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L^p BOUNDEDNESS FOR MAXIMAL SINGULAR INTEGRALS WITH MIXED HOMOGENEITY ALONG COMPOUNDS CURVES

SHAOYONG HE* AND YAN XU

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Abstract. In this paper, we study the maximal truncated singular integral operators with rough kernel along certain compound curves, which contain many classical model examples. We prove the L^p boundedness of such maximal singular integral operators under very weak conditions on the integral kernels both on the unit sphere and the radial direction. The main results essentially improve and extend certain previous results.

1. Introduction and main results

Let $\mathbb{R}^n$, $n \geqslant 2$, be the n-dimension Euclidean space and let S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Now we give a function $F(x,t) := \sum_{i=1}^{n} x_i^2 t^{\alpha_i}$ where $(x,t) \in \mathbb{R}^n \times (0,\infty)$, and $\alpha_i \geqslant 1$ for $j = 1, \cdots, n$. It is clear that for each fixed $x \in \mathbb{R}^n$, the function $F(x,t)$ is a decreasing function in $t > 0$. Moreover, there is a unique function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $F(x,\rho(x)) = 1$. Fabes and Rivière [10] showed that $(\mathbb{R}^n, \rho)$ is a metric space which is often called the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^n$. For $\lambda > 0$, let A_λ be the diagonal $n \times n$ matrix, that is $A_\lambda = \mathrm{diag}\{\lambda^{\alpha_1}, \cdots, \lambda^{\alpha_n}\}$. For a function φ: $\mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = (0,\infty)$, we shall let $A_\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$A_\varphi(y) = A_{\varphi(\rho(y))} y'$$

where $y' = A_{\rho(y)^{-1}} y \in S^{n-1}$.

Note that the change of variables related to the spaces $(\mathbb{R}^n, \rho)$ is given by the transformation

$$\begin{aligned}
x_1 &= \rho^{\alpha_1} \cos\theta_1 \cdots \cos\theta_{n-2} \cos\theta_{n-1}, \\
x_2 &= \rho^{\alpha_2} \cos\theta_1 \cdots \cos\theta_{n-2} \sin\theta_{n-1}, \\
\cdots &\ \cdots\cdots\cdots\cdots, \\
x_{n-1} &= \rho^{\alpha_{n-1}} \cos\theta_1 \sin\theta_2, \\
x_n &= \rho^{\alpha_n} \sin\theta_1.
\end{aligned}$$

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* Corresponding author.

Therefore, it is easy to check that

$$dx = \rho^{\alpha-1} J(x') d\rho d\sigma(x'),$$

where $\rho^{\alpha-1}J(x')$ is the Jacobian of the above transform and $\alpha = \sum_{j=1}^{n} \alpha_j$. Furthermore, we can check that $J(x') \in C^{\infty}(\mathbb{S}^{n-1})$ and there exists $M > 0$ such that

$$1 \leqslant J(x') \leqslant M, \ \ x' \in S^{n-1}.$$

We would like to mention that if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then $\rho(x) = |x|$.

Let Ω be a real valued and measurable function on $\mathbb{R}^n$ with $\Omega \in L^1(S^{n-1})$ and satisfy

$$\Omega(A_\lambda x) = \Omega(x), \forall \lambda > 0, \text{ and } \int_{S^{n-1}} \Omega(y')J(y')d\sigma(y') = 0. \tag{1.1}$$

For a suitable function h on $(0,\infty)$, we define the parabolic singular integral operator T_h by

$$T_h(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha}} h(\rho(y)) f(x-y) dy. \tag{1.2}$$

For $h(t) \equiv 1$, we denote $T_h = T$. For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, the operator T reduced to the classical singular integral operator, which has been received an increasing interest in recent years, see e.g., [5, 6, 9, 11, 12, 15, 16, 18, 20] et al. For $\alpha_i \geqslant 1$, $i = 1, \cdots, n$, Fabes and Rivière [10] first initiated the singular integrals with mixed homogeneity and established the L^p boundedness of these singular integral operators for $1 < p < \infty$ when $\Omega \in C^1(S^{n-1})$. In 1976, Nagel, Rivière and Wainger [19] improved the result of [10] to the case $\Omega \in L\log^+ L(S^{n-1})$. Inspired by the ideas in [12], Chen, Ding and Fan [7] extended further the condition to the case $\Omega \in H^1(S^{n-1})$. To this end, Chen, Wang and Yu proved that T is bounded on $L^p(\mathbb{R}^n)$ for $\frac{2\beta}{2\beta-1} < p < 2\beta$ provided that $\Omega \in \mathcal{F}_\beta(S^{n-1})$ for some $\beta > 1$, where

$$\mathcal{F}_\beta(S^{n-1}) := \left\{ \Omega \in L^1(S^{n-1}) : \sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\langle y', \xi' \rangle|} \right)^{\beta} d\sigma(y') < +\infty \right\}.$$

We would like to remark that the function class $\mathcal{F}_\beta(S^{n-1})$ was originally introduced in Walsh's paper [22] and developed by Grafakos and Stefanov [15].

For the general operator T_h, in the Euclidean setting, that is $\alpha_1 = \cdots = \alpha_n = 1$, Fefferman first studied the singular integral operator T_h and established the L^p boundedness of T_h for $1 < p < \infty$, provided that $\Omega \in \mathrm{Lip}(S^{n-1})$ and $h \in L^\infty(\mathbb{R}^+)$. Subsequently, many works on the L^p boundedness for $1 < p < \infty$ for T_h were obtained [9, 12, 13, 20]. For example, Duoandikoetxea and Francia [9] showed that T_h is of type (p,p) for $1 < p < \infty$, provided that $\Omega \in L^q(S^{n-1})$ and $h \in \Delta_2(\mathbb{R}^+)$, where $\Delta_\gamma(\mathbb{R}^+)$ denotes the set of all measurable functions h defined on $\mathbb{R}^+$ satisfying the condition

$$\|h\|_{\Delta_\gamma} := \sup_{R>0} \left(R^{-1} \int_0^R |h(r)|^\gamma dr \right)^{1/\gamma} < \infty.$$

