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CONSISTENCY OF ESTIMATOR FOR NONPARAMETRIC REGRESSION UNDER NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES

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Abstract. In this article, we discuss the complete consistency and strong consistency of wavelet estimators in nonparametric regression model with negatively superadditive dependent random variables, which improve and extend some existing ones. Finally, some numerical simulations are carried out to confirm the theoretical results.

1. Introduction

Consider the estimation of a standard nonparametric regression model involving a regression function $g(\cdot)$ which is defined on $[0,1]$:

$$Y_{ni} = g(t_{ni}) + \varepsilon_{ni}, \qquad i = 1,2,\dots,n,\ n \geqslant 1, \tag{1.1}$$

where t_{ni} are nonrandom design points, t_{ni}'s are denoted $t_{(ni)}$ and taken to be ordered $0 \leqslant t_{(n1)} \leqslant \cdots \leqslant t_{(nn)} \leqslant 1$, ε_{ni} are random errors. For each $n \geqslant 1$, $(\varepsilon_{n1}, \varepsilon_{n2}, \dots, \varepsilon_{nn})$ have the same distribution as $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$.

In order to introduce wavelet estimator, we state two definitions here.

DEFINITION 1.1. S_r is said to be Schwarz space, if S_r is continuous differentiable for r-times function space, and the function in S_r rapidly decreasing at infinity, i.e. for $h \in S_r$, there exists a constant $C_{pr} > 0$ such that

$$\left|h^{(k)}(t)\right| \leqslant C_{pk}(1+|t|)^{-p}, \quad k = 0,1,\cdots,r,\ \ p \in \mathbb{Z},\ \ t \in \mathbb{R}.$$

DEFINITION 1.2. A function space $H^{\gamma}(\gamma \in \mathbb{R})$ is said to be Sobolev space with order γ, i.e. if $h \in H^{\gamma}$ then

$$\int \left|\widehat{h}(\omega)\right|^2 (1+\omega^2)^{\gamma} d\omega < \infty,$$

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where $\widehat{h}$ is the Fourier transform of h.

Let $\phi(\cdot)$ be a given scaling function in the Schwarz space with order l. A multiresolution analysis of $L^2(\mathbb{R})$ consists of an increasing sequence of the closed subspace $\{V_m, m \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the integer set and $L^2(\mathbb{R})$ is a set of square integral functions over the real line. Since $\{\phi(x-k), k \in \mathbb{Z}\}$ is an orthogonal family of $L^2(\mathbb{R})$ and V_0 is the subspace spanned, if we define

$$\phi_{mk}(x) = 2^{m/2}\phi(2^m x - k),\ k \in \mathbb{Z},$$

then $\{\phi_{0k}, k \in \mathbb{Z}\}$ is an orthogonal basis of V_0 , and $\{\phi_{mk}, k \in \mathbb{Z}\}$ is an orthogonal basis of V_m. The associated integral kernel of V_m is given by

$$E_m(t,s) = 2^m E_0(2^m t, 2^m s) = 2^m \sum_{k\in\mathbb{Z}} \phi(2^m t - k)\phi(2^m s - k).$$

For model (1.1), a natural nonparametric wavelet estimator of $g(\cdot)$ is defined as

$$g_n(t) = \sum_{i=1}^{n} Y_{ni} \int_{A_{ni}} E_m(t,s)ds, \tag{1.2}$$

where $A_{ni} = [s_{n(i-1)}, s_{ni}]$, $s_{n0} = 0$, $s_{nn} = 1$, $s_{ni} = (t_{ni} + t_{n(i+1)})/2$, $i = 1,\dots,n$. Hence $t_{(ni)} \in A_{ni}$ for $1 \leqslant i \leqslant n$.

As we know that regression model (1.1) has many applications in practical problems, such as, filtering and prediction in communications and control systems, pattern recognition, classification and econometrics, and an important tool of data analysis. The weighted function estimates have been investigated to estimate the regression function $g(\cdot)$. For instance, see, Priestley and Chao (1972), Stone (1977), Georgiev (1988) and the references therein for the independent case; Roussas and Tran (1992), Liang and Jing (2005), Wang et al. (2014), Shen et al. (2015) and Yang et al. (2018), Ding (2020), Shen et al. (2021), Ding et al. (2022) for the various dependence cases.

The wavelet method in nonparametric curve estimation has become a well-known technique. The major advantage of the wavelet method is its adaptability to the degree of smoothness of the underlying unknown function. Due to its ability to adapt to local features of unknown curves, many authors have applied wavelet procedures to estimate the nonparametric model without repeated measurements. So in order to meet practical demands, since the 90s of the twentieth century, some authors have considered using wavelet method to estimate $g(\cdot)$. See recent works, for example, Antoniadis et al. (1994) under independent errors; Li et al. (2011), Shen et al. (2021) and Ding et al. (2022) under φ-mixing errors, Zhou and Lin (2014) under α-mixing errors, Wang et al. (2021) under extended negatively dependent errors and so on.

DEFINITION 1.3. (c.f. Kemperman 1977) A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called superadditive if $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geqslant \phi(\mathbf{x}) + \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\vee$ stands for componentwise maximum, and $\wedge$ denotes componentwise minimum.

DEFINITION 1.4. (c.f. Hu 2000) A random vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is said to be negatively superadditive dependent (NSD) if

$$\mathrm{E}\phi(X_1, X_2, \cdots, X_n) \leqslant \mathrm{E}\phi(X_1^*, X_2^*, \cdots, X_n^*), \tag{1.1}$$

where $X_1^*, X_2^*, \cdots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i, and ϕ is a superadditive function such that the expectations in (1.1) exist.

A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be NSD if for all $n \geqslant 1$, $(X_1, X_2, \cdots, X_n)$ is NSD.

