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NEW CHEBYSHEV–TYPE INEQUALITIES FOR THE GENERALIZED RIEMANN–LIOUVILLE FRACTIONAL INTEGRAL WITH RESPECT TO AN INCREASING FUNCTION

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Abstract. We obtain several results for the generalized Riemann-Liouville fractional integrals whose orders are variable. We prove Chebyshev-type inequalities and consider the log-convexity of a function whose variable is the order of the generalized Riemann-Liouville fractional integral. Obtained results are applied to some special kinds of fractional integrals.

1. Introduction

From the early days of the discovery of fractional integrals until today, this topic has attracted a lot of attention. Various properties of fractional integrals and their use in many mathematical disciplines are considered. Our attention has been devoted to establishing inequalities involving fractional integrals.

The main subject of this study is the generalized Riemann-Liouville fractional integral with respect to an increasing function which is defined as the following, ([4, p. 99], [7, p. 325]):

DEFINITION 1. Let $\Lambda : [u,v] \to \mathbb{R}$ be an increasing function having a continuous derivative on (u,v) and let $f : [u,v] \to \mathbb{R}$ be an integrable function. The left-sided generalized Riemann-Liouville fractional integral of a function f with respect to a function Λ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{u+;\Lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x \Lambda'(t)\Big(\Lambda(x) - \Lambda(t)\Big)^{\alpha-1} f(t)dt, \quad x > u.$$

The right-sided generalized Riemann-Liouville fractional integral of a function f with respect to a function Λ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{v-;\Lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v \Lambda'(t)\Big(\Lambda(t) - \Lambda(x)\Big)^{\alpha-1} f(t)dt, \quad x < v.$$

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Sometimes the above-defined integrals are shortly named as Λ-Riemann-Liouville fractional integrals.

Let us highlight two particular cases of the above-defined fractional integrals which have been intensively studied. The first of them is the Riemann-Liouville fractional integral.

DEFINITION 2. Let $\alpha > 0$ and f be integrable on $[u,v]$. The left-sided Riemann-Liouville fractional integral of a function f of order α is defined by:

$$I_{u+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_u^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > u.$$

The right-sided Riemann-Liouville fractional integral of a function f of order α is defined by:

$$I_{v-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t)\, dt, \quad x < v.$$

The second of them is the Hadamard fractional integral.

DEFINITION 3. Let $\alpha > 0$ and let f be integrable on $[u,v] \subseteq (0,\infty)$. The left-sided and the right-sided Hadamard fractional integrals of a function f of order α are defined by:

$$_H J_{u+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_u^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt, \quad u < x,$$

$$_H J_{v-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^v \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t}\, dt, \quad x < v,$$

respectively.

As is known, in the literature one can find many results for fractional integrals related to the Chebyshev, Gruss, and other inequalities, see [1], [2], [3], [5], [8] and references therein.

For example, the Chebyshev inequality for the left-sided Riemann-Liouville fractional integral operator states: *If f and g are two similarly ordered functions, p is a non-negative weight function, $\alpha, \beta > 0$, then*

$$I_{0+}^{\alpha} p(t) I_{0+}^{\beta} pfg(t) + I_{0+}^{\alpha} pfg(t) I_{0+}^{\beta} p(t) \geqslant I_{0+}^{\alpha} pf(t) I_{0+}^{\beta} pg(t) + I_{0+}^{\alpha} pg(t) I_{0+}^{\beta} pf(t).$$

Since every fractional integral can be understood as a positive linear functional, the study of similar Chebyshev inequalities in which different fractional integrals appear was completed in [5], where the following general result was obtained.

Let A and B be two positive linear functionals on L, let $p,q \in L$ be non-negative functions and $f,g \in L$ be two functions such that $pf, pg, pfg, qf, qg, qfg \in L$. If f and g are two similarly ordered functions, then

$$A(pfg)B(q) + A(p)B(qfg) \geqslant A(pf)B(qg) + A(pg)B(qf).$$

If f and g are two oppositely ordered functions, then the reversed inequality holds.

In all articles published so far, the function on which the integral acts was considered as a variable. Now we will look at integrals where the order of that integral acts as a variable.