It is easy to see that $L^\infty = \Delta_\infty \subsetneq \Delta_{\gamma_2} \subsetneq \Delta_{\gamma_1}$ for $0 < \gamma_1 < \gamma_2 < \infty$. Fan and Pan [12] extended the result of [9] to the singular integrals along polynomial mappings provided that $\Omega \in H^1(S^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ with $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, where γ' denotes its dual exponent. In 2009, Fan and Sato[13] obtained that T_h is bounded on $L^p(\mathbb{R}^n)$ for some $\beta > \max\{\gamma', 2\}$ with $|1/p - 1/2| < \min\{1/\gamma', 1/2\} - 1/\beta$, provided that $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ and Ω satisfies the following size condition:

$$\sup_{\xi' \in S^{n-1}} \iint_{S^{n-1}\times S^{n-1}} |\Omega(x')\,\Omega(y')| \left(\log \frac{1}{|\langle x' - y', \xi'\rangle|}\right)^\beta d\sigma(x')d\sigma(y') < +\infty. \tag{1.3}$$

For the sake of simplicity, we denote

$$\mathcal{WF}_\beta(S^{n-1}) := \{\Omega \in L^1(S^{n-1}) : \Omega \text{ satisfies } (1.3)\}.$$

It should be pointed out that the condition (1.3) was originally introduced by Fan and Sato in more general form in [13]. Moreover, $\mathcal{F}_\beta(S^{n-1}) \subset W\mathcal{F}_\beta(S^{n-1})$ was proved for $n = 2$.

Very recently, Liu and Wu [18] considered a family of operators which is broader than T_h. To be more precise, let P_N be a non-negative polynomial on $\mathbb{R}$ with $P_N(0) = 0$, where N is the degree of P_N. For suitable mappings h and ϕ defined on $\mathbb{R}^+$, they studied the parabolic singular integral operators $T_{h,\Omega,P,\phi}$ along the compound curves $\{A_{P_N(\phi)}(u) : u \in \mathbb{R}^n\}$ defined by

$$T_{h,\Omega,P,\phi}(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)h(\rho(y))}{\rho(y)^\alpha} h(\rho(y)) f(x - A_{P_N(\phi)}(y))dy. \tag{1.4}$$

Clearly, T_h is the special case of $T_{h,\Omega,P,\phi}$ for $P_N(t) = \phi(t) = t$. In particular, for $\alpha_1 = \cdots = \alpha_n = 1$, $A_{P_N(\phi)}(u) = P_N(\phi(|u|))u'$. Moreover, they established the following result.

THEOREM A. ([18]) *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N satisfying $P_N(0) = 0$ with $P_N(t) > 0$ for $t \neq 0$, and let $\phi \in \Im$, where $\Im$ is the set of functions ϕ satisfying the following conditions:*

1. *ϕ is continuous strictly increasing function on $(0, \infty)$ and $\phi \in \mathcal{C}^1$;*
2. *there exists C_ϕ, $c_\phi > 0$ such that $t\phi'(t) \geqslant C_\phi \phi(t)$ and $\phi(2t) \leqslant c_\phi \phi(t)$ for all $t > 0$.*

Suppose that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in \mathcal{WF}_\beta(S^{n-1})$ for $\beta > \max\{\gamma', 2\}$ and satisfies (1.1). Then $T_{h,\Omega,P,\phi}$ defined as in (1.4) is bound on $L^p(\mathbb{R}^n)$ for p satisfying in $|1/p - 1/2| < 1/\max\{\gamma', 2\} - 1/\beta$.

As is well-known, the maximal truncated singular integrals is concerned to the existence a.e. of pointwise limit. Our main purpose of this paper is to resolve this issue by establishing the L^p boundedness of $T^*_{h,\Omega,P,\phi}$, where

$$T^*_{h,\Omega,P,\phi}(f)(x) = \sup_{\varepsilon>0} \Big| \int_{\rho(y)>\varepsilon} \frac{\Omega(y')}{\rho(y)^\alpha} h(\rho(y)) f(x - P_N(\phi(y))y')dy \Big|. \tag{1.5}$$

For $\varepsilon > 0$, denote by

$$T^{\varepsilon}_{h,\Omega,P,\phi}(f)(x) = \int_{\rho(y)>\varepsilon} \frac{\Omega(y')}{\rho(y)^{\alpha}} h(\rho(y)) f(x - P_N(\phi(y))y')dy$$

the truncated singular integral operator associated with $T_{h,\Omega,P,\phi}$. Such operator in the Euclidean setting, have been studied for example in R. Fefferman [14], Duoandikoetxea and Francia [9], Fan and Pan [12], where the L^p boundedness for $1 < p < \infty$ under different assumptions of Ω. It should be pointed that the authors in [1, 4] studied the boundedness of maximal truncated singular integral operators defined by polynomial mappings and rough kernels on product spaces.

We have the following results concerning $T^{*}_{h,\Omega,P,\phi}$.

THEOREM 1.1. *Let $T^{*}_{h,\Omega,P,\phi}$ be given by (1.5). Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N with $P_N(0) = 0$ and $P_N(t) > 0$ for $t \neq 0$, and let $\phi \in \Im$. Assume that $h \in \Delta_\gamma$ for $\gamma \geqslant 2$ and $\Omega \in \mathcal{WF}_\beta(S^{n-1})$ for $\beta > 3$ and satisfies (1.1). Then for $\frac{\gamma'(\beta-1)}{\gamma'+\beta-3} < p < \beta - 1$ there exists a constant $C > 0$ such that*

$$\|T^{*}_{h,\Omega,P,\phi}(f)\|_{L^p} \leqslant C\|f\|_{L^p}.$$

The bound C may depend on N, ϕ, γ and β, but it is independent of the coefficients of P_N .

THEOREM 1.2. *Let P_N, ϕ be as in Theorem 1.1. Suppose that $h \in \Delta_\gamma$ for $1 < \gamma < 2$ and $\Omega \in \mathcal{WF}_\beta(S^{n-1})$ for $\beta > 3\gamma'/2$ satisfying (1.1). Then $T^{*}_{h,\Omega,P,\phi}$ defined as in (1.5) is bound on $L^p(\mathbb{R}^n)$ for p satisfying $\frac{2\beta^2+\beta\gamma-\gamma^2}{2\beta^2-2\beta\gamma+\gamma^2} < p < 2(\frac{\beta}{\gamma} - \frac{1}{2}))$. The bound is independent of coefficients of P_N ,but may depend on N, ϕ, γ, β.*

REMARK 1.1. It should be pointed out the introduce of the compound curves $P_N(\phi(t))$ originates from Al-Salman's works [1, 2]. In the current paper, our theorems show that the L^p-boundedness of the maximal singular integral operator, whose kernel has the additional roughness in the radial direction due to the presence of h, depends on the index γ, which characterize the roughness of h. Moreover, that our results are new even for the case $\alpha_1 = \cdots \alpha_n = 1$, the Euclidean setting.

