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GENERALIZED NUMERICAL RADIUS INEQUALITIES INVOLVING POSITIVE SEMIDEFINITE BLOCK MATRICES

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Abstract. In this paper, we are interested in generalized numerical radius inequalities for the off-diagonal part of a positive semidefinite block matrix. These inequalities produce a new set of inequalities for products and sums of matrices and some inequalities related to sectorial matrices.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. A Hermtian matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called positive semidefinite if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathbb{C}^n$. We write $A \geqslant 0$ to mean that A is positive semidefinite. For Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we write $A \geqslant B$ to mean that $A - B$ is positive semidefinite. A real-valued function $f(t)$ on $[0, \infty)$ is called matrix monotone if for all $A, B \in \mathbb{M}_n(\mathbb{C})$, $A \geqslant B \geqslant 0$ implies $f(A) \geqslant f(B)$.

Due to the importance of comparing matrices in many areas of mathematics, including operator theory, mathematical analysis, and mathematical physics, this topic has been the focus of the current study. However, as the above order is a partial order on $\mathbb{M}_n(\mathbb{C})$, researchers have referred to numerical values related to elements in $\mathbb{M}_n(\mathbb{C})$. Although these comparisons are weaker than the partial order mentioned before, they nonetheless show their use and value.

A norm $N(.)$ on $\mathbb{M}_n(\mathbb{C})$ is said to be unitarily invariant if it has the basic property $N(UAV) = N(A)$, where $A \in \mathbb{M}_n(\mathbb{C})$ and $U, V \in \mathbb{M}_n(\mathbb{C})$ are unitary, it is called weakly unitarily invariant if $N(UAU^*) = N(A)$, where $A \in \mathbb{M}_n(\mathbb{C})$ and $U \in \mathbb{M}_n(\mathbb{C})$ is unitary, and it is called normalized if $N(diag(1,0,\ldots,0)) = 1$. Examples of such norms are the usual operator norm defined by $\|A\| = \max\limits_{\|x\|=1} \|Ax\| = s_1(A)$ and the Schatten p-norms defined by $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}}$, $p \geqslant 1$, where $s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity.

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For $A \in \mathbb{M}_n(\mathbb{C})$, the numerical range of A is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

Notice that $A \geqslant 0$ if and only if $W(A) \subseteq [0,\infty)$.

The numerical radius of A is defined by

$$w(A) = \max\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

It is well known that $w(.)$ defines a norm on $\mathbb{M}_n(\mathbb{C})$. In fact, for every $A \in \mathbb{M}_n(\mathbb{C})$, we have

$$w(A) \leqslant \|A\| \leqslant 2w(A),$$

which indicates that the numerical radius and the operator norm are equivalent. The norm $w(.)$ is self-adjoint and weakly unitarily invariant, but it is not unitarily invariant.

A useful formula for the numerical radius of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ was given in [17] as follows:

$$w(A) = \max_{\theta \in R} \left\| \mathrm{Re}(e^{i\theta}A) \right\|.$$

Abu-Omar and Kittaneh [1] defined the generalized numerical radius induced by a norm $N(.)$ on $\mathbb{M}_n(\mathbb{C})$) by

$$w_N(A) = \max_{\theta \in R} N\left(\mathrm{Re}(e^{i\theta}A)\right)$$

for every $A \in \mathbb{M}_n(\mathbb{C})$.

Several generalizations of the numerical radius have been discussed in [3], [4], [7], [10], [16], [19], and references therein.

One of the topics that has attracted the attention of researchers in recent years is proving matrix inequalities involving positive semidefinite block matrices of the form

$$T = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \text{ where } A, B, C \in \mathbb{M}_n(\mathbb{C}).$$

An estimation of the numerical radius of the off-diagonal part of T was given by Burqan and Al-Saafin [12] as follows:

$$w(B) \leqslant \frac{1}{2}\|A + C\|. \tag{1.1}$$

Burqan and Abu-Rahma [13] generalized the inequality (1.1) as follows:

$$w^r(B) \leqslant \frac{1}{2}\|A^r + C^r\| \text{ for } r \geqslant 1. \tag{1.2}$$

An interesting generalization of the inequality (1.2) introduced by Yang [18] is as follows:

$$f(w(B)) \leqslant \frac{1}{2}\|f(A) + f(C)\|,$$

where f is an increasing geometrically convex function on $[0,\infty)$.

