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A CLASS OF HALF–DISCRETE HILBERT–TYPE INEQUALITIES IN THE WHOLE PLANE INVOLVING SOME CLASSICAL KERNELS

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Abstract. In this work, we first construct a half-discrete kernel function, which is defined in the whole plane and involves both the homogeneous and the non-homogeneous cases. By employing the method of weight coefficient and some classical techniques of real analysis, a class of half-discrete Hilbert-type inequalities with the newly constructed kernel as well as the equivalent inequalities of Hardy's type are established. In addition, we prove that all the constant factors in the newly established inequalities are the best possible. Lastly, assigning special values to the parameters, and using the partial fraction expansions of cotangent function and cosecant function, some new half-discrete Hilbert-type inequalities with special kernels defined in the whole plane are presented at the end of the paper.

1. Introduction

In this paper, it is assumed that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\mathbb{Z}^0:=\mathbb{Z}\setminus\{0\}$,

$$\Omega:=\left\{x: x=\frac{2l+1}{2m+1}, l,m\in\mathbb{Z}\right\},$$

$$\Theta:=\left\{x: x=\frac{2l}{2m+1}, l,m\in\mathbb{Z}\right\}.$$

Let $\boldsymbol{a}=\{a_m\}_{m=1}^{\infty}\in l_2$, $\boldsymbol{b}=\{b_n\}_{n=1}^{\infty}\in l_2$ be two real number sequences, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2, \tag{1.1}$$

where the constant factor π is the best possible.

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Inequality (1.1) is normally named as Hilbert double series inequality [3], which was first put forward by the famous mathematician D. Hilbert in 1908. Schur established an integral analogy of (1.1) in 1911, that is,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \mathrm{d}x\mathrm{d}y < \pi \|f\|_2 \|g\|_2, \tag{1.2}$$

where $f,g \in L_2(\mathbb{R}^+)$, and the constant factor π in (1.2) is also the best possible.

For more than one hundred years, especially since the 1990s, the study of Hilbert inequality has been a hot topic for researchers of analysis, and a variety of extended forms of (1.1) and (1.2) were established, such as the following one provided by M. Krnić and J. Pečarić [4]:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n)^\beta} < B\left(\frac{\beta}{2}, \frac{\beta}{2}\right) \|\boldsymbol{a}\|_{p,\mu} \|\boldsymbol{b}\|_{q,\nu}, \tag{1.3}$$

where $0 < \beta \leqslant 4$, $\mu_m = m^{p(1-\beta/2)-1}$, $\nu_n = n^{q(1-\beta/2)-1}$, and $B(x,y)$ is the beta function [11]. Additionally, an extension of (1.2) was established by Yang [16], that is,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\beta + y^\beta} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\beta \sin \beta_1 \pi} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{1.4}$$

where $\beta, \beta_1, \beta_2 > 0$, $\beta_1 + \beta_2 = 1$, $\mu(x) = x^{p(1-\beta_1\beta)-1}$, and $\nu(y) = y^{q(1-\beta_2\beta)-1}$.

Such inequalities as (1.3) and (1.4) are commonly named as Hilbert-type inequalities. With regard to other extended forms of (1.1) and (1.2), we refer to [5, 15, 19, 24, 25, 6, 17]. Furthermore, by introducing new kernel functions, and considering the homogeneous and the non-homogeneous cases, high-dimensional extension, reverse inequality as well as the more accurate form, a variety of new Hilbert-type inequalities were established in the past 20 years (see [13, 14, 8, 9, 27, 26, 23, 21, 2]). It should be pointed out that such type of inequalities have already grown into a vast theoretical system and are crucial to the research of analysis.

Generally, if a integral Hilbert-type inequality involving a homogeneous kernel holds, then it can be obtained that a Hilbert-type inequality involving a corresponding non-homogeneous kernel holds, such as the following one which is the non-homogeneous form of inequality (1.2) [17]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} \mathrm{d}x\mathrm{d}y < \pi \|f\|_2 \|g\|_2, \tag{1.5}$$

where the constant factor π is the best possible. The non-homogeneous form of (1.1) can also be proved, but the constant factor is not yet to be proved to be the best possible (see [17], p. 315). In 2005, Yang [12] established the half-discrete form of (1.5) and the constant factor is proved to be the best possible, that is,

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{1+nx} \mathrm{d}x < \pi \|f\|_2 \|\boldsymbol{a}\|_2. \tag{1.6}$$

For some other half-discrete Hilbert-type inequalities with new kernels and best possible constant factors, we refer to [1, 7, 20, 18, 10, 22].