The structure of the paper is the following: after this introductory section we proved the main result, i.e. the Chebyshev-type inequality and its consequences. The third section is devoted to the log-convexity of the function whose variable is the order of the generalized Riemann-Liouville fractional integrals. Finally, we apply the obtained results to various types of fractional integrals.

2. Main results

The following theorem gives Chebyshev-type inequalities for the generalized Riemann-Liouville fractional integrals with respect to an increasing function.

THEOREM 1. *Let α,β,γ be positive numbers. Let Λ be an increasing function, having a continuous derivative on $[u,v]$ and let $f_1,f_2:[u,v]\to[0,\infty)$ be functions monotone in the same direction.*

(i) If f_1 and f_2 are increasing, then

$$\begin{aligned}
&I^{\gamma}_{u+;\Lambda}f_1(x)I^{\alpha+\beta+\gamma}_{u+;\Lambda}f_2(x)+I^{\alpha+\beta+\gamma}_{u+;\Lambda}f_1(x)I^{\gamma}_{u+;\Lambda}f_2(x)\\
&\quad\geqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\times\\
&\quad\times\left(I^{\alpha+\gamma}_{u+;\Lambda}f_1(x)I^{\beta+\gamma}_{u+;\Lambda}f_2(x)+I^{\beta+\gamma}_{u+;\Lambda}f_1(x)I^{\alpha+\gamma}_{u+;\Lambda}f_2(x)\right),\quad x>u
\end{aligned}\tag{1}$$

and

$$\begin{aligned}
&I^{\gamma}_{v-;\Lambda}f_1(x)I^{\alpha+\beta+\gamma}_{v-;\Lambda}f_2(x)+I^{\alpha+\beta+\gamma}_{v-;\Lambda}f_1(x)I^{\gamma}_{v-;\Lambda}f_2(x)\\
&\quad\leqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\times\\
&\quad\times\left(I^{\alpha+\gamma}_{v-;\Lambda}f_1(x)I^{\beta+\gamma}_{v-;\Lambda}f_2(x)+I^{\beta+\gamma}_{v-;\Lambda}f_1(x)I^{\alpha+\gamma}_{v-;\Lambda}f_2(x)\right),\quad x<v,
\end{aligned}\tag{2}$$

provided all integrals exist.

(ii) If f_1 and f_2 are decreasing, then the reverse inequality in (1) and (2) holds.

Proof. Without loss of generality we may suppose that $f_i(x)\neq f_i(u)$, $i=1,2$.

(i) Firstly, we prove the statement of the theorem for the left-sided generalized Riemann-Liouville fractional integrals in which increasing functions f_1 and f_2 appear. The following abbreviations are used:

$$l(t):=\Lambda(x)-\Lambda(t),\quad L:=\Lambda(x)-\Lambda(u).$$

Also, we often use $\int g\,df$ instead of $\int_u^x g(t)\,df(t)$.

Using integration by parts, we get

$$\begin{aligned}\Gamma(\alpha+1)I^{\alpha}_{u+;\Lambda}f_i(x) &= \alpha\int_u^x \Lambda'(t)(\Lambda(x)-\Lambda(t))^{\alpha-1}f_i(t)dt\\ &= f_i(u)(\Lambda(x)-\Lambda(u))^{\alpha}+\int_u^x(\Lambda(x)-\Lambda(t))^{\alpha}df_i(t)\\ &= f_i(u)L^{\alpha}+\int_u^x l^{\alpha}(t)df_i(t),\end{aligned}\tag{3}$$

where we use the above-described abbreviations.