In an attempt to extend the definition of positive definite matrices, sectorial matrices were established in [6]. For $\alpha \in [0,\frac{\pi}{2})$, let

$$S_\alpha = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, |\mathrm{Im}(z)| \leqslant \tan(\alpha)\mathrm{Re}(z)\}.$$

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ whose numerical range is a subset of a sector S_α, for some $\alpha \in [0,\frac{\pi}{2})$, is called a sectorial matrix. In this case, the smallest possible such α is called the index of sectoriality. Notice that a sectorial matrix A with $\alpha = 0$ is a positive definite matrix. Alakhrass and Sababheh [2] have presented a special treatment of sectorial matrices under Lieb functions. This treatment has produced several inequalities for the blocks of sectorial matrices.

In this paper, we are interested in establishing new upper bounds for $w_N(.)$ of the off-diagonal part of the positive semidefinite block matrix T. These bounds produce a new set of inequalities related to the generalized numerical radius, including products and sums of matrices that imply many well known results in the literature.

2. Lemmas

The following lemmas are essential to obtain and prove our results. The first lemma follows from the fact that $A+C \geqslant \pm(B+B^*)$ for any positive semidefinite block matrix of the form $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$. This lemma was proved in [11]. The second lemma is a norm inequality for matrix monotone functions and can be found in [14]. The third lemma has been proved in [5]. The fourth lemma can be found in [15], and the fifth lemma is a new characterization of the sectorial matrices in terms of certain positive semidefinite blocks (see [2]).

LEMMA 2.1. *Let $A,B,C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$ and let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then*

$$N(B+B^*) \leqslant N(A+C).$$

LEMMA 2.2. *Let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$ and let $N(.)$ be a normalized unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then for every $A \in \mathbb{M}_n(\mathbb{C})$,*

$$f(N(A)) \leqslant N(f(|A|)).$$

LEMMA 2.3. *Let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$ and let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then for every positive semidefinite $A,B \in \mathbb{M}_n(\mathbb{C})$,*

$$N(f(A+B) \leqslant N(f(A)+f(B)).$$

LEMMA 2.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $0 < \nu < 1$. Then*

$$\begin{bmatrix} |A|^{2\nu} & A \\ A^* & |A^*|^{2(1-\nu)} \end{bmatrix} \geqslant 0.$$

LEMMA 2.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $\alpha \in [0, \frac{\pi}{2})$. Then the following statements are equivalent:*

i. $W(A) \subset S_\alpha$.

ii. $\begin{bmatrix} \sec(\alpha)\mathrm{Re}(A) & A \\ A^* & \sec(\alpha)\mathrm{Re}(A) \end{bmatrix} \geqslant 0.$

3. Main results

At the beginning of this section, we introduce an estimation for the generalized numerical radius of the off-diagonal part of the positive semidefinite block matrix T.

THEOREM 3.1. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$ and let $f(t)$ be a nonnegative matrix monotone function on $[0, \infty)$. Then*

$$f(2w_N(B)) \leqslant N\left(f(A) + f(C)\right)$$

for every normalized unitarily invariant norm $N(.)$.

Proof. Since $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$, we have $\begin{bmatrix} A & e^{i\theta}B \\ e^{-i\theta}B^* & C \end{bmatrix} \geqslant 0$ for all $\theta \in R$. In fact, if $U = \begin{bmatrix} I & 0 \\ 0 & e^{i\theta}I \end{bmatrix}$, then U is unitary and

$$\begin{bmatrix} A & e^{i\theta}B \\ e^{-i\theta}B^* & C \end{bmatrix} = U \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} U^* \geqslant 0.$$

Now, by Lemma 2.1, we have

$$N(e^{i\theta}B + e^{-i\theta}B^*) \leqslant N(A + C).$$

Since f is a matrix monotone function, we have

$$f\left(2N(\mathrm{Re}(e^{i\theta}B))\right) \leqslant f\left(N(A + C)\right),$$

and so by Lemma 2.2, it follows that

$$f\left(2N(\mathrm{Re}(e^{i\theta}B))\right) \leqslant N(f\left(A + C\right)).$$

Thus, in view of Lemma 2.3, we have

$$\begin{aligned} f(2w_N(B)) &= f\left(2\max_{\theta\in R} N(\mathrm{Re}(e^{i\theta}B))\right) \\ &= \max_{\theta\in R} f\left(2N(\mathrm{Re}(e^{i\theta}B))\right) \\ &\leqslant N(f(A+C)), \\ &\leqslant N(f(A)+f(C)). \end{aligned}$$

This completes the proof. □

Using the fact that if $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$, then

$$\begin{bmatrix} XAX^* & XBY \\ Y^*B^*X^* & Y^*CY \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y^* \end{bmatrix}^* \geqslant 0$$

for all $X,Y \in \mathbb{M}_n(\mathbb{C})$, we have the following corollary.