REMARK 1.2. For any $\phi \in \Im$, it is easy to check that there exists a constant B_ϕ such that $\phi(2r) \geqslant B_\phi \phi(r)$ for all $r > 0$ (see [2, 3, 17, 21]). We note that model examples for functions $\phi \in \Im$ are $t^{\sigma}(\sigma > 0)$, $t\ln(1+t)$, $t\ln\ln(e+t)$ and real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ (see [2]).

The paper is organized as follows. In Section 2 we will introduce some notations and give some technical lemmas. The proofs of our main results will be given in Section 3. Before proving the theorem, we want to say a few words. Although we can follow the ideas from [9, 11, 12, 15, 16, 18], for instance, the Littlewood-Paley theory without any modifications. We understand that the underlying space in this note is a special homogeneous group. Moreover, we find that the methods and techniques are not an easy process of copy and paste, which may be more complex, such as the proof of Lemmas 2.4 and 2.5.

Throughout this paper, the letter C, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. In what follows, we let p' denote the conjugate index of p which satisfies $\frac{1}{p}+\frac{1}{p'}=1$ for $p\geqslant 1$. For a measure σ, we denote by $|\sigma|$ the total variation of σ.

2. Some notations and preliminary lemmas

In this section, we will establish some necessary notations and lemmas. We first introduce some relevant notations and definitions. For given positive polynomial $P_N(t)=\sum_{i=1}^{N}\beta_i t^i$, we let

$$(P_N(t))^{\alpha_k}=\sum_{i=1}^{N\alpha_k}a_{i,k}t^i \ \text{ for } k\in\{1,2,\cdots,n\}$$

for $x,\xi\in\mathbb{R}^n$,

$$A_{P_N(\phi)}(x)\cdot\xi=\sum_{k=1}^{n}(P_N(\phi(p(x))))^{\alpha_k}x'_k\cdot\xi_k=\sum_{k=1}^{n}\sum_{i=1}^{N\alpha_k}a_{i,k}\phi(p(x))^i x'_k\cdot\xi_k$$

where $\phi\in\mathfrak{I}$. We denote $\mathcal{N}=\max\{N\alpha_k:1\leqslant k\leqslant n\}$ and $\alpha_{i,k}=0$ whenever $i>N\alpha_k$. Hence, we can write

$$A_{P_N(\phi)}(x)\cdot\xi=\sum_{k=1}^{n}\sum_{i=1}^{N\alpha_k}a_{i,k}\phi(p(x))^i x'_k\cdot\xi_k=\sum_{i=1}^{N}\big(L_i(\xi)x'\big)\phi(p(x))^i,$$

where $L_i(\xi)=(a_{i,1}\xi_1,a_{i,2}\xi_2,\cdots,a_{i,n}\xi_n)$. For $\nu\in\{0,1,\cdots,\mathcal{N}\}$, we set

$$Q_\nu(x)=\Big(\sum_{i=1}^{\nu}a_{i,1}\phi(p(x))^i x'_1,\sum_{i=1}^{\nu}a_{i,2}\phi(p(x))^i x'_2,\cdots,\sum_{i=1}^{\nu}a_{i,n}\phi(p(x))^i x'_n\Big).$$

Therefore,

$$Q_\nu(x)\cdot\xi=\sum_{i=1}^{\nu}\big(L_i(\xi)x'\big)\phi(p(x))^i.$$

Let $E_j=\{\xi\in\mathbb{R}^n:2^j<\rho(\xi)\leqslant 2^{j+1}\}$. For any $\nu\in\{1,2,\cdots,\mathcal{N}\}$, we define the measures $\{\sigma_r^\nu\}$ and $\{|\sigma_r^\nu|\}$ as follows,

$$\widehat{\sigma_j^\nu}(\xi)=\int_{E_j}\frac{\Omega(u)}{\rho(u)^\alpha}h(\rho(u))e^{-i\xi\cdot Q_\nu(u)}du,$$

$$\widehat{\left|\sigma_j^\nu\right|}(\xi)=\int_{E_j}\frac{|\Omega(u)h(\rho(u))|}{\rho(u)^\alpha}e^{-i\xi\cdot Q_\nu(u)}du. \tag{2.1}$$

It is easy to check that $\widehat{\sigma_j^0}(\xi)=0$ and

$$T_{h,\Omega,P,\phi}(f)(x)=\sum_{j\in\mathbb{Z}}\sigma_j^{\mathcal{N}}*f(x).$$

Now we introduce some lemmas, which will play key roles in the proofs of our main theorems.

LEMMA 2.1. [18] *Let $h\in\Delta_\gamma$ for $1<\gamma\leqslant\infty$. Suppose that $\Omega\in\mathcal{WF}_\beta(S^{n-1})$ for some $\beta>0$ and satisfies (1.1). Then for $j\in\mathbb{Z}$, $\nu\in\{1,2,\ldots,\mathcal{N}\}$ and $\xi\in\mathbb{R}^n$, there exists a $C>0$ such that*

1. $\sup_{j\in\mathbb{Z}}|\widehat{\sigma_j^\nu}(\xi)|\leqslant C$;
2. $|\widehat{\sigma_j^\nu}(\xi)-\widehat{\sigma_j^{\nu-1}}(\xi)|\leqslant C|\phi(2^{j+1})^\nu L_\nu(\xi)|$;
3. *for* $\left|\phi(2^{j+1})^\nu L_\nu(\xi)\right|\geqslant 1$,

$$|\widehat{\sigma_j^\nu}(\xi)|\leqslant C\left(\log\left|(\phi(2^{j+1}))^\nu L_\nu(\xi)\right|\right)^{-\beta/\gamma'}\quad \textit{for}\quad 1<\gamma\leqslant 2,$$
$$|\widehat{\sigma_j^\nu}(\xi)|\leqslant C\left(\log\left|\phi(2^{j+1})^\mu L_\nu(\xi)\right|\right)^{-\beta/2}\quad \textit{for}\quad \gamma>2.$$