Let us consider a difference D between the left-hand side and the right-hand side of (1) multiplied by the denominator $\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)$. Using formula (3), after some calculations we get the following expressions:

$$\begin{aligned}D =& \left(L^{\gamma}f_1(u)+\int l^{\gamma}df_1\right)\left(L^{\alpha+\beta+\gamma}f_2(u)+\int l^{\alpha+\beta+\gamma}df_2\right)\\ &+\left(L^{\alpha+\beta+\gamma}f_1(u)+\int l^{\alpha+\beta+\gamma}df_1\right)\left(L^{\gamma}f_2(u)+\int l^{\gamma}df_2\right)\\ &-\left(L^{\alpha+\gamma}f_1(u)+\int l^{\alpha+\gamma}df_1\right)\left(L^{\beta+\gamma}f_2(u)+\int l^{\beta+\gamma}df_2\right)\\ &-\left(L^{\beta+\gamma}f_1(u)+\int l^{\beta+\gamma}df_1\right)\left(L^{\alpha+\gamma}f_2(u)+\int l^{\alpha+\gamma}df_2\right)\\ =& f_1(u)L^{\gamma}\int l^{\gamma}(l^{\alpha}-L^{\alpha})(l^{\beta}-L^{\beta})df_2+f_2(u)L^{\gamma}\int l^{\gamma}(l^{\alpha}-L^{\alpha})(l^{\beta}-L^{\beta})df_1\\ &+\int\int l^{\gamma}(t)l^{\gamma}(s)(l^{\alpha}(s)-l^{\alpha}(t))(l^{\beta}(s)-l^{\beta}(t))df_1(t)df_2(s).\end{aligned}$$

Since products $(l^{\alpha}-L^{\alpha})(l^{\beta}-L^{\beta})$ and $(l^{\alpha}(s)-l^{\alpha}(t))(l^{\beta}(s)-l^{\beta}(t))$ are non-negative, f_1 and f_2 are non-negative increasing and l is non-negative, it follows that $D\geqslant 0$ and the proof for this case is complete.

(ii) Suppose that f_1 and f_2 are decreasing. Using notations: $\bar{f}_i := -f_i$, $\bar{F}_i := \dfrac{f_i(u)}{f_i(u)-f_i(x)}$, $i=1,2$, the integral $\Gamma(\alpha+1)I^{\alpha}_{u+;\Lambda}f_i(x)$ becomes

$$\Gamma(\alpha+1)I^{\alpha}_{u+;\Lambda}f_i(x)=f_i(u)L^{\alpha}-\int_u^x l^{\alpha}(t)d\bar{f}_i(t)=\int_u^x\left(\bar{F}_iL^{\alpha}-l^{\alpha}(t)\right)d\bar{f}_i(t).$$

Then the difference between the left-hand side and the right-hand side of (1) multiplied by $\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)$ becomes:

$$\begin{aligned}D =& \int\left(L^{\gamma}\bar{F}_1-l^{\gamma}\right)d\bar{f}_1\int\left(L^{\alpha+\beta+\gamma}\bar{F}_2-l^{\alpha+\beta+\gamma}\right)d\bar{f}_2\\ &+\int\left(L^{\alpha+\beta+\gamma}\bar{F}_1-l^{\alpha+\beta+\gamma}\right)d\bar{f}_1\int\left(L^{\gamma}\bar{F}_2-l^{\gamma}\right)d\bar{f}_2\\ &-\int\left(L^{\alpha+\gamma}\bar{F}_1-l^{\alpha+\gamma}\right)d\bar{f}_1\int\left(L^{\beta+\gamma}\bar{F}_2-l^{\beta+\gamma}\right)d\bar{f}_2\\ &-\int\left(L^{\beta+\gamma}\bar{F}_1-l^{\beta+\gamma}\right)d\bar{f}_1\int\left(L^{\alpha+\gamma}\bar{F}_2-l^{\alpha+\gamma}\right)d\bar{f}_2\end{aligned}$$

$$
\begin{aligned}
&= -\int\int L^{\gamma}\bar{F}_2 l^{\gamma}(t)(l^{\alpha}(t)-L^{\alpha})(l^{\beta}(t)-L^{\beta})d\bar{f}_1(t)\,d\bar{f}_2(s)\\
&\quad -\int\int L^{\gamma}\bar{F}_1 l^{\gamma}(s)(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(s)-L^{\beta})d\bar{f}_1(t)\,d\bar{f}_2(s)\\
&\quad +\int\int l^{\gamma}(t)l^{\gamma}(s)(l^{\alpha}(s)-l^{\alpha}(t))(l^{\beta}(s)-l^{\beta}(t))\,d\bar{f}_1(t)\,d\bar{f}_2(s).
\end{aligned}
$$