The definition of NSD random variable sequence was introduced by Hu (2000), which was based on the class of superadditive functions. Hu (2000) presented an example illustrating that NSD does not imply NA, and subsequently presented an open problem whether NA implies NSD. Christofides and Vaggelatou (2004) solved this open problem and indicated that NA implies NSD. The concept of NSD extends the concept of NA. Eghbal et al. (2010) derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables. Shen (2012) studied moment inequality, Kolmogorov-type inequality and Hájek-Rényi-type inequalities for NSD sequence. As a consequence, she obtained Khintchine-Kolmogorov convergence theorem, three series theorem and Marcinkiewicz strong law of large numbers, extending the corresponding results for independent sequence and NA sequence. Shen et al. (2013) obtained almost sure convergence theorem and strong stability for weighted sums of NSD random variables. Shen et al. (2014) gave the Rosenthal-type inequality for NSD random variables and its applications. Wang et al. (2014) investigated complete convergence for arrays of rowwise NSD random variables and gave its applications to nonparametric regression model (1.1). Zheng et al. (2015) obtained some results on the complete convergence for sequences of NSD random variables by using some inequalities and the truncated method, which extended the corresponding conclusions for weighted sums of NA random variables with identical distribution to the case of sequences of NSD random variables with nonidentical distribution. Xue et al. (2015) investigated the complete moment convergence for maximal partial sum of NSD random variables under some more general conditions. Wang and Hu (2015) discussed the strong consistency of M-estimates of the regression parameters in a linear model with NSD random errors. The result improves the moment condition and generalises the case of independent random errors to that of NSD random errors. Wang et al. (2015) presented some basic properties for NSD random variables, and then studied the complete convergence for weighted sums of NSD random variables and applied it to obtain the complete consistency for the LS estimators in the EV regression model with NSD errors under mild conditions, and so on. However, there are very few literatures on consistency for the wavelet estimator of nonparametric regression model (1.1) based on NSD random errors.

By using the wavelet method, we continue to discuss the consistency properties for the estimator of nonparametric regression model with NSD errors. The complete consistency and strong consistency of the wavelet estimator of nonparametric regression model are presented in this article.

The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows: a sequence $\{X_n, n \geqslant 1\}$ of random variables converges completely to a constant C if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}(|X_n - C| > \varepsilon) < \infty.$$

By the Borel-Cantelli lemma, this implies that $X_n \to C$ *a.s.*, and so complete convergence is a stronger concept than almost sure convergence.

The structure of the rest article is as follows. In Section 2, we give some basic assumptions and main results. Some preliminary lemmas are stated in Section 3. Proofs of the main results are provided in Section 4. Some numerical simulations are presented in Section 5.

Throughout the paper for any function g, we use $c(g)$ to denote all continuity points of the regression function g on $[0,1]$. Let C denote positive constants which may be different in various places. $a_n = O(b_n)$ stands for $a_n \leqslant Cb_n$. Let $I(\cdot)$ be the indicator function. All limits are taken as the sample size n tends to ∞, unless specified otherwise.

2. Assumptions and main results

For easy reference, the assumptions used in this article are listed below:

(A1) $g(\cdot) \in H^{\nu}$, $\nu > 3/2$, and $g(\cdot)$ satisfies the Lipschitz condition of order 1;

(A2) $\phi(\cdot) \in S_r$, and $\phi(\cdot)$ satisfies the Lipschitz condition with order 1 and $|\widehat{\phi}(\varepsilon) - 1| = O(\varepsilon)$ as $\varepsilon \to 0$, where $\widehat{\phi}(\cdot)$ is the Fourier transform of $\phi(\cdot)$;

(A3) $\max_{1\leqslant i\leqslant n} |s_{ni} - s_{n(i-1)}| = O(n^{-1})$. When $n \to \infty$, then $m \to \infty$, $n2^{-m} \to \infty$.

REMARK 2.1. Assumptions (A1)–(A3) are general basic assumption conditions of wavelet estimation, which have been used by many authors, one can refer to Antoniadis et al. (1994), Li et al. (2011), Zhou and Lin (2014), Wang et al. (2021) or Shen et al. (2021) so we can see that the assumptions in this article are suitable and reasonable.

Our main results can be given below:

THEOREM 2.1. *Let* $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of zero mean NSD random variables, which is stochastically dominated by a random variable* X *with* $\mathrm{E}|X|^{5/2} < \infty$. *Assume that assumptions* (A1)–(A3) *hold. Suppose further that* $2^m = O(n^{\frac{1}{3}})$. *Then for any* $t \in c(g)$,

$$g_n(t) \to g(t) \ \text{ completely, as } n \to \infty.$$

THEOREM 2.2. *Let* $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of zero mean NSD random variables, which is stochastically dominated by a random variable* X *with* $\mathrm{E}|X|^{2/p+1} < \infty$ *for some* $0 < p < 1$. *Assume that assumptions* (A1)–(A3) *hold. Suppose further that* $2^m = O(n^{1-p}\log^{-\delta} n)$ *for some* $\delta > 1$. *Then for any* $t \in c(g)$ *and every* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}\left(|g_n(t) - g(t)| \geqslant \varepsilon\right) < \infty,$$

and thus,

$$g_n(t) \to g(t) \ \ a.s., \ \text{ as } n \to \infty.$$

REMARK 2.2. The moment condition $\mathrm{E}|X|^{5/2} < \infty$ in Theorem 2.1 is weaker than $\mathrm{E}|X|^3 < \infty$ in Ding (2020). Therefore, Theorem 2.1 extends and improves the corresponding ones in Ding (2020).

3. Some lemmas

In this section, we will present some important lemmas which will be used to prove the above main results. The first one comes from Hu (2000).

LEMMA 3.1. *If* $(X_1, X_2, \cdots, X_n)$ *is NSD and* $g_1, g_2, \cdots, g_n$ *are nondecreasing functions, then* $(g_1(X_1), g_2(X_2), \cdots, g_n(X_n))$ *is NSD.*

The next one was provided by Li et al. (2011).

LEMMA 3.2. *Assume that the assumptions* (A1)–(A3) *are satisfied. Then*
(i) $\left|\int_{A_{ni}} E_m(t,s)ds\right| = O(2^m/n),\ i = 1,2,\cdots,n;$
(ii) $\sum\limits_{i=1}^{n} \left|\int_{A_{ni}} E_m(t,s)ds\right| \leqslant C,\ i = 1,2,\cdots,n;$
(iii) $\sum\limits_{i=1}^{n} \left(\int_{A_{ni}} E_m(t,s)ds\right)^2 = O(2^m/n),\ i = 1,2,\cdots,n.$

The Rosenthal-type inequality for NSD random variables comes from Hu (2000).