In this work, the main objective is to establish a class of half-discrete Hilbert-type inequalities with the kernel functions defined in the whole plane and involving both the homogeneous and the non-homogeneous cases, such as

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\alpha}+(xn)^{\alpha}+n^{2\alpha}} \mathrm{d}x < \gamma_0 \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{1.7}$$

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{[1\pm(xn)^{\alpha}+(xn)^{2\alpha}]\max\left\{1,(xn)^{6\alpha}\right\}} \mathrm{d}x < \left(\gamma_0 - \frac{8}{5\alpha}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu} \ (\alpha\in\Omega), \tag{1.8}$$

where $\mu(x)=|x|^{p(1-\alpha)-1}$, $\nu_n=|n|^{q(1-\alpha)-1}$, $0<\alpha<\frac{1}{2}$, $\gamma_0=\frac{2\sqrt{3}\pi}{3\alpha}$ for $\alpha\in\Omega$, and $\gamma_0=\frac{4\sqrt{3}\pi}{9\alpha}$ for $\alpha\in\Theta$.

In what follows, we will construct a more general kernel function with several parameters, which includes the kernels in (1.7) and (1.8). And then, a half-discrete Hilbert-type inequality and its equivalent forms are established. The paper is organized as follows: detailed lemmas are presented in Section 2, and main theorems and some corollaries are presented in Section 3 and Section 4, respectively.

2. Definitions and lemmas

LEMMA 2.1. *Assume that $\tau\in\{1,-1\}$, $\kappa\in(0,1)$, $\gamma\in\mathbb{R}^+\cup\{0\}$, and $\alpha,\beta\in\Omega$. Let $0<\alpha<\beta$ and $\alpha+\kappa<1$. Define*

$$K(z) := \frac{1+\tau z^{\alpha}}{(1+\tau z^{\beta})\max\{1,|z|^{\gamma}\}}, \tag{2.1}$$

where $z\in\mathbb{R}\setminus\{1\}$ for $\tau=-1$, and $z\in\mathbb{R}\setminus\{-1\}$ for $\tau=1$. Let $K(1):=\frac{\alpha}{\beta}$ for $\tau=-1$, and $K(-1):=\frac{\alpha}{\beta}$ for $\tau=1$. Then

$$H(z) := K(z)\,|z|^{\kappa-1}$$

decreases monotonically with z for $z\in\mathbb{R}^+$, and increases monotonically with z for $z\in\mathbb{R}^-$.

Proof. To begin with, we consider the case where $\tau=1$ and $z\in(0,1)$. Then

$$H(z) = \frac{z^{\kappa-1}+z^{\alpha+\kappa-1}}{1+z^{\beta}}. \tag{2.2}$$

Taking the derivative of (2.2), and observing that $\kappa\in(0,1)$ and $\alpha+\kappa<1$, we have

$$\frac{\mathrm{d}H}{\mathrm{d}z}=\frac{z^{\kappa-2}}{\left(1+z^{\beta}\right)^{2}}\big[(\kappa-1)+(\kappa-1+\alpha)z^{\alpha} \\ +(\kappa-1-\beta)z^{\beta}+(\kappa-1+\alpha-\beta)z^{\alpha+\beta}\big]<0. \tag{2.3}$$

In addition, if $\tau=1$ and $z\in(1,\infty)$, in view of $\gamma\in\mathbb{R}^{+}\cup\{0\}$, then it can also be proved that $\frac{\mathrm{d}H}{\mathrm{d}z}<0$. Therefore, the continuous function $H(z)(z\in\mathbb{R}^{+})$ decreases monotonically with z $(z\in\mathbb{R}^{+})$ for $\tau=1$.

Furthermore, we will prove that $H(z)$ increases monotonically with z $(z\in\mathbb{R}^{-})$ for $\tau=1$. In fact, setting $u=-z$, and observing that $\alpha,\beta\in\Omega$, we have

$$\frac{1+z^{\alpha}}{1+z^{\beta}}=\frac{1-u^{\alpha}}{1-u^{\beta}}:=L(u)\ \ (u\in\mathbb{R}^{+}\setminus\{1\}). \tag{2.4}$$

Taking the derivative of $L(u)$, we have

$$\frac{\mathrm{d}L}{\mathrm{d}u}=\frac{-u^{\alpha-1}}{\left(1-u^{\beta}\right)^{2}}\left[\alpha+(\beta-\alpha)u^{\beta}-\beta u^{\beta-\alpha}\right]:=\frac{-u^{\alpha-1}}{\left(1-u^{\beta}\right)^{2}}g(u).$$

It is obvious that

$$\frac{\mathrm{d}g}{\mathrm{d}u}=(\beta^{2}-\alpha\beta)u^{\beta-\alpha-1}(u^{\alpha}-1). \tag{2.5}$$