Taking into account that $\int\left(L^{\gamma}\bar{F}_1-l^{\gamma}(t)\right)d\bar{f}_1(t)\geqslant 0$, multiplying it with a non-negative expression $l^{\gamma}(s)(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(s)-L^{\beta})$ and integrating over $[u,x]$, we get

$$
\int\int l^{\gamma}(s)(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(s)-L^{\beta})\left(L^{\gamma}\bar{F}_1-l^{\gamma}(t)\right)d\bar{f}_1(t)\,d\bar{f}_2(s)\geqslant 0.
$$

From that inequality, we arrive at the relation:

$$
\begin{aligned}
&\int\int L^{\gamma}\bar{F}_1 l^{\gamma}(s)(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(s)-L^{\beta})\,d\bar{f}_1(t)\,d\bar{f}_2(s)\\
&\quad\geqslant\int\int l^{\gamma}(t)l^{\gamma}(s)(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(s)-L^{\beta})\,d\bar{f}_1(t)\,d\bar{f}_2(s).
\end{aligned}\tag{4}
$$

Similarly, we get:

$$
\begin{aligned}
&\int\int L^{\gamma}\bar{F}_2 l^{\gamma}(t)(l^{\alpha}(t)-L^{\alpha})(l^{\beta}(t)-L^{\beta})\,d\bar{f}_1(t)\,d\bar{f}_2(s)\\
&\quad\geqslant\int\int l^{\gamma}(t)l^{\gamma}(s)(l^{\alpha}(t)-L^{\alpha})(l^{\beta}(t)-L^{\beta})\,d\bar{f}_1(t)\,d\bar{f}_2(s).
\end{aligned}\tag{5}
$$

Taking into account (4) and (5), a term D is estimated as follows:

$$
\begin{aligned}
D &\leqslant -\int\int l^{\gamma}(t)l^{\gamma}(s)\Big[(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(s)-L^{\beta})+(l^{\alpha}(t)-L^{\alpha})(l^{\beta}(t)-L^{\beta})\\
&\quad -(l^{\alpha}(s)-l^{\alpha}(t))(l^{\beta}(s)-l^{\beta}(t))\Big]d\bar{f}_1(t)\,d\bar{f}_2(s)\\
&= -\int\int l^{\gamma}(t)l^{\gamma}(s)\Big[(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(t)-L^{\beta})\\
&\quad +(l^{\alpha}(t)-L^{\alpha})(l^{\beta}(s)-L^{\beta})\Big]d\bar{f}_1(t)\,d\bar{f}_2(s).
\end{aligned}
$$

Since products $(l^{\alpha}(s)-L^{\alpha})(l^{\beta}(t)-L^{\beta})$, $(l^{\alpha}(t)-L^{\alpha})(l^{\beta}(s)-L^{\beta})$ are non-negative, $\bar{f}_1$ and $\bar{f}_2$ are increasing and l is non-negative, we have $D\leqslant 0$. The proof is complete for the left-sided generalized Riemann-Liouville fractional integrals.

The statement for the right-sided generalized Riemann-Liouville fractional integrals is proved in a similar way. □

The result of the following corollary is the reason why inequalities in this section are called Chebyshev-type inequalities.

COROLLARY 1. *Let α,β,γ be positive numbers. Let Λ be an increasing function having a continuous derivative on $[u,v]$.*

(i) If $f:[u,v]\to[0,\infty)$ is increasing, then

$$I^{\gamma}_{u+;\Lambda}f(x)I^{\alpha+\beta+\gamma}_{u+;\Lambda}f(x)\geqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}I^{\alpha+\gamma}_{u+;\Lambda}f(x)\,I^{\beta+\gamma}_{u+;\Lambda}(x) \tag{6}$$

and

$$I^{\gamma}_{v-;\Lambda}f(x)I^{\alpha+\beta+\gamma}_{v-;\Lambda}f(x)\leqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}I^{\alpha+\gamma}_{v-;\Lambda}f(x)\,I^{\beta+\gamma}_{v-;\Lambda}f(x), \tag{7}$$

provided all integrals exist.