LEMMA 3.3. *Let* $p \geqslant 1$ *and* $\{X_n, n \geqslant 1\}$ *be a sequence of NSD random variables with* $\mathrm{E}|X_i|^p < \infty$ *for each* $i \geqslant 1$. *Then for all* $n \geqslant 1$,

$$\mathrm{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leqslant 2^{3-p}\sum_{i=1}^{n}\mathrm{E}|X_i|^p, \quad \text{for } 1 < p \leqslant 2,$$

and

$$\mathrm{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leqslant 2\left(\frac{15p}{\ln p}\right)^p\left\{\sum_{i=1}^{n}\mathrm{E}|X_i|^p + \left(\sum_{i=1}^{n}\mathrm{E}|X_i|^2\right)^{p/2}\right\}, \quad \text{for } p > 2.$$

The Bernstein-type inequality for NSD random variables was used in Wang and Hu (2015).

LEMMA 3.4. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of NSD random variables such that* $\mathrm{E}X_n = 0$ *and* $|X_n| \leqslant b$ *a.s. for all* $n \geqslant 1$, *and some* $b > 0$. *Let* $B_n^2 = \sum_{i=1}^{n}\mathrm{E}X_i^2$. *Then for all* $\varepsilon > o$,

$$\mathrm{P}\left(\left|\sum_{i=1}^{n} X_i\right| > \varepsilon\right) \leqslant 2\exp\left\{-\frac{\varepsilon^2}{2(2B_n^2 + b\varepsilon)}\right\}.$$

For convenience, we give the definition of stochastic domination here.

DEFINITION 3.1. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$\mathrm{P}(|X_n| > x) \leqslant C\mathrm{P}(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

By the definition of stochastic domination and integration by parts, we can get the following property for stochastic domination. For the details of the proof, one can refer to Wu (2006).

LEMMA 3.5. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then, for any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$\mathrm{E}|X_n|^{\alpha} I(|X_n| \leqslant b) \leqslant C_1 \left(\mathrm{E}|X|^{\alpha} I(|X| \leqslant b) + b^{\alpha}\mathrm{P}(|X| > b)\right),$$

$$\mathrm{E}|X_n|^{\alpha} I(|X_n| > b) \leqslant C_2 \mathrm{E}|X|^{\alpha} I(|X| > b),$$

where C_1 and C_2 are positive constants. Consequently, $\mathrm{E}|X_n|^{\alpha} \leqslant C\mathrm{E}|X|^{\alpha}$, where C is a positive constant.

The last one can be found in Walter (1994).

LEMMA 3.6. *Assume that $\phi \in S_r$, then*
(i) $\int_0^1 E_m(x,y)dy \to 1$ as $x \in [0,1]$ for $m \to \infty$;
(ii) $\sup\limits_{m \geqslant 1} \int_0^1 |E_m(x,y)|dy < \infty$;
(iii) $\int_0^1 |E_m(x,y)| I(|x-y| > \varepsilon)dy \to 0$ as $x \in [0,1]$ for $\forall\, \varepsilon > 0$, $m \to \infty$.

4. Proofs of the main results

Proof of Theorem 2.1. Let $\phi \in S_r$, $g \in L^1(\mathbb{R})$, when $m \to \infty$, we have

$$\int_0^1 E_m(t,s)g(s)ds \to g(t), \quad \text{as } t \in [0,1]. \tag{4.1}$$

And since g is continuous in $[0,1]$, g is uniformly continuous in $[0,1]$. For given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|t-s| \leqslant \delta$, $|g(t) - g(s)| \leqslant \varepsilon$ for any $|t-s| \leqslant \delta$. Denote $M = \sup_{t \in [0,1]} g(t)$, and hence $M < \infty$.

$$\begin{aligned}
&\left|\int_0^1 E_m(t,s)g(s)ds - g(t)\right| \\
&= \left|\int_0^1 E_m(t,s)g(s)ds - \int_0^1 E_m(t,s)g(t)ds + \int_0^1 E_m(t,s)g(t)ds - g(t)\right| \\
&\leqslant \int_0^1 |E_m(t,s)||g(s)-g(t)|ds + M\left|\int_0^1 E_m(t,s)ds - 1\right| \\
&\doteq T_1 + T_2,
\end{aligned}$$

by Lemma 3.6(i), when $m \to \infty$, we have $T_2 \to 0$ as $t \in [0,1]$.

$$\begin{aligned}
T_1 &= \int_0^1 |E_m(t,s)||g(s)-g(t)| I(|t-s| \leqslant \delta)ds + \int_0^1 |E_m(t,s)||g(s)-g(t)| I(|t-s| > \delta)ds \\
&\doteq T_3 + T_4,
\end{aligned}$$

by Lemma 3.6(iii), when $m \to \infty$, we have $T_4 \leqslant 2M \int_0^1 |E_m(t,s)| I(|t-s| > \delta)ds \to 0$ as $t \in [0,1]$, and we have that $T_3 \leqslant \varepsilon \int_0^1 |E_m(t,s)|ds$ by choosing δ. By Lemma 3.6(ii), there exists a constant $C > 0$ such that $\int_0^1 |E_m(t,s)|ds \leqslant C$ as $t \in [0,1]$. Therefore, $T_3 \leqslant C\varepsilon$ as $t \in [0,1]$. Consequently, (4.1) is proved.

