_{m=1}^{\infty} \varphi(m)|a_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}_{n=1}^{\infty}; ||b||_{q,\psi} := \left(\sum_{n=1}^{\infty} \psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{n=1}^{\infty}; ||c||_{p,\psi^{1-p}} := \left(\sum_{n=1}^{\infty} \psi^{1-p}(n)|c_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{v})^\beta) a_m, \ n \in \mathbf{N},$$

we can rewrite (19) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

DEFINITION 2. Define a more accurate Hilbert-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$, satisfying for any $n \in \mathbf{N}, Ta(n) = c_n$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta, b) := \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{v})^\beta) a_m \right) b_n = I,$$

$$||T|| := \sup_{a(\neq 0)\in l_{p,\varphi}} \frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

THEOREM 3. *If $a \in l_{p,\varphi}$, $b \in l_{q,\psi}$, $||a||_{p,\varphi}$, $||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta,b) < \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}||b||_{q,\psi}, \tag{28}$$

$$||Ta||_{p,\psi^{1-p}} < \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}. \tag{29}$$

Moreover, if $U(\infty) = V(\infty) = \infty$, then, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (28) (resp. (29)) is the best possible, namely,

$$||T|| = \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}. \tag{30}$$

EXAMPLE 1. We set $k_\lambda(x,y) = \frac{1}{(cx+y)^\lambda}$ $(c,\lambda > 0; x,y > 0)$. Then we find

$$k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n, -\widehat{v})^\beta) = \frac{1}{[c(U_m - \widehat{\mu})^\alpha + (V_n - \widehat{v})^\beta]^\lambda}.$$

For $0 < \alpha, \beta \leqslant 1$, $0 < \lambda_1$, $\lambda - \lambda_2 \leqslant \frac{1}{\alpha}$, $0 < \lambda_2$, $\lambda - \lambda_1 \leqslant \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is a strictly decreasing and strictly convex function with respect to $x, y > 0$, and for $\gamma = \lambda_1$, $\lambda - \lambda_2$

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}}{(cu+1)^\lambda}du = \frac{1}{c^\gamma}B(\gamma, \lambda - \gamma) \in \mathbf{R}_+.$$

In view of Theorem 3, it follows that if $U(\infty) = V(\infty) = \infty$, then, $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{1}{c^\gamma}B(\lambda_1, \lambda_2).$$

EXAMPLE 2. We set $k_\lambda(x,y) = \frac{\ln(cx/y)}{(cx)^\lambda - y^\lambda}$ $(c > 0, 0 < \lambda \leqslant 1; x,y > 0)$. Then we find

$$k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n, -\widehat{v})^\beta) = \frac{\ln[c(U_m - \widehat{\mu})^\alpha/(V_n - \widehat{v})^\beta]}{[c(U_m - \widehat{\mu})^\alpha]^\lambda - [(V_n - \widehat{v})^\beta]^\lambda}.$$

For $0 < \alpha, \beta \leqslant 1$, $0 < \lambda_1$, $\lambda - \lambda_2 \leqslant \frac{1}{\alpha}$, $0 < \lambda_2$, $\lambda - \lambda_1 \leqslant \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is a strictly decreasing and

strictly convex function with respect to $x, y > 0$ (cf. [25], Example 2.2.1), and for $\gamma = \lambda_1,\ \lambda - \lambda_2$

$$k_\lambda(\gamma) = \int_0^\infty \frac{\ln(cu)}{(cu)^\lambda - 1} u^{\gamma - 1} du = \frac{1}{c^\gamma}\left[\frac{\pi}{\sin(\pi\gamma/\lambda)}\right]^2 \in \mathbf{R}_+.$$

In view of Theorem 3, it follows that if $U(\infty) = V(\infty) = \infty$, then, $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}} \frac{1}{c^\gamma}\left[\frac{\pi}{\sin(\pi\gamma/\lambda)}\right]^2.$$

EXAMPLE 3. For $s \in \mathbf{N}$, we set $k_\lambda(x,y) = \dfrac{1}{\prod\limits_{k=1}^{s}(x^{\lambda/s} + c_k y^{\lambda/s})}$ $(0 < c_1 \leqslant \cdots \leqslant c_s,$