Let $|\sigma_j^\nu|$ be defined as in (2.1), we define the maximal function by

$$\sigma_\nu^*f(x)=\sup_{\mathrm{j}\in\mathbb{Z}}\left|\left|\sigma_j^\nu\right|*f(x)\right|. \tag{2.2}$$

LEMMA 2.2. [18] *Let $h\in\Delta_\gamma(\mathbb{R}^+)$ for some $\gamma>1$, and let $\Omega\in L^1(S^{n-1})$. Then the maximal function defined in (2.2) is bounded on $L^p(\mathbb{R}^n)$ for $p>\gamma'$.*

Clearly, Lemma 2.2 reveals that for $1<\gamma<2$, the range of p is shrunk to $p>\gamma'>2$. It is natural to enlarge its range. We first recall the following lemma.

LEMMA 2.3. *If $\|\sigma_\nu^*f\|_{L^s}\leqslant C\|f\|_{L^s}$ and $\frac{1}{2s}=\left|\frac12-\frac1q\right|$, then for arbitrary function sequence $\{g_j\}$ we have*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\left|\sigma_j^\nu*g_j\right|^2\right)^{\frac12}\right\|_{L^q}\leqslant C\left\|\left(\sum_{j\in\mathbb{Z}}\left|g_j\right|^2\right)^{\frac12}\right\|_{L^q}.$$

LEMMA 2.4. *Let $h\in\Delta_\gamma(\mathbb{R}^+)$ for $1<\gamma<2$. Suppose that $\Omega\in\mathcal{WF}_\beta(S^{n-1})$ for $\beta>3\gamma'/2$. Then for $\nu\in\{0,1,\cdots,\mathcal{N}\}$, the operator σ_ν^* satisfies*

$$\|\sigma_\nu^*f\|_{L^p}\leqslant C\|f\|_{L^p} \tag{2.3}$$

for $\frac{\beta+\gamma'}{\beta}<p<\frac{\beta+\gamma'}{\gamma'}$.

Proof. We prove this lemma by induction on ν.

Case 1. It is easy to check that $\sigma_0^*(f)(x) \leqslant C|f(x)|$, which yields the estimate (2.3) for $\nu = 0$.

Case 2. Let $m \in \{0,1,\cdots,\mathcal{N}\}$ and suppose that (2.3) holds for $\nu = m-1$. We will prove (2.3) for $\nu = m$. Let $\psi \in S(\mathbb{R}^n)$ such that $\psi(t) \equiv 1$ for $|t| \leqslant 1$ and $\psi(t) \equiv 0$ for $|t| > 2$. Define the measures $\{\tau_j^\nu\}$ by

$$\widehat{\tau_j^\nu}(\xi) = \widehat{\sigma_j^\nu}(\xi) - \psi(|\phi(2^{j+1})^\nu L_\nu(\xi)|)\widehat{\sigma_j^{\nu-1}}(\xi)$$

for $\nu \in \{1,\cdots,\mathcal{N}\}$. By Lemma 2.1, we obtain that

$$\left|\widehat{\tau_j^\nu}(\xi)\right| \leqslant C \min\{1, \left(\log\left|(\phi(2^{j+1}))^\nu L_\nu(\xi)\right|\right)^{-\beta/\gamma'}\}. \tag{2.4}$$

Obviously, we have

$$\begin{aligned}\sigma_m^* f(x) &\leqslant \sup_{j\in\mathbb{Z}}\left|\tau_j^m * |f|(x)\right| + C\mathcal{M}\left(\sigma_{m-1}^* f\right)(x)\\ &\leqslant \Big(\sum_{j\in\mathbb{Z}} |\tau_j^m * f(x)|^2\Big)^{\frac{1}{2}} + C\mathcal{M}\left(\sigma_{m-1}^* f\right)(x),\end{aligned}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. It follows from our assumption and the L^p mapping properties of $\mathcal{M}$ that

$$\|\mathcal{M}\left(\sigma_{m-1}^* f\right)(x)\|_{L^p} \leqslant C\|f\|_{L^p}$$

for $\frac{\beta+\gamma'}{\beta} < p < \frac{\beta+\gamma'}{\gamma'}$. Thus, it suffices to prove

$$\|\Big(\sum_{j\in\mathbb{Z}} |\tau_j^m * f(x)|^2\Big)^{\frac{1}{2}}\|_{L^p} \leqslant C\|f\|_{L^p} \tag{2.5}$$

for $\frac{\beta+\gamma'}{\beta} < p < \frac{\beta+\gamma'}{\gamma'}$. By the well-known Rademacher's function, (2.5) follows from the following lemma. □

LEMMA 2.5. *Let* $V_\varepsilon^m(f)(x) = \sum_{j\in\mathbb{Z}} \varepsilon_j \left(\tau_j^m * f(x)\right)$ *with* $\varepsilon = \{\varepsilon_k\}$, $\varepsilon_j = 1$ *or* -1. *Then*

$$\|V_\varepsilon^m f\|_{L^p} \leqslant C\|f\|_{L^p}$$

for $\frac{\beta+\gamma'}{\beta} < p < \frac{\beta+\gamma'}{\gamma'}$.

Proof. For any $\nu \in \{1,2,\cdots,\mathcal{N}\}$, we choose a sequence of nonnegative functions $\{\Psi_k\}_{k\in\mathbb{Z}}$ in $C_0^\infty(\mathbb{R}^+)$ such that

1. $\mathrm{supp}(\Psi_k) \subset \left[\phi(2^{k+1})^{-\nu}, \phi(2^{k-1})^{-\nu}\right]$;

2. $0 \leqslant \Psi_k \leqslant 1$, $\sum_{k \in Z} \Psi_k(t)^2 = 1$, $\left|(d/dt)^j \Psi_k\right| \leqslant C|t|^{-j}$, for all $t > 0$ and $j = 1, 2, \cdots$.