(ii) If f is decreasing, then inequalities (6) and (7) are reversed.

Proof. Putting in Theorem 1 $f_1=f_2=f$, we obtain (6), (7) and its reversed versions. □

Inequalities from the previous corollary can be considered as multiplicative-type inequalities. The following corollary gives additive-type inequalities.

COROLLARY 2. *Let α,β,γ be positive numbers. Let Λ be an increasing function having a continuous derivative on $[u,v]$.*

(i) If $f:[u,v]\to[0,\infty)$ is increasing, then

$$\begin{aligned}&L^{\alpha+\beta}\Gamma(\gamma+1)I^{\gamma}_{u+;\Lambda}f(x)+\Gamma(\alpha+\beta+\gamma+1)I^{\alpha+\beta+\gamma}_{u+;\Lambda}f(x)\\&\quad\geqslant L^{\beta}\Gamma(\alpha+\gamma+1)I^{\alpha+\gamma}_{u+;\Lambda}f(x)+L^{\alpha}\Gamma(\beta+\gamma+1)I^{\beta+\gamma}_{u+;\Lambda}f(x)\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\tilde{L}^{\alpha+\beta}\Gamma(\gamma+1)I^{\gamma;\Lambda}_{v-}f(x)+\Gamma(\alpha+\beta+\gamma+1)I^{\alpha+\beta+\gamma;\Lambda}_{v-}f(x)\\&\quad\leqslant\tilde{L}^{\beta}\Gamma(\alpha+\gamma+1)I^{\alpha+\gamma;\Lambda}_{v-}f(x)+\tilde{L}^{\alpha}\Gamma(\beta+\gamma+1)I^{\beta+\gamma;\Lambda}_{v-}f(x),\end{aligned} \tag{9}$$

where $L:=\Lambda(x)-\Lambda(u)$, $\tilde{L}:=\Lambda(v)-\Lambda(x)$, provided all integrals exist.

(ii) If f is decreasing, then inequalities (8) and (9) are reversed.

Proof. Putting in Theorem 1 $f_1=f$ and $f_2=1$, we get the statement of this corollary. □

3. Log-convexity

THEOREM 2. *Let Λ be a strictly increasing function, having a continuous derivative on $[u,v]$ and let $f_1,f_2:[u,v]\to[0,\infty)$ be positive functions.*

(i) If a function f is increasing, then a function

$$\varphi(\alpha):=\Gamma(\alpha+1)I^{\alpha}_{u+;\Lambda}f(x)$$

is log-convex. If f is decreasing, then φ is log-concave.

(ii) If a function f is increasing, then a function

$$\psi(\alpha) := \Gamma(\alpha+1) I^{\alpha}_{v-;\Lambda} f(x)$$

is log-concave. If f is decreasing, then ψ is log-convex.

Proof. From assumptions, we conclude that $\varphi(\alpha)$ and $\psi(\alpha)$ are positive numbers for $\alpha > 0$.

(i) Let us suppose that f is increasing. Let $\alpha, \beta > 0$ and $r, s \in [0,1]$ such that $r+s=1$. Using the same notations as in Theorem 1, we get

$$\begin{aligned}
\varphi^{r}(\alpha)\varphi^{s}(\beta) &= \left(\Gamma(\alpha+1) I^{\alpha}_{u+;\Lambda} f(x)\right)^{r} \left(\Gamma(\beta+1) I^{\beta}_{u+;\Lambda} f(x)\right)^{s} \\
&= \left(L^{\alpha} f(u) + \int_u^x l^{\alpha}(t)\,df(t)\right)^{r} \left(L^{\beta} f(u) + \int_u^x l^{\beta}(t)\,df(t)\right)^{s} \\
&\geqslant f(u) L^{r\alpha} L^{s\beta} + \left(\int_u^x l^{\alpha}(t)\,df(t)\right)^{r} \left(\int_u^x l^{\beta}(t)\,df(t)\right)^{s} \\
&\geqslant f(u) L^{r\alpha+s\beta} + \int_u^x l^{r\alpha+s\beta}(t)\,df(t) \\
&= \Gamma(r\alpha+s\beta+1) I^{r\alpha+s\beta}_{u+;\Lambda} f(x) = \varphi(r\alpha+s\beta),
\end{aligned}$$