$0 < \lambda \leqslant 1;\ x, y > 0)$. Then we find

$$k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n, -\widehat{v})^\beta) = \frac{1}{\prod\limits_{k=1}^{s}[(U_m - \widehat{\mu})^{\alpha\lambda/s} + c_k(V_n - \widehat{v})^{\beta\lambda/s}]}.$$

For $0 < \alpha, \beta \leqslant 1,\ 0 < \lambda_1,\ \lambda - \lambda_2 \leqslant \frac{1}{\alpha}, 0 < \lambda_2, \lambda - \lambda_1 \leqslant \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is a strictly decreasing and strictly convex function with respect to $x, y > 0$, and for $\gamma = \lambda_1,\ \lambda - \lambda_2$, by Example 1 of [29], it follows that

$$k_\lambda^{(s)}(\gamma) = \int_0^\infty \frac{u^{\gamma - 1}}{\prod\limits_{k=1}^{s}(u^{\lambda/s} + c_k)} du$$

$$= \frac{\pi s}{\lambda\sin(\frac{\pi s\gamma}{\lambda})} \sum_{k=1}^{s} c_k^{\frac{s\gamma}{\lambda} - 1} \prod_{j=1(j \neq k)}^{s} \frac{1}{c_j - c_k} \in \mathbf{R}_+.$$

In view of Theorem 3, it follows that if $U(\infty) = V(\infty) = \infty$, then, $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = \frac{k_\lambda^{(s)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}} \frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})} \sum_{k=1}^{s} c_k^{\frac{s\lambda_1}{\lambda} - 1} \prod_{j=1(j \neq k)}^{s} \frac{1}{c_j - c_k}.$$

In particular, for $c_1 = \cdots = c_s = c$, we have $k_\lambda(x,y) = \frac{1}{(x^{\lambda/s} + c y^{\lambda/s})^s}$ and

$$k_\lambda^{(s)}(\gamma) = \int_0^\infty \frac{t^{\gamma - 1}}{(t^{\lambda/s} + c)^s} dt = \frac{s}{\lambda c^{\lambda_2 s/\lambda}} B\left(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}\right).$$

If $s = 1$, then we have $k_\lambda(x,y) = \frac{1}{x^\lambda + c y^\lambda}$,

$$k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n, -\widehat{v})^\beta) = \frac{1}{(U_m - \widehat{\mu})^{\alpha\lambda} + c(V_n - \widehat{v})^{\beta\lambda}}.$$

and

$$||T|| = \frac{k_\lambda^{(s)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{\pi}{\lambda c^{\lambda_2/\lambda}\sin(\frac{\pi\lambda_1}{\lambda})}.$$

5. Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate Hardy-Hilbert-type inequality with the general homogeneous kernel and the discrete intermediate variables is obtained, which is a more accurate extension of inequality (4). The equivalent forms are given in Theorem 1. The equivalent statements of the best possible constant factor related to some parameters are considered in Theorem 2. Some particular cases are given in Remark 3. As applications, the operator expressions and some examples are given in Theorem 3 and Example 1–3. The lemmas and theorems provide an extensive account of this type of inequalities.

REFERENCES

[1] L. E. AZAR, *The connection between Hilbert and Hardy inequalities*, Journal of Inequalities and Applications (2013), 2013: 452.
[2] V. ADIYASUREN, T. BATBOLD, AND M. KRNIC, *Hilbert-type inequalities involving differential operators, the best constants and applications*, Math. Inequal. Appl., **18** (1) (2015), 111–124.
[3] G. H. HARDY, J. E. LITTLEWOOD, AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[4] Q. L. HUANG, *A new extension of Hardy-Hilbert-type inequality*, Journal of Inequalities and Applications (2015), 2015: 397.
[5] B. HE, *A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor*, Journal of Mathematical Analysis and Applications, **431** (2015), 990–902.
[6] Z. X. HUANG, AND B. C. YANG, *On a half-discrete Hilbert-type inequality similar to Mulholland's inequality*, Journal of Inequalities and Applications (2013), 2013: 290.
[7] Y. HONG, AND Y. M. WEN, *A necessary and Sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor*, Annals Mathematica, **37A** (3) (2016), 329–336.
[8] Y. HONG, *On the structure character of Hilbert's type integral inequality with homogeneous kernel and applications*, Journal of Jilin University (Science Edition), **55** (2) (2017), 189–194.
[9] Y. HONG, Q. L. HUANG, B. C. YANG, AND J. Q. LIAO, *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, Journal of Inequalities and Applications (2017), 2017: 316.
[10] Y. HONG, B. HE, AND B. C. YANG, *Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory*, Journal of Mathematics Inequalities, **12** (3) (2018), 777–788.
[11] Z. X. HUANG, AND B. C. YANG, *Equivalent property of a half-discrete Hilbert's inequality with parameters*, Journal of Inequalities and Applications (2018) 2018: 333.
[12] M. KRNIC, AND J. PECARIC, *General Hilbert's and Hardy's inequalities*, Mathematical Inequalities, **8** (2005), 29-51.
[13] J. C. KUANG, *Applied inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.