Define the operator S_k by

$$\widehat{S_k(f)}(\xi) = \Psi_k\left(|L_\nu(\xi)|\right) \widehat{f}(\xi).$$

Then

$$\begin{aligned} V_\varepsilon^m(f)(x) &= \sum_{j \in \mathbb{Z}} \varepsilon_j \left(\tau_j^m * f(\xi)\right) \\ &= \sum_{j \in \mathbb{Z}} \varepsilon_j \left(\tau_j^m * \sum_{k \in \mathbb{Z}} S_{j+k} S_{j+k} f(x)\right) \\ &= \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \varepsilon_j S_{j+k} \left(\tau_j^m * S_{j+k} f\right)(x) \\ &= \sum_{k \in \mathbb{Z}} V_k^m(f)(x). \end{aligned} \tag{2.6}$$

By the Littlewood-Paley theory and Plancherel's theorem, we have

$$\begin{aligned} \|V_k^m(f)\|_{L^2} &\leqslant C \Big\| \Big(\sum_{j \in \mathbb{Z}} |\tau_j^m * S_{j+k} f|^2\Big)^{\frac{1}{2}} \Big\|_{L^2} \\ &\leqslant C \sum_{j \in \mathbb{Z}} \int_{\Lambda_{j+k}} |\widehat{\tau_j^m}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi, \end{aligned}$$

where

$$\Lambda_{j+k} = \{\xi \in \mathbb{R}^n : \phi(2^{j+k+1})^{-\nu} \leqslant |L_\nu(\xi)| \leqslant \phi(2^{j+k-1})^{-\nu}\}.$$

By (2.4), we get

$$\|V_k^m(f)\|_{L^2} \leqslant C B_k \|f\|_{L^2}, \tag{2.7}$$

where

$$B_k = \begin{cases} |k|^{-\beta/\gamma'}, & k \leqslant -2 \\ 1, & -2 < k \leqslant 2 \\ B_\phi^{(2-k)\nu}, & k > 2. \end{cases}$$

This together with (2.6) yields

$$\|V_\varepsilon^m(f)\|_{L^2} \leqslant C \|f\|_{L^2}.$$

Hence,

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |\tau_j^m * f(x)|^2\Big)^{\frac{1}{2}} \Big\|_{L^2} \leqslant C \|f\|_{L^2}.$$

Applying Lemma 2.3 with $s=2$, and $q=q_0=4$, we obtain that

$$\|V_k^m f\|_{L^{q_0}} \leqslant C\|(\sum_{j\in\mathbb{Z}}|\tau_j^m * S_{j+k}f|^2)^{\frac{1}{2}}\|_{L^{q_0}} \leqslant C\|(\sum_{j\in\mathbb{Z}} S_{j+k}f|^2)^{\frac{1}{2}}\| \leqslant C\|f\|_{L^{q_0}}, \tag{2.8}$$

where the middle inequality is a application of Lemma 2.3 and the first and last inequalities follow from the Littlewood-Paley theorem.

Interpolating between estimates (2.7) and (2.8) we obtain that

$$\|V_k^m f\|_{L^p} \leqslant CB_k^{\theta_p}\|f\|_{L^p},$$

where $1/p=\theta_p/2+(1-\theta_p)/q_0$. Observe that $V_\varepsilon^m=\sum_k V_k^m$ maps $L^p\to L^p$ for all p's for which $p_1'<p<p_1$, where $p_1=\frac{4\beta}{\beta+\gamma'}$. Then

$$\|\big(\sum_{j\in\mathbb{Z}}|\tau_j^m * f(x)|^2\big)^{\frac{1}{2}}\|_{L^{p_1}} \leqslant C\|f\|_{L^{p_1}}, \text{ for } p\in(p_1',p_1).$$

Now continuing this way, together with bootstrapping argument, we can obtain ultimately that

$$\|V_\varepsilon^m f\|_{L^p} \leqslant C\|f\|_{L^p}, \text{ for } \frac{\beta+\gamma'}{\beta}<p<\frac{\beta+\gamma'}{\gamma'}.$$

This completes the proof of Lemma 2.5. $\square$

Now we take a radial Schwartz function $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that $\varphi(t)\equiv 1$ for $|t|\leqslant 1$ and $\varphi(t)\equiv 0$ for $|t|>2$. Define the measure $\{\omega_j^\nu\}$ by

$$\widehat{\omega_j^\nu}(\xi)=\widehat{\sigma_j^\nu}(\xi)\Pi_\nu(\xi)-\widehat{\sigma_j^{\nu-1}}(\xi)\Pi_{\nu-1}(\xi) \tag{2.9}$$

for $\nu\in\{1,2,\cdots,\mathcal{N}\}$, where $\Pi_\nu(\xi)=\Pi_{k=\nu+1}^{\mathcal{N}}\varphi(\phi(2^{j+1})^k L_k(\xi))$. It is easy to see that

$$\sigma_j^{\mathcal{N}}=\sum_{\nu=1}^{\mathcal{N}}\omega_j^\nu. \tag{2.10}$$

LEMMA 2.6. [16] *For $j\in\mathbb{Z}$ and $\nu\in\{1,2,\cdots,\mathcal{N}\}$, there exists a constant $C>0$, which is independent of the coefficients of P_N, such that*

1.

$$|\widehat{\omega_j^\nu}(\xi)| \leqslant C|\phi(2^{j+1})^\nu L_\nu(\xi)|;$$

2. for $\left|\phi(2^{j+1})^\nu L_\nu(\xi)\right|\geqslant 1$,

$$|\widehat{\omega_j^\nu}(\xi)| \leqslant C\left(\log\left|(\phi(2^{j+1}))^\nu L_\nu(\xi)\right|\right)^{-\beta/\gamma'} \quad \textit{for} \quad 1<\gamma\leqslant 2,$$

$$|\widehat{\omega_j^\nu}(\xi)(\xi)| \leqslant C\left(\log\left|\phi(2^{j+1})^\mu L_\nu(\xi)\right|\right)^{-\beta/2} \quad \textit{for} \quad \gamma>2.$$

For any fixed $\nu \in \{1,2,\cdots,\mathcal{N}\}$, the operator ω_ν^* is defined by

$$\omega_\nu^* f(x) = \sup_{j\in Z} \big||\omega_j^\nu| * f(x)\big|$$

Applying Lemmas 2.2 and 2.4, it is easy to establish the following lemma.