COROLLARY 3.2. *Let $A,B,C,X,Y \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$ and let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$. Then*

$$f(2w_N(XBY)) \leqslant N\left(f(XAX^*)+f(Y^*CY)\right)$$

for every normalized unitarily invariant norm $N(.)$.

4. Inequalities for sums and products of matrices

In this section, we introduce several new inequalities for the generalized numerical radii associated with products and sums of matrices.

Using the fact

$$\begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} = \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix} \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}^* \geqslant 0,$$

Theorem 3.1 produces the following result.

THEOREM 4.1. *Let $A,B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices and let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$. Then*

$$f(2w_N(A^{1/2}B^{1/2})) \leqslant N\left(f(A)+f(B)\right)$$

for every normalized unitarily invariant norm $N(.)$.

In particular, for $f(t)=t$ and $N(.)=\|.\|$, we have

$$w(A^{1/2}B^{1/2}) \leqslant \frac{1}{2}\|A+B\|,$$

which can be also concluded from the arithemetic-geometric mean inequality for matrices (see, e.g., [8]).

Since the sum of positive semidefinite matrices is also positive semidefinite, it follows by Lemma 2.4 that

$$\begin{bmatrix} |A|^{2\nu}+|B|^{2\nu} & A+B \\ A^*+B^* & |A^*|^{2(1-\nu)}+|B^*|^{2(1-\nu)} \end{bmatrix} \geqslant 0.$$

Thus, Theorem 3.1 yields the following result.

THEOREM 4.2. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be positive semidefinite and let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$. Then*

$$f(2w_N(A+B)) \leqslant N\left(f(|A|^{2\nu}+|B|^{2\nu})+f(|A^*|^{2(1-\nu)}+|B^*|^{2(1-\nu)})\right)$$

for $0<\nu<1$ and every normalized unitarily invariant norm $N(.)$.

In the following theorem, we establish a generalized numerical radius inequality for matrices that produces a new inequality for the generalized numerical radii of commutators.

THEOREM 4.3. *Let $A,B,C,D,X,Y\in\mathbb{M}_n(\mathbb{C})$ and let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$. Then*

$$f(2w_N(X(AC^*+BD^*)Y)) \leqslant N\left(f(X(AA^*+BB^*)X^*)+f(Y^*(CC^*+DD^*)Y)\right)$$

for every normalized unitarily invariant norm $N(.)$.

Proof. We know that

$$\begin{bmatrix} AA^*+BB^* & AC^*+BD^* \\ CA^*+DB^* & CC^*+DD^* \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \geqslant 0$$

for all $A,B,C,D\in\mathbb{M}_n(\mathbb{C})$. So, by Corollary 3.2, we have

$$f(2w_N(X(AC^*+BD^*)Y)) \leqslant N\left(f(X(AA^*+BB^*)X^*)+f(Y^*(CC^*+DD^*)Y)\right)$$

for every $X,Y\in\mathbb{M}_n(\mathbb{C})$ and every normalized unitarily invariant norm $N(.)$. $\square$

By letting $X=Y=I$, $C^*=B$ and $D^*=\pm A$ in Theorem 4.3, we get the following generalized numerical radius inequality for commutators

$$f(2w_N((AB\pm BA) \leqslant N\left(f((AA^*+BB^*)+f(B^*B+A^*A)\right).$$

It is known (see, e.g., [9, p. 120]) that a nonnegative function $f(t)$ on $[0,\infty)$ is matrix monotone if and only if it is matrix concave. Consequently, for such a function, we have

$$f(\gamma X) \leqslant \gamma f(X) \tag{4.1}$$

for all $X \geqslant 0$ and $\gamma \geqslant 1$.

In view of Lemma 2.5 and the inequality (4.1), Theorem 3.1 can be employed to prove the following result.

THEOREM 4.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ and let $f(t)$ be a nonnegative matrix monotone function on $[0,\infty)$. Then*

$$f(2w_N(A)) \leqslant 2\sec(\alpha) N\left(f(\operatorname{Re} A)\right)$$

for every normalized unitarily invariant norm $N(.)$.

In particular for $N(.) = \|.\|$ and $f(t) = t$, we have

$$w(A) \leqslant \sec(\alpha)\,\|\operatorname{Re} A\|,$$

which has been obtained earlier in [2], using a different analysis.

Conflict of interest. The authors declare that they have no conflict of interest.

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