Moreover, since $\phi \in S_r$, $g \in L^1(\mathbb{R})$, $\max_{1\leqslant i\leqslant n}(s_{ni}-s_{n(i-1)}) = O(n^{-1})$, we have

$$\mathrm{E}g_n(t) - g(t) = \sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)dsg(t_{ni}) - g(t) \to 0, \ \text{ as } \ t \in [0,1], \ \ n \to \infty, \ \ m \to \infty. \tag{4.2}$$

In fact, note that

$$\begin{aligned}
&\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)dsg(t_{ni}) - g(t)\right| \\
&= \left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)dsg(t_{ni}) - \int_0^1 E_m(t,s)g(s)ds + \int_0^1 E_m(t,s)g(s)ds - g(t)\right| \\
&\leqslant \left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)dsg(t_{ni}) - \int_0^1 E_m(t,s)g(s)ds\right| + \left|\int_0^1 E_m(t,s)g(s)ds - g(t)\right| \\
&\doteq T_5 + T_6.
\end{aligned}$$

Since g is uniformly continuous in $[0,1]$, for any $\varepsilon > 0$, there exists a $\delta > 0$ and $|s-t| \leqslant \delta$ such that $|g(s)-g(t)| \leqslant \varepsilon$. Note that $\max_{1\leqslant i\leqslant n}(s_{ni}-s_{n(i-1)}) = O(n^{-1})$, there exists a N such that $\max_{1\leqslant i\leqslant n}(s_{ni}-s_{n(i-1)}) \leqslant \delta$ for all $n \geqslant N$. Then for $n \geqslant N$ we have

$$T_5 = \left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)(g(s)-g(t_{ni}))ds\right| \leqslant \varepsilon \int_0^1 |E_m(t,s)|ds \leqslant \varepsilon C, \ \text{ as } \ t \in [0,1].$$

Meanwhile, it follows by (4.1) that $T_6 \to 0$ as $t \in [0,1]$. Hence, (4.2) is proved. That is to say

$$|\mathrm{E}g_n(t) - g(t)| \to 0, \ \text{ as } \ t \in [0,1], \ \ n \to \infty. \tag{4.3}$$

For fixed design point $t \in [0,1]$, without loss of generality, we suppose that the weights $\int_{A_{ni}} E_m(t,s)ds > 0$ in what follows (otherwise, we use $\left(\int_{A_{ni}} E_m(t,s)ds\right)^+$ and $\left(\int_{A_{ni}} E_m(t,s)ds\right)^-$ instead of $\int_{A_{ni}} E_m(t,s)ds$, respectively, and note that $\int_{A_{ni}} E_m(t,s)ds = \left(\int_{A_{ni}} E_m(t,s)ds\right)^+ - \left(\int_{A_{ni}} E_m(t,s)ds\right)^-$). Noting that

$$\begin{aligned}
|g_n(t) - g(t)| &\leqslant |g_n(t) - \mathrm{E}g_n(t)| + |\mathrm{E}g_n(t) - g(t)| \\
&= \left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| + |\mathrm{E}g_n(t) - g(t)|.
\end{aligned} \tag{4.4}$$

Hence, by (4.4), we can see that in order to prove the main result, we only need to show that

$$\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni} \to 0 \ \text{ completely } \ \text{as } \ n \to \infty. \tag{4.5}$$

Namely, it suffices to show that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| > \varepsilon\right) < \infty. \tag{4.6}$$

For every fixed $n \geqslant 1$, take

$$X_{ni} = -I\left(\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni} < -1\right) + \int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni} I\left(\left|\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| \leqslant 1\right) \\ +I\left(\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni} > 1\right), \quad i = 1,2,\ldots,n.$$

We can easily obtain that for any given $\varepsilon > 0$,

$$\left(\left|\sum_{i=1}^{n}\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| > \varepsilon\right) \subset \left(\max_{1\leqslant i\leqslant n}\left|\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| > 1\right) \bigcup \left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right),$$

this yields

$$\begin{aligned} \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| > \varepsilon\right) &\leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{P}\left(\left|\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| > 1\right) \\ &\quad + \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right) \\ &\doteq J + K. \end{aligned} \tag{4.7}$$

Hence, to prove (4.6), it suffices to prove that $J < \infty$ and $K < \infty$.

By Lemma 3.2(ii) and $\mathrm{E}|X|^{5/2} < \infty$, we know that

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{P}\left(\left|\int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| > 1\right) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{P}\left(\left|\int_{A_{ni}} E_m(t,s)dsX\right| > 1\right) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \left|\int_{A_{ni}} E_m(t,s)ds\right| \mathrm{E}|X| I\left(\left|\int_{A_{ni}} E_m(t,s)dsX\right| > 1\right) \\ &\leqslant C\sum_{n=1}^{\infty} \mathrm{E}|X| I\left(|X| > n^{2/3}\right) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mathrm{E}|X| I\left(k^{2/3} \leqslant |X| < (k+1)^{2/3}\right) \\ &= C\sum_{k=1}^{\infty}\sum_{n=1}^{k} \mathrm{E}|X| I\left(k^{2/3} \leqslant |X| < (k+1)^{2/3}\right) \\ &= C\sum_{k=1}^{\infty} k\mathrm{E}|X| I\left(k^{2/3} \leqslant |X| < (k+1)^{2/3}\right) \\ &\leqslant C\sum_{k=1}^{\infty} \mathrm{E}|X|^{5/2} I\left(k^{2/3} \leqslant |X| < (k+1)^{2/3}\right) \\ &\leqslant C\mathrm{E}|X|^{5/2} < \infty, \end{aligned} \tag{4.8}$$

implying that $J < \infty$.

Next, we will consider $K < \infty$. Firstly, we will prove that

$$\left|\sum_{i=1}^{n} \mathrm{E}X_{ni}\right| \to 0, \quad \text{as } n \to \infty. \tag{4.9}$$

Clearly, from $\mathrm{E}\varepsilon_{ni}=0,\ 2^m=O(n^{\frac{1}{3}})$, Lemmas 3.2, 3.5 and $\mathrm{E}|X|^{5/2}<\infty$, one has