LEMMA 2.7. *Let* $\Omega \in \mathcal{WF}_\beta(S^{n-1})$, $h \in \Delta_\gamma(\mathbb{R}^+)$. ω_ν^* *satisfies*

$$\|\omega_\nu^* f\|_{L^p} \leqslant C\|f\|_{L^p}, \quad p > \gamma'.$$

Furthermore, when $1 < \gamma < 2$, *if* $\beta > 3\gamma'/2$, *then we have*

$$\|\omega_\nu^* f\|_{L^p} \leqslant C\|f\|_{L^p}, \quad p \in \left(\frac{\beta+\gamma'}{\beta}, \frac{\beta+\gamma'}{\gamma'}\right).$$

Proof. By the definition of ω_j^ν in (2.9), we have

$$w_j^\nu = \sigma_j^\nu * \varphi_\nu - \sigma_j^{\nu-1} * \varphi_{\nu-1},$$

where $\widehat{\varphi_\nu}(\xi) = \Pi_\nu(\xi)$. As $\varphi_\nu \in \mathcal{S}(\mathbb{R}^n)$, there holds

$$\big||\omega_j^\nu| * f(x)\big| \leqslant C\sigma_\nu^* * \mathcal{M}f(x) + C\sigma_{\nu-1}^* * \mathcal{M}f(x),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. By Lemmas 2.2 and 2.4, Lemma 2.7 is proved. $\square$

3. Proofs of main results

Proof of Theorm 1.1. For any $\varepsilon > 0$, there exists an integer k such that $2^{k-1} \leqslant \varepsilon < 2^k$. Then by (2.10)

$$\begin{aligned}|\mathrm{T}^{\varepsilon}_{h,\Omega,P,\phi} f(x)| &\leqslant \Big|\int_{\varepsilon\leqslant\rho(y)<2^k} \frac{\Omega(y)}{\rho(y)^\alpha} h(\rho(y)) f(x - A_{P_N(\phi)}(y))dy\Big| + \sup_{k\in\mathbb{Z}}\Big|\sum_{j\geqslant k} \sigma_j^\nu * f(x)\Big| \\ &\leqslant \sigma_\nu^*(|f|)(x) + \sum_{\nu=1}^{\mathcal{N}} \sup_{k\in\mathbb{Z}}\Big|\sum_{j\geqslant k} \omega_j^\nu * f(x)\Big|.\end{aligned}$$

By Lemma 2.2, it suffices to obtain that

$$\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{j\geqslant k} w_j^\nu * f\Big|\Big\|_{L^p} \leqslant C\|f\|_{L^p}$$

for $\frac{\gamma'(\beta-1)}{\gamma'+\beta-3} < p < \beta - 1$. Take a radial function $\Phi \in \mathcal{S}(\mathbb{R})$ such that $\Phi(\xi) = 1$ when $|\xi| < 1$ and $\Phi(\xi) = 0$ when $|\xi| > B_\phi$. Let $\widehat{\Phi_k}(\xi) = \Phi(\phi(2^k)^\nu |L_\nu(\xi)|)$. Then

$$\begin{aligned}\sum_{j\geqslant k} \omega_j^\nu * f(x) &= \Phi_k * \sum_{j\in\mathbb{Z}} w_j^\nu * f(x) - \Phi_k * \sum_{j=-\infty}^{k-1} w_j^\nu * f(x) + (\delta - \Phi_k) * \sum_{j\geqslant k} \omega_j^\nu * f(x) \\ &=: J_{1,k}f(x) + J_{2,k}f(x) + J_{3,k}f(x),\end{aligned}$$

where δ is the Dirac delta function. Thus,

$$\sup_{k\in\mathbb{Z}}\left|\sum_{j\geqslant k}\omega_j^\nu * f(x)\right| \leqslant \sup_{k\in\mathbb{Z}}|J_{1,k}f(x)| + \sup_{k\in\mathbb{Z}}|J_{2,k}f(x)| + \sup_{k\in\mathbb{Z}}|J_{3,k}f(x)|.$$

We obtain from Theorem A that

$$\left\|\sup_{k\in\mathbb{Z}}|J_{1,k}f|\right\|_{L^p} \leqslant C\|\mathcal{M}(\sum_{j\in\mathbb{Z}} w_j^\nu * f)\|_{L^p} \leqslant C\|f\|_{L^p} \tag{3.1}$$

for $\frac{\beta}{\beta-1} < p < \beta$. Next, we estimate $\sup_{k\in\mathbb{Z}}|J_{2,k}f|$. It holds that

$$\begin{aligned}\sup_{k\in\mathbb{Z}}|J_{2,k}f(x)| &= \sup_{k\in\mathbb{Z}}\left|\Phi_k * \sum_{j=1}^{+\infty}\omega_{k-j}^\nu * f(x)\right| \\ &\leqslant \sum_{j=1}^{+\infty}\sup_{k\in\mathbb{Z}}\left|\Phi_k * \omega_{k-j}^\nu * f(x)\right| \\ &=: \sum_{j=1}^{+\infty} H_j^\nu f(x).\end{aligned}$$

By Lemma 2.7, we have

$$\left\|H_j^\nu f\right\|_{L^p} \leqslant C\|f\|_{L^p}, \tag{3.2}$$

for each $p > \gamma'$.