where the second inequality follows from the Hölder inequality for integrals and the first inequality follows from the following discrete Hölder inequality for non-negative numbers a_1, a_2, b_1, b_2, w_1 and w_2:

$$w_1 a_1 b_1 + w_2 a_2 b_2 \leqslant \left(w_1 a_1^{1/r} + w_2 a_2^{1/r}\right)^{r} \left(w_1 b_1^{1/s} + w_2 b_2^{1/s}\right)^{s}$$

with substitutions

$$w_1 = f(u), \quad w_2 = 1, \quad a_1 = L^{r\alpha}, \quad b_1 = L^{s\beta},$$

$$a_2 = \left(\int_u^x l^{\alpha}(t)\,df(t)\right)^{r}, \quad b_2 = \left(\int_u^x l^{\beta}(t)\,df(t)\right)^{s}.$$

So, we conclude that a function φ is log-convex if f is increasing.

If f is decreasing, then a procedure is very similar. Namely, we have the following

$$\begin{aligned}
\varphi^{r}(\alpha)\varphi^{s}(\beta) &= \left(L^{\alpha} f(u) - \int_u^x l^{\alpha}(t)\,d\bar{f}(t)\right)^{r} \left(L^{\beta} f(u) - \int_u^x l^{\beta}(t)\,d\bar{f}(t)\right)^{s} \\
&\leqslant f(u) L^{r\alpha} L^{s\beta} - \left(\int_u^x l^{\alpha}(t)\,d\bar{f}(t)\right)^{r} \left(\int_u^x l^{\beta}(t)\,d\bar{f}(t)\right)^{s} \\
&\leqslant f(u) L^{r\alpha+s\beta} - \int_u^x l^{r\alpha+s\beta}(t)\,d\bar{f}(t) \\
&= \Gamma(r\alpha+s\beta+1) I^{r\alpha+s\beta}_{u+;\Lambda} f(x) = \varphi(r\alpha+s\beta),
\end{aligned}$$

where the second inequality follows from the Hölder inequality for integrals and the first inequality follows from the following discrete Popoviciu inequality for $a_i, b_i \geqslant 0$, $w_i > 0$, $i = 1,2$, ([6, p. 125]):

$$w_1 a_1 b_1 - w_2 a_2 b_2 \geqslant \left(w_1 a_1^{1/r} - w_2 a_2^{1/r}\right)^r \left(w_1 b_1^{1/s} - w_2 b_2^{1/s}\right)^s$$

with substitutions

$$w_1 = f(u), \quad w_2 = 1, \quad a_1 = L^{r\alpha}, \quad b_1 = L^{s\beta},$$

$$a_2 = \left(\int_u^x l^{\alpha}(t) df(t)\right)^r, \quad b_2 = \left(\int_u^x l^{\beta}(t) df(t)\right)^s.$$

So, we conclude that a function φ is log-concave if f is decreasing.

(ii) This case is done in a similar way. $\square$

As a simple consequence of log-convexity or log-concavity of the above-considered functions, we have the following Lyapunov-type inequalities.

COROLLARY 3. *Let functions Λ and f satisfy assumptions of Theorem 2.*

(i) If a function f is increasing, then for $p > q > r > 0$ the following holds:

$$\left(I^p_{u+;\Lambda} f(x)\right)^{q-r} \left(I^r_{u+;\Lambda} f(x)\right)^{p-q} \geqslant \frac{\Gamma^{p-r}(q+1)}{\Gamma^{q-r}(p+1)\Gamma^{p-q}(r+1)} \left(I^q_{u+;\Lambda} f(x)\right)^{p-r}. \tag{10}$$

If f is decreasing, then inequality (10) is reversed.

(ii) If a function f is increasing, then for $p > q > r > 0$ the following holds:

$$\left(I^p_{v-;\Lambda} f(x)\right)^{q-r} \left(I^r_{v-;\Lambda} f(x)\right)^{p-q} \leqslant \frac{\Gamma^{p-r}(q+1)}{\Gamma^{q-r}(p+1)\Gamma^{p-q}(r+1)} \left(I^q_{v-;\Lambda} f(x)\right)^{p-r}. \tag{11}$$

If f is decreasing, then inequality (11) is reversed.