$$\begin{aligned}\left|\sum_{i=1}^{n}\mathrm{E}X_{ni}\right| &\leqslant \left|\sum_{i=1}^{n}\mathrm{E}\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}I\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|\leqslant 1\right)\right| \\ &\quad+\sum_{i=1}^{n}\mathrm{P}\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|>1\right) \\ &=\left|\sum_{i=1}^{n}\mathrm{E}\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}I\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|>1\right)\right| \\ &\quad+\sum_{i=1}^{n}\mathrm{P}\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|>1\right) \\ &\leqslant C\sum_{i=1}^{n}\mathrm{E}\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|^{5/2}I\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|>1\right) \\ &\leqslant C\sum_{i=1}^{n}\left|\int_{A_{ni}}E_m(t,s)ds\right|^{5/2}\mathrm{E}|X|^{5/2}I\left(\left|\int_{A_{ni}}E_m(t,s)dsX\right|>1\right) \\ &\leqslant C\left|\int_{A_{ni}}E_m(t,s)ds\right|^{3/2}\sum_{i=1}^{n}\left|\int_{A_{ni}}E_m(t,s)ds\right|\mathrm{E}|X|^{5/2}I(|X|>n^{2/3}) \\ &\leqslant Cn^{-1}\mathrm{E}|X|^{5/2}I(|X|>n^{2/3})\to 0,\ \text{ as } n\to\infty,\end{aligned}\tag{4.10}$$

which leads to (4.9). Thus, to verify $K<\infty$, we need to prove only that for all $\varepsilon>0$,

$$K^{*}=\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}(X_{ni}-\mathrm{E}X_{ni})\right|>\varepsilon/2\right)<\infty.\tag{4.11}$$

It follows from Markov's inequality, Lemma 3.3, C_r inequality and Jensen's inequality that for $M\geqslant 2$

$$\begin{aligned}K^{*}&\leqslant C\sum_{n=1}^{\infty}\mathrm{E}\left(\left|\sum_{i=1}^{n}(X_{ni}-\mathrm{E}X_{ni})\right|^{M}\right)\leqslant C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\mathrm{E}|X_{ni}|^{2}\right)^{M/2}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathrm{E}|X_{ni}|^{M} \\ &\doteq K_1+K_2.\end{aligned}\tag{4.12}$$

Taking $M>3$, so that $-M/3<-1$ and $-2(M-1)/3<-1$. Hence, we have by C_r inequality and Lemma 3.5 that

$$\begin{aligned}K_1\leqslant C\sum_{n=1}^{\infty}&\left\{\sum_{i=1}^{n}\mathrm{P}\left(\left|\int_{A_{ni}}E_m(t,s)dsX\right|>1\right)\right. \\ &\left.+\sum_{i=1}^{n}\mathrm{E}\left|\int_{A_{ni}}E_m(t,s)dsX\right|^{2}I\left(\left|\int_{A_{ni}}E_m(t,s)dsX\right|\leqslant 1\right)\right\}^{M/2}.\end{aligned}\tag{4.13}$$

It follows from $2^m=O(n^{\frac{1}{3}})$, Markov's inequality, Lemma 3.2(i)(ii) and $\mathrm{E}|X|^{5/2}<\infty$

that

$$
\begin{aligned}
K_1 &\leqslant C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\left|\int_{A_{ni}}E_m(t,s)ds\right|^2\mathrm{E}|X|^2\right)^{M/2}\\
&\leqslant C\sum_{n=1}^{\infty}\left(\left|\int_{A_{ni}}E_m(t,s)ds\right|\sum_{i=1}^{n}\left|\int_{A_{ni}}E_m(t,s)ds\right|\right)^{M/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{-M/3}<\infty.
\end{aligned}
\tag{4.14}
$$

Combining (4.13) and (4.14), we have $K_1<\infty$.

Applying the C_r inequality and Lemma 3.5, we get that

$$
\begin{aligned}
K_2 &\leqslant C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left\{\mathrm{E}\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|^M I\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|\leqslant 1\right)\right.\\
&\qquad\left.+\mathrm{P}\left(\left|\int_{A_{ni}}E_m(t,s)ds\varepsilon_{ni}\right|>1\right)\right\}\\
&\leqslant C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathrm{P}\left(\left|\int_{A_{ni}}E_m(t,s)dsX\right|>1\right)\\
&\qquad+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathrm{E}\left|\int_{A_{ni}}E_m(t,s)dsX\right|^M I\left(\left|\int_{A_{ni}}E_m(t,s)dsX\right|\leqslant 1\right)\\
&\doteq K_3+K_4.
\end{aligned}
\tag{4.15}
$$

It is easy to know that $K_3<\infty$ by (4.8).

In what follows, we will prove that $K_4<\infty$. Write

$$
J_{nj}=\left\{i:[n(j+1)]^{-2/3}<\int_{A_{ni}}E_m(t,s)ds\leqslant(nj)^{-2/3}\right\},\quad n\geqslant 1,\quad j\geqslant 1. \tag{4.16}
$$

On the other hand, we find that $J_{nk}\bigcap J_{nj}=\emptyset$ for $k\neq j$ and $\bigcup_{j=1}^{\infty}J_{nj}=\{1,2,\ldots,n\}$ for all $n\geqslant 1$. Writing $\sharp M$ for the cardinality of a set M, it follows

$$
\begin{aligned}
K_4 &\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i\in J_{nj}}\mathrm{E}\left|\int_{A_{ni}}E_m(t,s)dsX\right|^M I\left(\left|\int_{A_{ni}}E_m(t,s)dsX\right|\leqslant 1\right)\\
&\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp J_{nj})(nj)^{-2M/3}\mathrm{E}|X|^M I\left(|X|\leqslant[n(j+1)]^{2/3}\right)\\
&\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp J_{nj})(nj)^{-2M/3}\sum_{k=0}^{n(j+1)}\mathrm{E}|X|^M I\left(k\leqslant|X|^{3/2}<k+1\right)\\
&= C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp J_{nj})(nj)^{-2M/3}\sum_{k=0}^{2n}\mathrm{E}|X|^M I\left(k\leqslant|X|^{3/2}<k+1\right)\\
&\qquad+C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp J_{nj})(nj)^{-2M/3}\sum_{k=2n+1}^{n(j+1)}\mathrm{E}|X|^M I\left(k\leqslant|X|^{3/2}<k+1\right)\\
&\doteq K_5+K_6.
\end{aligned}
\tag{4.17}
$$