On the other hand, one can easily check that

$$H_j^\nu f(x) \leqslant (\sum_{k\in\mathbb{Z}}|\Phi_k * \omega_{k-j}^\nu * f(x)|^2)^{\frac{1}{2}}.$$

Hence, by the Plancherel's theorem, we have

$$\begin{aligned}\left\|H_j^\nu f\right\|_{L^2}^2 &\leqslant C\left\|\left(\sum_{k\in\mathbb{Z}}\left|\Phi_k * \omega_{k-j}^\nu * f\right|^2\right)^{\frac{1}{2}}\right\|_{L^2}^2 \\ &\leqslant C\sum_{k\in\mathbb{Z}}\int_{|(\phi(2^k))^\nu L_\nu(\xi)|<B_\phi}|\widehat{\omega_{k-j}^\nu(\xi)}|^2|\widehat{f(\xi)}|^2 d\xi \\ &\leqslant C\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}|\widehat{\omega_{k-j}^\nu(\xi)}|^2|\widehat{f(\xi)}|^2\chi_{\{|(\phi(2^k))^\nu L_\nu(\xi)|<B_\phi\}}d\xi \\ &\leqslant C\sup_{\xi\in\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\phi(2^{k-j+1})^\nu L_\nu(\xi)|^2\chi_{\{|(\phi(2^k))^\nu L_\nu(\xi)|<B_\phi\}}\|f\|_{L^2}^2 \\ &\leqslant C{B_\phi}^{2(-j+1)}\|f\|_{L^2}^2,\end{aligned}$$

where the last inequality is obtained by the properties of lacunary sequence. So

$$\left\|H_j^\nu f\right\|_{L^2} \leqslant C{B_\phi}^{-j+1}\|f\|_{L^2}. \tag{3.3}$$

By interpolation theorem between (3.2) and (3.3), we find a positive number θ_2 such that

$$\left\|H_j^\nu f\right\|_{L^p} \leqslant C {B_\phi}^{-j\theta_2} \|f\|_{L^p}$$

for any $p > \gamma'$. Thus,

$$\left\|\sup_{k\in\mathbb{Z}} |J_{2,k} f|\right\|_{L^p} \leqslant C \|f\|_{L^p},$$

for $p > \gamma'$. Finally, we estimate $\sup_{k\in\mathbb{Z}} |J_{3,k} f|$. We have

$$\sup_{k\in Z} \left|(\delta - \Phi_k) * \sum_{j\geqslant k} \omega_j^\nu * f(x)\right| \leqslant \sum_{j\geqslant 0} \sup_{k\in\mathbb{Z}} \left|(\delta - \Phi_k) * \omega_{j+k}^\nu * f(x)\right| := \sum_{j\geqslant 0} \Lambda_j(f)(x). \tag{3.4}$$

By Lemma 2.7, we obtain

$$\left\|\Lambda_j(f)\right\|_p \leqslant C \|\omega_\nu^*\|_p \leqslant C\|f\|_p \tag{3.5}$$

for $p > \gamma'$. Also, we have

$$\begin{aligned}
&\|\Lambda_j(f)\|_2^2 \\
&\leqslant C \left\| \left(\sum_{k\in\mathbb{Z}} \left|(\delta - \Phi_k) * \omega_{k+j}^\nu * f\right|^2 \right)^{\frac{1}{2}} \right\|_2^2 \\
&\leqslant C \sum_{k\in\mathbb{Z}} \int_{|L_\nu(\xi)|\geqslant(\phi(2^k))^{-\nu}} \left|\widehat{\omega_{k+j}^\nu}(\xi)\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=0}^{+\infty} \int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}} \left|\widehat{\omega_{k+j}^\nu}(\xi)\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=0}^{+\infty} \int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}} \left(\log\left|(\phi(2^{k+j+1}))^\nu L_\nu(\xi)\right|\right)^{-\beta} \left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=0}^{+\infty} \int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}} \left(\log\left|(\phi(2^{k-i-1}))^{-\nu}(\phi(2^{k+j+1}))^\nu\right|\right)^{-\beta} \\
&\quad \times \left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=0}^{+\infty} \int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}} \left(\log {B_\phi}^{(i+j+2)\nu}\right)^{-\beta} \left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=0}^{+\infty} \left(\frac{1}{i+j+2}\right)^\beta \int_{(\phi(2^{k+i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k+i-1}))^{-\nu}} \left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C j^{1-\beta} \|f\|_2^2.
\end{aligned}$$

Then

$$\left\|\Lambda_j(f)\right\|_2 \leqslant Cj^{\frac{1-\beta}{2}}\|f\|_2. \tag{3.6}$$

It follows from (3.4) and interpolation between (3.5) and (3.6) that

$$\left\|\sup_{k\in\mathbb{Z}}|J_{3,k}f|\right\|_{L^p} \leqslant C\|f\|_{L^p}$$

for any $\beta > 3$, $p \in (\frac{\gamma'(\beta-1)}{\gamma'+\beta-3}, \beta-1)$. This proves Theorem 1.1. □

Proof of Theorem 1.2. Following the same arguments as in Theorem 1.1,

$$\begin{aligned}|\mathrm{T}^{\varepsilon}_{h,\Omega,P,\phi}f(x)| &\leqslant \Big|\int_{\varepsilon\leqslant\rho(y)<2^k}\frac{\Omega(y)}{\rho(y)^{\alpha}}h(\rho(y))f(x-A_{P_N(\phi)}(y))dy\Big| + \sup_{k\in\mathbb{Z}}\Big|\sum_{j\geqslant k}\sigma_j^{\nu}*f(x)\Big| \\ &\leqslant \sigma_\nu^*(|f|)(x) + \sum_{\nu=1}^{\mathcal{N}}\sup_{k\in\mathbb{Z}}\Big|\sum_{j\geqslant k}\omega_j^{\nu}*f(x)\Big|.\end{aligned}$$

By Lemma 2.2, it suffices to obtain that

$$\left\|\sup_{k\in\mathbb{Z}}\Big|\sum_{j\geqslant k}w_j^{\nu}*f\Big|\right\|_{L^p} \leqslant C\|f\|_{L^p}$$

for $\frac{2\beta^2+\beta\gamma-\gamma^2}{2\beta^2-2\beta\gamma+\gamma^2} < p < 2(\frac{\beta}{\gamma'}-\frac{1}{2}))$. Take a radial function $\Phi \in \mathcal{S}(\mathbb{R})$ such that $\Phi(\xi) = 1$ when $|\xi| < 1$ and $\Phi(\xi) = 0$ when $|\xi| > B_\phi$. Let $\widehat{\Phi_k}(\xi) = \Phi(\phi(2^k)^{\nu}|L_\nu(\xi)|)$. As the proof of Theorem 1.1,

$$\begin{aligned}\sum_{j\geqslant k}\omega_j^{\nu}*f(x) &= \Phi_k*\sum_{j\in\mathbb{Z}}w_j^{\nu}*f(x) - \Phi_k*\sum_{j=-\infty}^{k-1}w_j^{\nu}*f(x) + (\delta-\Phi_k)*\sum_{j\geqslant k}\omega_j^{\nu}*f(x) \\ &=: J_{1,k}f(x) + J_{2,k}f(x) + J_{3,k}f(x),\end{aligned}$$

where δ is the Dirac delta function. Then

$$\sup_{k\in\mathbb{Z}}\left|\sum_{j\geqslant k}\omega_j^{\nu}*f(x)\right| \leqslant \sup_{k\in\mathbb{Z}}|J_{1,k}f(x)| + \sup_{k\in\mathbb{Z}}|J_{2,k}f(x)| + \sup_{k\in\mathbb{Z}}|J_{3,k}f(x)|.$$