Proof. (i) Let us suppose that f is increasing and $p > q > r > 0$. Since a function φ, defined in Theorem 2, is log-convex, then for $\alpha, \beta > 0$ and $r, s \in [0,1]$, $r + s = 1$ we get:

$$\varphi(r\alpha + s\beta) \leqslant \varphi^r(\alpha)\varphi^s(\beta). \tag{12}$$

Putting in (12)

$$r = \frac{q-r}{p-r}, \quad s = \frac{p-q}{p-r}, \quad \alpha = p, \quad \beta = r,$$

we get:

$$\varphi(q) \leqslant \varphi^{\frac{q-r}{p-r}}(p)\varphi^{\frac{p-q}{p-r}}(r).$$

Taking the $(p-r)$th power of the above inequality we have:

$$\varphi^{p-r}(q) \leqslant \varphi^{q-r}(p)\varphi^{p-q}(r)$$

and after using a definition of a function φ, we obtain inequality (10).

Other cases are done in a similar way. $\square$

4. Applications on the Hadamard and Riemann-Liouville fractional integrals

In this section, we present results for the most common kinds of the generalized Riemann-Liouville fractional integrals with respect to a function. Namely, results for the Riemann-Liouville fractional integral and Hadamard fractional integral are given.

THEOREM 3. *Let α,β,γ be positive numbers. Let $f_1,f_2:[u,v]\to[0,\infty)$ be functions monotone in the same direction.*

(i) If f_1 and f_2 are increasing, then

$$\begin{aligned}&I^{\gamma}_{u+}f_1(x)I^{\alpha+\beta+\gamma}_{u+}f_2(x)+I^{\alpha+\beta+\gamma}_{u+}f_1(x)I^{\gamma}_{u+}f_2(x)\\&\quad\geqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\left(I^{\alpha+\gamma}_{u+}f_1(x)I^{\beta+\gamma}_{u+}f_2(x)+I^{\beta+\gamma}_{u+}f_1(x)I^{\alpha+\gamma}_{u+}f_2(x)\right),\end{aligned}\tag{13}$$

and

$$\begin{aligned}&I^{\gamma}_{v-}f_1(x)I^{\alpha+\beta+\gamma}_{v-}f_2(x)+I^{\alpha+\beta+\gamma}_{v-}f_1(x)I^{\gamma}_{v-}f_2(x)\\&\quad\leqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\left(I^{\alpha+\gamma}_{v-}f_1(x)I^{\beta+\gamma}_{v-}f_2(x)+I^{\beta+\gamma}_{v-}f_1(x)I^{\alpha+\gamma}_{v-}f_2(x)\right).\end{aligned}\tag{14}$$

(ii) If f_1 and f_2 are decreasing, then the reverse signs of inequalities in (13) *and* (14) *are valid.*

Proof. If $\Lambda(t)=t$, then the generalized Riemann-Liouville fractional integral $I^{\alpha}_{u+;\Lambda}$ collapses to the classical Riemann-Liouville fractional integral. Applying Theorem 1, we get statements of this theorem. □

COROLLARY 4. *Let α,β,γ be positive numbers.*

(i) If $f:[u,v]\to[0,\infty)$ is increasing, then

$$I^{\gamma}_{u+}f(x)I^{\alpha+\beta+\gamma}_{u+}f(x)\geqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}I^{\alpha+\gamma}_{u+}f(x)\,I^{\beta+\gamma}_{u+}f(x),$$

$$\begin{aligned}&(x-u)^{\alpha+\beta}\Gamma(\gamma+1)I^{\gamma}_{u+}f(x)+\Gamma(\alpha+\beta+\gamma+1)I^{\alpha+\beta+\gamma}_{u+}f(x)\\&\quad\geqslant(x-u)^{\beta}\Gamma(\alpha+\gamma+1)I^{\alpha+\gamma}_{u+}f(x)+(x-u)^{\alpha}\Gamma(\beta+\gamma+1)I^{\beta+\gamma}_{u+}f(x),\end{aligned}$$