Since $0<\alpha<\beta$, it follows that $\frac{\mathrm{d}g}{\mathrm{d}u}<0$ when $u\in(0,1)$, and $\frac{\mathrm{d}g}{\mathrm{d}u}>0$ when $u\in(1,\infty)$. Hence, $g(u)\geqslant g(1)=0$, and it implies that $\frac{\mathrm{d}L}{\mathrm{d}u}<0$ $(u\neq 1)$. Let $L(1):=\frac{\alpha}{\beta}$, then $L(u)$ is continuous on $\mathbb{R}^{+}$, and decreases monotonically with u $(u\in\mathbb{R}^{+})$. Therefore, by (2.4), it is obvious that $\frac{1+z^{\alpha}}{1+z^{\beta}}$ increases monotonically with z $(z\in\mathbb{R}^{-})$. Additionally, since $\kappa\in(0,1]$ and $\gamma\in\mathbb{R}^{+}\cup\{0\}$, it can be shown that $|z|^{\kappa-1}$ and $|z|^{\kappa-\gamma-1}$ increases monotonically with z $(z\in\mathbb{R}^{-})$, and it follows therefore that $H(z)$ increases monotonically with z $(z\in\mathbb{R}^{-})$.

Lemma 2.1 is proved for $\tau=1$. Furthermore, based on the above discussions, it can also be proved that Lemma 2.1 holds true for $\tau=-1$. $\square$

LEMMA 2.2. *Assume that* $\tau\in\{1,-1\}$, $\kappa\in(0,1)$, $\gamma\in\mathbb{R}^{+}\cup\{0\}$ *and* $\alpha,\beta\in\Theta$. *Let* $0<\alpha<\beta$ *for* $\tau=-1$. *Let* $0\leqslant\alpha<\beta$ *and* $\alpha+\kappa<1$ *for* $\tau=1$. *Define*

$$K(z):=\frac{1+\tau z^{\alpha}}{\left(1+\tau z^{\beta}\right)\max\{1,|z|^{\gamma}\}}, \tag{2.6}$$

where $z\in\mathbb{R}$ *for* $\tau=1$, $z\in\mathbb{R}\setminus\{1,-1\}$ *for* $\tau=-1$, *and* $K(1)=K(-1):=\frac{\alpha}{\beta}$ *when* $\tau=-1$. *Then*

$$H(z):=K(z)\,|z|^{\kappa-1}$$

decreases monotonically with z *for* $z\in\mathbb{R}^{+}$, *and increases monotonically with* z *for* $z\in\mathbb{R}^{-}$.

Proof. Since $\alpha,\beta\in\Theta$, it is easy to show that $K(z)$ is an even function. If $\tau=-1$, and $z\in\mathbb{R}^+$, then

$$H(z)=\frac{1-z^{\alpha}}{1-z^{\beta}}\frac{z^{\kappa-1}}{\max\{1,z^{\gamma}\}}.$$

From the discussions in Lemma 2.1, it can be proved that $H(z)$ decreases with z $(z\in\mathbb{R}^+)$. And it is obvious that $H(z)$ increases with z $(z\in\mathbb{R}^-)$ according to the symmetry of even function. Lemma 2.2 is proved for $\tau=-1$. Similarly, Lemma 2.2 can easily be proved for $\tau=1$. $\square$

LEMMA 2.3. *Assume that $\tau\in\{1,-1\}$, $\kappa\in(0,1)$, $\gamma\in\mathbb{R}^+\cup\{0\}$ and $\alpha,\beta\in\Omega$. Suppose that $0<\alpha<\beta$, $\alpha+\kappa<\beta+\gamma$, and $K(z)$ is defined by (2.1). Define*

$$\begin{aligned}C(\alpha,\beta,\gamma,\kappa)=&\,2\sum_{i=0}^{\infty}\left(\frac{1}{2i\beta+\kappa}-\frac{1}{2i\beta+2\beta+\gamma-\kappa}\right)\\&+2\sum_{i=0}^{\infty}\left(\frac{1}{2i\beta+\beta+\gamma-\alpha-\kappa}-\frac{1}{2i\beta+\alpha+\beta+\kappa}\right).\end{aligned}\tag{2.7}$$

Then

$$\int_{-\infty}^{\infty}K(z)\,|z|^{\kappa-1}\,\mathrm{d}z=C(\alpha,\beta,\gamma,\kappa).\tag{2.8}$$