It is easy to check that for all $m \geqslant 1$,

$$\begin{aligned} C &\geqslant \sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds = \sum_{j=1}^{\infty} \sum_{i \in J_{nj}} \int_{A_{ni}} E_m(t,s)ds \geqslant \sum_{j=1}^{\infty} (\sharp J_{nj})[n(j+1)]^{-2/3} \\ &\geqslant \sum_{j=m}^{\infty} (\sharp J_{nj})[n(j+1)]^{-2/3} \geqslant \sum_{j=m}^{\infty} (\sharp J_{nj})[n(j+1)]^{-2/3} \left[\tfrac{n(m+1)}{n(j+1)}\right]^{2(M-1)/3} \\ &\geqslant \sum_{j=m}^{\infty} (\sharp J_{nj})[n(j+1)]^{-2M/3}[n(m+1)]^{-2(M-1)/3}, \end{aligned}$$

it is enough to show that for all $m \geqslant 1$,

$$\sum_{j=m}^{\infty} (\sharp J_{nj})(nj)^{-2M/3} \leqslant Cn^{-2(M-1)/3}m^{-2(M-1)/3}. \tag{4.18}$$

So that

$$\begin{aligned} K_5 &= C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} (\sharp J_{nj})(nj)^{-2M/3} \sum_{k=0}^{2n} \mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{n=1}^{\infty} n^{-2(M-1)/3} \sum_{k=0}^{2n} \mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{k=0}^{2}\sum_{n=1}^{\infty} n^{-2(M-1)/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\quad + C\sum_{k=2}^{\infty}\sum_{n=[k/2]}^{\infty} n^{-2(M-1)/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C + C\sum_{k=2}^{\infty} k^{1-2(M-1)/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C + C\sum_{k=2}^{\infty} \mathrm{E}|X|^{M+3/2-(M-1)} I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C + C\mathrm{E}|X|^{5/2} < \infty, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} K_6 &= C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} (\sharp J_{nj})(nj)^{-2M/3} \sum_{k=2n+1}^{n(j+1)} \mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{k=2n+1}^{\infty}\sum_{j \geqslant k/n-1} (\sharp J_{nj})(nj)^{-2M/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{k=2n+1}^{\infty} n^{-2(M-1)/3}(k/n)^{-2(M-1)/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{k=2}^{\infty}\sum_{n=1}^{[k/2]} k^{-2(M-1)/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{k=2}^{\infty} k^{1-2(M-1)/3}\mathrm{E}|X|^M I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\sum_{k=2}^{\infty} \mathrm{E}|X|^{M+3/2-(M-1)} I\left(k \leqslant |X|^{3/2} < k+1\right) \\ &\leqslant C\mathrm{E}|X|^{5/2} < \infty. \end{aligned} \tag{4.20}$$

Thus, we obtain the desired inequality (4.11) through (4.12)–(4.15), (4.17), (4.19) and (4.20) immediately. The proof is completed. □

Proof of Theorem 2.2. Without loss of generality, we assume that $\int_{A_{ni}} E_m(t,s)ds \geqslant 0$. Otherwise, we will use $\left(\int_{A_{ni}} E_m(t,s)ds\right)^+$ and $\left(\int_{A_{ni}} E_m(t,s)ds\right)^-$ instead of

$$\int_{A_{ni}} E_m(t,s)ds$$

respectively. Combining (4.3) with (4.4), we can see that in order to prove the main result, we only need to show that

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni}\right| \geqslant \varepsilon/2\right) < \infty. \tag{4.21}$$

Denote for $1 \leqslant i \leqslant n$ and $n \geqslant 1$ that

$$\varepsilon'_{ni} = -n^{p/2}I(\varepsilon_{ni} < -n^{p/2}) + \varepsilon_{ni}I(|\varepsilon_{ni}| \leqslant n^{p/2}) + n^{p/2}I(\varepsilon_{ni} > n^{p/2}),$$

$$\varepsilon''_{ni} = \varepsilon_{ni} - \varepsilon'_{ni} = (\varepsilon_{ni} - n^{p/2})I(\varepsilon_{ni} > n^{p/2}) + (\varepsilon_{ni} + n^{p/2})I(\varepsilon_{ni} < -n^{p/2}),$$

$$\varepsilon'_{i} = -n^{p/2}I(\varepsilon_{i} < -n^{p/2}) + \varepsilon_{i}I(|\varepsilon_{i}| \leqslant n^{p/2}) + n^{p/2}I(\varepsilon_{i} > n^{p/2}),$$

$$\varepsilon''_{i} = \varepsilon_{i} - \varepsilon'_{i} = (\varepsilon_{i} - n^{p/2})I(\varepsilon_{i} > n^{p/2}) + (\varepsilon_{i} + n^{p/2})I(\varepsilon_{i} < -n^{p/2}).$$

Since $\mathrm{E}\varepsilon_{ni} = \mathrm{E}\varepsilon_i = 0$, then

$$\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds\varepsilon_{ni} = \sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds(\varepsilon'_{ni} - \mathrm{E}\varepsilon'_{ni}) + \sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds(\varepsilon''_{ni} - \mathrm{E}\varepsilon''_{ni}).$$

Hence, to obtain (4.21), it suffices to show that

$$J_1 \doteq \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds(\varepsilon'_{ni} - \mathrm{E}\varepsilon'_{ni})\right| \geqslant \varepsilon/4\right) < \infty, \tag{4.22}$$

and

$$J_2 \doteq \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds(\varepsilon''_{ni} - \mathrm{E}\varepsilon''_{ni})\right| \geqslant \varepsilon/4\right) < \infty. \tag{4.23}$$

It follows by Lemmas 3.2, 3.4, $2^m = O(n^{1-p}\log^{-\delta} n)$ and $\mathrm{E}|X|^{2/p+1} < \infty$ that

$$\left|\int_{A_{ni}} E_m(t,s)ds(\varepsilon'_{ni} - \mathrm{E}\varepsilon'_{ni})\right| \leqslant Cn^{-p/2}\log^{-\delta} n, \tag{4.24}$$

and

$$\begin{aligned} B_n^2 &= \sum_{i=1}^{n} \mathrm{E}\left|\int_{A_{ni}} E_m(t,s)ds(\varepsilon'_{ni} - \mathrm{E}\varepsilon'_{ni})\right|^2 \\ &\leqslant \sum_{i=1}^{n} \left(\int_{A_{ni}} E_m(t,s)ds\right)^2 \mathrm{E}|\varepsilon'_i|^2 \\ &\leqslant \sum_{i=1}^{n} \left(\int_{A_{ni}} E_m(t,s)ds\right)^2 \mathrm{E}|\varepsilon_i|^2 \\ &\leqslant C \max_{1\leqslant i\leqslant n} \left|\int_{A_{ni}} E_m(t,s)ds\right| \sum_{i=1}^{n} \left|\int_{A_{ni}} E_m(t,s)ds\right| \\ &\leqslant Cn^{-p}\log^{-\delta} n. \end{aligned} \tag{4.25}$$