$$I^{\gamma}_{v-}f(x)I^{\alpha+\beta+\gamma}_{v-}f(x)\leqslant\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}I^{\alpha+\gamma}_{v-}f(x)\,I^{\beta+\gamma}_{v-}f(x),$$

and

$$\begin{aligned}&(v-x)^{\alpha+\beta}\Gamma(\gamma+1)I^{\gamma}_{v-}f(x)+\Gamma(\alpha+\beta+\gamma+1)I^{\alpha+\beta+\gamma}_{v-}f(x)\\&\quad\leqslant(v-x)^{\beta}\Gamma(\alpha+\gamma+1)I^{\alpha+\gamma}_{v-}f(x)+(v-x)^{\alpha}\Gamma(\beta+\gamma+1)I^{\beta+\gamma}_{v-}f(x),\end{aligned}$$

provided all integrals exist.

(ii) If f is decreasing, then inequalities in (i) are reversed.

Proof. It is a consequence of Corollaries 1 and 2 for $\Lambda(t) = t$. $\square$

The following statements hold for the left-sided Hadamard fractional integral. The corresponding results for the right-sided Hadamard fractional integral can be stated and proved in a similar way.

THEOREM 4. *Let α, β, γ be positive numbers. Let f_1, f_2 be non-negative functions monotone in the same direction on $[u, v] \subseteq (0, \infty)$.*

If f_1 and f_2 are increasing, then

$$\begin{aligned} &{}_HJ^{\gamma}_{u+}f_1(x)\,{}_HJ^{\alpha+\beta+\gamma}_{u+}f_2(x) + {}_HJ^{\alpha+\beta+\gamma}_{u+}f_1(x)\,{}_HJ^{\gamma}_{u+}f_2(x) \\ &\quad \geqslant \frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)} \times \\ &\quad \times \left({}_HJ^{\alpha+\gamma}_{u+}f_1(x)\,{}_HJ^{\beta+\gamma}_{u+}f_2(x) + {}_HJ^{\beta+\gamma}_{u+}f_1(x)\,{}_HJ^{\alpha+\gamma}_{u+}f_2(x) \right), \quad x > u. \end{aligned} \tag{15}$$

If f_1 and f_2 are decreasing, then the reverse inequality in (15) holds.

Proof. If $\Lambda(t) = \log t$, then the generalized Riemann-Liouville fractional integral $I^{\alpha}_{u+;\Lambda}$ becomes the Hadamard fractional integral. Applying Theorem 1, we get inequality (15). $\square$

COROLLARY 5. *Let α, β, γ be positive numbers and let f be a non-negative function on $[u, v] \subseteq (0, \infty)$.*

(i) If f is increasing, then for $x > u$ we get

$${}_HJ^{\gamma}_{u+}f(x)\,{}_HJ^{\alpha+\beta+\gamma}_{u+}f(x) \geqslant \frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\,{}_HJ^{\alpha+\gamma}_{u+}f(x)\,{}_HJ^{\beta+\gamma}_{u+}(x)$$

and

$$\begin{aligned} &\left(\log\frac{x}{u}\right)^{\alpha+\beta}\Gamma(\gamma+1)\,{}_HJ^{\gamma}_{u+}f(x) + \Gamma(\alpha+\beta+\gamma+1)\,{}_HJ^{\alpha+\beta+\gamma}_{u+}f(x) \\ &\quad \geqslant \left(\log\frac{x}{u}\right)^{\beta}\Gamma(\alpha+\gamma+1)\,{}_HJ^{\alpha+\gamma}_{u+}f(x) \\ &\qquad + \left(\log\frac{x}{u}\right)^{\alpha}\Gamma(\beta+\gamma+1)\,{}_HJ^{\beta+\gamma}_{u+}f(x). \end{aligned}$$

(ii) If f is decreasing, then inequalities from (i) are reversed.

Proof. Putting in Corollaries 1 and 2 $\Lambda(t) = \log t$, we get statements of this corollary. $\square$

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