By Theorem A,

$$\left\|\sup_{k\in\mathbb{Z}}|J_{1,k}f|\right\|_{L^p} \leqslant C\|\mathcal{M}(\sum_{j\in\mathbb{Z}}w_j^{\nu}*f)\|_{L^p} \leqslant C\|f\|_{L^p} \tag{3.7}$$

for $|\frac{1}{p}-\frac{1}{2}|<\frac{1}{\gamma'}-\frac{1}{\beta}$. For the second term, it holds that

$$\begin{aligned}\sup_{k\in\mathbb{Z}}|J_{2,k}f(x)| &= \sup_{k\in\mathbb{Z}}\left|\Phi_k * \sum_{j=1}^{+\infty}\omega_{k-j}^{\nu} * f(x)\right|\\ &\leqslant \sum_{j=1}^{+\infty}\sup_{k\in\mathbb{Z}}\left|\Phi_k * \omega_{k-j}^{\nu} * f(x)\right|\\ &=: \sum_{j=1}^{+\infty} H_j^{\nu} f(x).\end{aligned}$$

From Lemma 2.7, we have

$$\left\|H_j^{\nu} f\right\|_{L^p} \leqslant C\|f\|_{L^p}, \tag{3.8}$$

for $p\in(\frac{\beta+\gamma'}{\beta},\frac{\beta+\gamma'}{\gamma'})$. By interpolation theorem between (3.8) and (3.3), there exists a positive number θ_2' such that

$$\left\|H_j^{\nu} f\right\|_{L^p} \leqslant C B_{\phi}{}^{-j\theta_2'}\|f\|_{L^p}$$

for any $p\in(\frac{\beta+\gamma'}{\beta},\frac{\beta+\gamma'}{\gamma'})$. Hence,

$$\left\|\sup_{k\in\mathbb{Z}}|J_{2,k}f|\right\|_{L^p} \leqslant C\|f\|_{L^p},$$

for $p\in(\frac{\beta+\gamma'}{\beta},\frac{\beta+\gamma'}{\gamma'})$. For the third term $\sup_{k\in\mathbb{Z}}|J_{3,k}f|$, we have

$$\sup_{k\in Z}\left|(\delta-\Phi_k) * \sum_{j\geqslant k}\omega_j^{\nu} * f(x)\right| \leqslant \sum_{j\geqslant 0}\sup_{k\in\mathbb{Z}}\left|(\delta-\Phi_k) * \omega_{j+k}^{\nu} * f(x)\right| := \sum_{j\geqslant 0}\Lambda_j(f)(x). \tag{3.9}$$

By Lemma 2.7, we obtain

$$\left\|\Lambda_j(f)\right\|_p \leqslant C\|\omega_\nu^*\|_p \leqslant C\|f\|_p \tag{3.10}$$

for $p\in(\frac{\beta+\gamma'}{\beta},\frac{\beta+\gamma'}{\gamma'})$. Moreover,

$$\begin{aligned}&\|\Lambda_j(f)\|_2^2\\ &\leqslant C\left\|\left(\sum_{k\in\mathbb{Z}}\left|(\delta-\Phi_k) * \omega_{k+j}^{\nu} * f\right|^2\right)^{\frac{1}{2}}\right\|_2^2\\ &\leqslant C\sum_{k\in\mathbb{Z}}\int_{|L_\nu(\xi)|\geqslant(\phi(2^k))^{-\nu}}\left|\widehat{\omega_{k+j}^{\nu}}(\xi)\hat{f}(\xi)\right|^2 d\xi\end{aligned}$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{+\infty}\int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}}\left|\widehat{\omega_{k+j}^{\nu}}(\xi)\hat{f}(\xi)\right|^2 d\xi$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{+\infty}\int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}}\left(\log\left|(\phi(2^{k+j+1}))^{\nu}L_\nu(\xi)\right|\right)^{-\frac{2\beta}{\gamma'}}\left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{+\infty}\int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}}\left(\log\left|(\phi(2^{k-i-1}))^{-\nu}(\phi(2^{k+j+1}))^{\nu}\right|\right)^{-\frac{2\beta}{\gamma'}}$$

$$\times\left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{+\infty}\int_{(\phi(2^{k-i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k-i-1}))^{-\nu}}\left(\log B_\phi{}^{(i+j+2)\nu}\right)^{-\frac{2\beta}{\gamma'}}\left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{+\infty}\left(\frac{1}{i+j+2}\right)^{\frac{2\beta}{\gamma'}}\int_{(\phi(2^{k+i}))^{-\nu}\leqslant|L_\nu(\xi)|\leqslant(\phi(2^{k+i-1}))^{-\nu}}\left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leqslant Cj^{1-\frac{2\beta}{\gamma'}}\|f\|_2^2.$$

Then

$$\left\|\Lambda_j(f)\right\|_2\leqslant Cj^{\frac{1-\beta/\gamma'}{2}}\|f\|_2. \tag{3.11}$$

It follows from (3.9) and interpolation between (3.10) and (3.11) that

$$\left\|\sup_{k\in\mathbb{Z}}|J_{3,k}f|\right\|_{L^p}\leqslant C\|f\|_{L^p}$$

for $\beta>3\gamma'/2$, $\frac{2\beta^2+\beta\gamma-\gamma^2}{2\beta^2-2\beta\gamma+\gamma^2}<p<2(\frac{\beta}{\gamma'}-\frac{1}{2}))$. This proves Theorem 1.1. □

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Shaoyong He
Department of Mathematics
Huzhou University
Huzhou, 313000, R. R. China
e-mail: hsyongmath@sina.com

Yan Xu
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing, 210016, P. R. China
e-mail: xuyan1@nuaa.edu.cn