Noting that $(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn})$ have the same distribution as $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ for each $n \geqslant 1$, It is easily seen that $\left\{\int_{A_{ni}} E_m(t,s)ds(\varepsilon'_i - \mathrm{E}\varepsilon'_i), 1 \leqslant i \leqslant n\right\}$ are zero mean NSD random variables by Lemma 3.1, we have by Lemma 3.4, (4.24) and (4.25) that

$$\begin{aligned} J_1 &= \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds(\varepsilon'_i - \mathrm{E}\varepsilon'_i)\right| \geqslant \varepsilon/4\right) \\ &\leqslant \sum_{n=1}^{\infty} 2\exp\left\{-\frac{\varepsilon^2/16}{2(2Cn^{-p}\log^{-\delta} n + Cn^{-p/2}\log^{-\delta} n\varepsilon)}\right\} \\ &\leqslant C\sum_{n=1}^{\infty} n^{-2} < \infty, \end{aligned}$$

which implies (4.22).

For J_2, we have by Markov's inequality, Lemmas 3.2, 3.5, $2^m = O(n^{1-p}\log^{-\delta} n)$ and $\mathrm{E}|X|^{2/p+1} < \infty$ that

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} \int_{A_{ni}} E_m(t,s)ds(\varepsilon''_i - \mathrm{E}\varepsilon''_i)\right| \geqslant \varepsilon/4\right) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \left|\int_{A_{ni}} E_m(t,s)ds\right| \mathrm{E}|\varepsilon_i| I(|\varepsilon_i| > n^{p/2}) \\ &\leqslant C\sum_{n=1}^{\infty} \mathrm{E}|X| I(|X| > n^{p/2}) \\ &= C\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mathrm{E}|X| I(k^{p/2} < |X| \leqslant (k+1)^{p/2}) \\ &= C\sum_{k=1}^{\infty} \mathrm{E}|X| I(k^{p/2} < |X| \leqslant (k+1)^{p/2}) \sum_{n=1}^{k} 1 \\ &= C\sum_{k=1}^{\infty} k\mathrm{E}|X| I(k^{p/2} < |X| \leqslant (k+1)^{p/2}) \\ &\leqslant C\sum_{k=1}^{\infty} \mathrm{E}|X|^{2/p+1} I(k^{p/2} < |X| \leqslant (k+1)^{p/2}) \\ &\leqslant C\mathrm{E}|X|^{2/p+1} < \infty, \end{aligned}$$

which yields (4.23). This completes the proof of the theorem. □

5. Numerical simulation

In this subsection, we will study the finite sample performance of the wavelet estimator $g_n(t)$. The data are generate from model (1.1). For any fixed $n \geqslant 3$, let normal random vector $(\varepsilon_{n1}, \varepsilon_{n2}, \cdots, \varepsilon_{nn}) \sim N_n(\mathbf{0}, \Lambda)$, where $\mathbf{0}$ represents zero vector and

$$\Lambda = \begin{pmatrix} 1.04 & -0.2 & 0 & \cdots & 0 & 0 & 0 \\ -0.2 & 1.04 & -0.2 & \cdots & 0 & 0 & 0 \\ 0 & -0.2 & 1.04 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1.04 & -0.2 & 0 \\ 0 & 0 & 0 & \cdots & -0.2 & 1.04 & -0.2 \\ 0 & 0 & 0 & \cdots & 0 & -0.2 & 1.04 \end{pmatrix}_{n\times n}.$$

It follows from Joag-Dev and Proschan (1983) that $(\varepsilon_{n1}, \varepsilon_{n2}, \cdots, \varepsilon_{nn})$ is a NA vector for each $n \geqslant 3$ with finite moment of any order, and thus a NSD vector. For simplicity, choose the scale function $\varphi(t) = I(0 \leqslant t \leqslant 1)$, $2^m = n^{1/3}$, $t_i = (i-0.5)/n$ and $s_i = i/n$ for $i = 1, 2, \ldots, n$. Taking the points $t = 0.25, 0.5, 0.75$ and the sample sizes n as $n = 100, 200, 400, 800, 1200$, we compute $g_n(t) - g(t)$ with $g(t) = \sin 2t$ for 500 times and obtain the boxplots of $g_n(t) - g(t)$ in Figure 1. It shows in the figures that the differences converge to zero and the fluctuation ranges become smaller as n increases. That is to say, the wavelet estimator $g_n(t)$ converges to the true function $g(t)$ as the

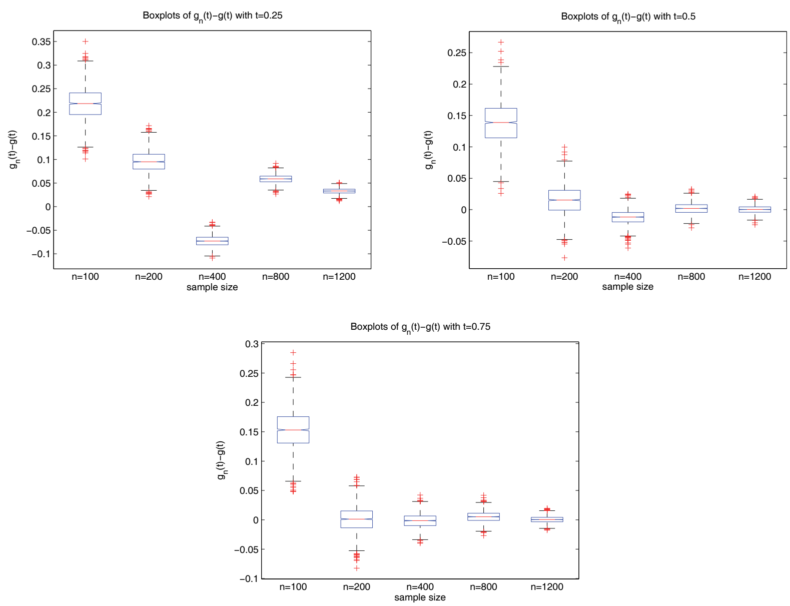

Figure 1: *The Boxplots of* $g_n(t) - g(t)$ *with* $g(t) = \sin 2t$ *by 500 times.*

sample n increases. These simulation results show a good fit of the theoretical results established in Section 2.