[14] J. C. KUANG, *Real analysis and functional analysis (continuation)* (sec. vol.), Higher Education Press, Beijing, China, 2015.

[15] I. PERIC, AND P. VUKOVIC, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach Journal of Mathematical Analysis, **5** (2) (2011), 33–43.

[16] M. TH. RASSIAS, AND B. C. YANG, *On half-discrete Hilbert's inequality*, Applied Mathematics and Computation, **220** (2013), 75–93.

[17] M. TH. RASSIAS, AND B. C. YANG, *A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function*, Applied Mathematics and Computation, **225** (2013), 263–277.

[18] M. TH. RASSIAS, AND B. C. YANG, *On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function*, Applied Mathematics and Computation, **242** (2013), 800–813.

[19] J. S. XU, *Hardy-Hilbert's inequalities with two parameters*, Advances in Mathematics, **36** (2) (2007), 63–76.

[20] Z. T. XIE, Z. ZENG, AND Y. F. SUN, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2, Advances and Applications in Mathematical Sciences, **12** (7) (2013), 391–401.

[21] D. M. XIN, *A Hilbert-type integral inequality with the homogeneous kernel of zero degree*, Mathematical Theory and Applications, **30** (2) (2010), 70–74.

[22] D. M. XIN, B. C. YANG, AND A. Z. WANG, *Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane*, Journal of Function Spaces, Vol. 2018, Article ID2691816, 8 pages.

[23] B. C. YANG, *On Hilbert's integral inequality*, J. Math. Anal. Appl., **220** (1998), 778–785.

[24] B. C. YANG, *A note on Hilbert's integral inequality*, Chinese Quarterly Journal of Mathematics, **13** (4) (1998), 83–86.

[25] B. C. YANG, *The norm of operator and Hilbert-type inequalities*, Science Press, Beijing, China, 2009.

[26] B. C. YANG, *Hilbert-type integral inequalities*, Bentham Science Publishers Ltd., The United Arab Emirates, 2009.

[27] B. C. YANG, AND M. KRNIC, *A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree* 0, Journal of Mathematical Inequalities, **6** (3) (2012), 401–417.

[28] B. C. YANG, AND L. DEBNATH, *Half-discrete Hilbert-type inequalities*, World Scientific Publishing, Singapore, 2014.

[29] B. C. YANG, *On a more accurate multidimensional Hilbert-type inequality with parameters*, Mathematical Inequalities and Applications, **18** (2) (2015), 429–441.

[30] Z. ZHENG, K. RAJA RAMA GANDHI, AND Z. T. XIE, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2 *and with the integral*, Bulletin of Mathematical Sciences and Applications, **3** (1) (2014), 11–20.

Yong Hong
Department of Applied Mathematics
Guangzhou Huashang and College
Guangdong, Guangzhou 511300, P. R. China
and
College of Mathematics and Statistics
Guangdong University and of Finance and Economics
Guangdong, Guangzhou 510320, P. R. China
e-mail: mathhongyong@163.com

Yanru Zhong
School of Computer Science and Information Security
Guilin and University of Electronic Technology
Guilin, Guangxi 541004, P. R. China
e-mail: bcyang@gdei.edu.cn

Bicheng Yang
School of Mathematics
Guangdong University of Education
Guangzhou, Guangdong 510303, P. R. China
e-mail: 18577399236@163.com

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com