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REFERENCES

[1] A. ANTONIADIS, G. GREGOIRE, I. W. MCKEAGUE, *Wavelet methods for curve estimation*, J. Am. Stat. Assoc. **89** (428): 1340–1353, (1994).

[2] T. C. CHRISTOFIDES, E. VAGGELATOU, *A connection between supermodular ordering and positive/negative association*, J. Multivar. Anal. **88**: 138–151, (2004).

[3] L. W. DING, *Almost sure and complete consistency of the estimator in nonparametric regression model for negatively orthant dependent random variables*, Bull. Korean Math. Soc. **57** (1): 51–68, (2020).

[4] L. W. DING, P. CHEN, Y. M. LI, *Statistical inference for a heteroscedastic regression model with φ-mixing errors*, Comm. Statist. Simul. Comput. **51** (8): 4658–4676, (2022).

[5] N. EGHBAL, M. AMINI, A. BOZORGNIA, *Some maximal inequalities for quadratic forms of negative superadditive dependence random variables*, Statist. Probab. Lett. **80**: 587–591, (2010).

[6] A. A. GEORGIEV, *Consistent nonparametric multiple regression: the fixed design case*, J. Multivar. Anal. **25** (1): 100–110, (1988).

[7] P. L. HSU, H. ROBBINS, *Complete convergence and the law of large numbers*, Proc. Natl. Acad. Sci. USA. **33** (2): 25–31, (1947).

[8] T. Z. HU, *Negatively superadditive dependence of random variables with applications*, Chin. J. Appl. Probab. Stat. **16**: 133–144, (2000).

[9] K. JOAG-DEV, F. PROSCHAN, *Negative association of random variables with applications*, Ann. Statist. **11** (1): 286–295, (1983).

[10] J. H. B. KEMPERMAN, *On the FKG-inequalities for measures on a partially ordered space*, Nederl Akad Wetensch Proc. Ser. A **80**: 313–331, (1977).

[11] Y. M. LI, C. D. WEI, G. D. XING, *Berry-Esseen bounds for wavelet estimator in a regression model with linear process errors*, Statist. Probab. Lett. **81**: 103–110, (2011).

[12] H. Y. LIANG, B. Y. JING, *Asymptotic properties for estimates of nonparametric regression models based on negatively associated sequences*, J. Multivar. Anal. **95**: 227–245, (2005).

[13] M. B. PRIESTLEY, M. T. CHAO, *Nonparametric function fitting*, J. R. Stat. Soc. Ser. B **34** (3): 385–392, (1972).

[14] G. G. ROUSSAS, L. T. TRAN, *Fixed design regression for time series: asymptotic normality*, J. Multivar. Anal. **40**: 262–291, (1992).

[15] A. T. SHEN, Y. ZHANG, A. VOLODIN, *Applications of the Rosenthal-type inequality for negatively superadditive dependent random variables*, Metrika **78**: 295–311, (2015).

[16] A. T. SHEN, X. LI, Y. J. ZHANG, Y. G. QIU, X. J. WANG, *Consistency and asymptotic normality of wavelet estimator in a nonparametric regression model*, Stochastics **93** (6): 868–885, (2021).

[17] Y. SHEN, *Strong stability and probability inequlities for some dependent sequences*, Ph.D. Dissertation, Anhui University, (2012).

[18] Y. SHEN, X. J. WANG, W. Z. YANG, S. H. HU, *Almost sure convergence theorem and strong stability for weighted sums of NSD random variables*, Acta Math. Sin. Engl. Ser. **29** (4): 743–756, (2013).

[19] C. J. STONE, *Consistent nonparametric regression*, Ann. Statist. **5**: 595–620, (1977).

[20] G. G. WALTER, *Wavelets and Orthogonal Systems with Applications*, CRC Press Inc., Florida, (1994).

[21] X. J. WANG, X. DENG, L. L. ZHENG, S. H. HU, *Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications*, Statistics **48** (4): 834–850, (2014).

[22] X. J. WANG, A. T. SHEN, Z. Y. CHEN, S. H. HU, *Complete convergence for weighted sums of NSD random variables and its application in the EV regression model*, TEST **24**: 166–184, (2015).

[23] X. J. WANG, Y. WU, R. WANG, S. H. HU, *On consistency of wavelet estimator in nonparametric regression models*, Stat. Pap. **62** (2): 935–962, (2021).

[24] X. H. WANG, S. H. HU, *On the strong consistency of M-estimates in linear models for negatively superadditive dependent errors*, Aust. NZ J. Stat. **57** (2): 259–274, (2015).
[25] Q. Y. WU, *Probability Limit Theory for Mixed Sequence*, Science Press of China, Beijing, (2006).
[26] Z. XUE, L. L. ZHANG, Y. J. LEI, Z. G. CHEN, *Complete moment convergence for weighted sums of negatively superadditive dependent random variables*, J. Inequal. Appl. **2015**: 117, (2015).
[27] W. Z. YANG, H. Y. XU, L. CHEN., S. H. HU, *Complete consistency of estimators for regression models based on extended negatively dependent errors*, Stat. Pap. **59** (2): 449–465, (2018).
[28] L. L. ZHENG, Q. WANG, X. J. WANG, *Strong convergence for weighted sums of negatively superadditive dependent random variables*, J. Univ. Sci. Tech. China. **45** (2): 101–106, (2015).
[29] X. C. ZHOU, J. G. LIN, *Wavelet estimator in nonparametric regression model with dependent error's structure*, Commun. Stat. Theory Methods **43** (22): 4707–4722, (2014).

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