Proof. we first consider the case where $\tau=-1$. Observing that $\alpha,\beta\in\Omega$, we have

$$\begin{aligned}\int_{-\infty}^{\infty}K(z)\,|z|^{\kappa-1}\,\mathrm{d}z&=\int_{[-1,1]}\frac{1-z^{\alpha}}{1-z^{\beta}}\,|z|^{\kappa-1}\,\mathrm{d}z+\int_{\mathbb{R}\setminus[-,1]}\frac{1-z^{\alpha}}{1-z^{\beta}}\,|z|^{\kappa-\gamma-1}\,\mathrm{d}z\\&=\int_0^1\left(\frac{1-z^{\alpha}}{1-z^{\beta}}+\frac{1+z^{\alpha}}{1+z^{\beta}}\right)z^{\kappa-1}\mathrm{d}z\\&\quad+\int_1^{\infty}\left(\frac{1-z^{\alpha}}{1-z^{\beta}}+\frac{1+z^{\alpha}}{1+z^{\beta}}\right)z^{\kappa-\gamma-1}\mathrm{d}z\\&=2\left[\int_0^1\frac{z^{\kappa-1}-z^{\alpha+\beta+\kappa-1}}{1-z^{2\beta}}\mathrm{d}z+\int_1^{\infty}\frac{z^{\kappa-\gamma-1}-z^{\alpha+\beta+\kappa-\gamma-1}}{1-z^{2\beta}}\mathrm{d}z\right]\\&=2\int_0^1\frac{z^{\kappa-1}-z^{2\beta+\gamma-\kappa-1}+z^{\beta+\gamma-\alpha-\kappa-1}-z^{\alpha+\beta+\kappa-1}}{1-z^{2\beta}}\mathrm{d}z.\end{aligned}\tag{2.9}$$

Expanding $\frac{1}{1-z^{2\beta}}$ $(z\in(0,1))$ into a power series at point $z=0$, employing Lebesgue term-by-term integration theorem, and observing that $\alpha+\kappa<\beta+\gamma$, it follows that

$$\begin{aligned}\int_0^1\frac{z^{\kappa-1}-z^{2\beta+\gamma-\kappa-1}}{1-z^{2\beta}}\mathrm{d}z&=\int_0^1\sum_{i=0}^{\infty}\left(z^{2i\beta+\kappa-1}-z^{2i\beta+2\beta+\gamma-\kappa-1}\right)\mathrm{d}z\\&=\sum_{i=0}^{\infty}\left(\frac{1}{2i\beta+\kappa}-\frac{1}{2i\beta+2\beta+\gamma-\kappa}\right).\end{aligned}\tag{2.10}$$

Similarly, it can also be obtained that

$$\int_0^1 \frac{z^{\beta+\gamma-\alpha-\kappa-1} - z^{\alpha+\beta+\kappa-1}}{1-z^{2\beta}} \mathrm{d}z = \sum_{i=0}^{\infty} \left(\frac{1}{2i\beta+\beta+\gamma-\alpha-\kappa} - \frac{1}{2i\beta+\alpha+\beta+\kappa} \right). \tag{2.11}$$

Applying (2.10) and (2.11) to (2.9), we arrive at (2.8). Lemma 2.3 is proved. □

LEMMA 2.4. *Assume that* $\tau \in \{1,-1\}$, $\kappa \in (0,1)$, $\gamma \in \mathbb{R}^+ \cup \{0\}$ *and* $\alpha, \beta \in \Theta$. *Let* $0 \leqslant \alpha < \beta$, $\alpha + \kappa < \beta + \gamma$, *and* $\alpha \neq 0$ *for* $\tau = -1$. *Suppose that* $K(z)$ *is defined by* (2.6), *and*

$$c(\alpha,\beta,\gamma,\kappa) = 2\sum_{i=0}^{\infty} \left(\frac{(-\tau)^i}{i\beta+\kappa} + \frac{\tau(-\tau)^i}{i\beta+\beta+\gamma-\kappa} \right) + 2\sum_{i=0}^{\infty} \left(\frac{(-\tau)^i}{i\beta+\beta+\gamma-\alpha-\kappa} + \frac{\tau(-\tau)^i}{i\beta+\alpha+\kappa} \right). \tag{2.12}$$

Then

$$\int_{-\infty}^{\infty} K(z)\,|z|^{\kappa-1}\,\mathrm{d}z = c(\alpha,\beta,\gamma,\kappa). \tag{2.13}$$

Proof. Firstly, consider the case where $\tau = -1$. It is obvious that $K(z)$ is an even function owing to $\alpha, \beta \in \Theta$, and therefore we have

$$\begin{aligned} \int_{-\infty}^{\infty} K(z)\,|z|^{\kappa-1}\,\mathrm{d}z &= 2\int_0^1 \frac{1-z^\alpha}{1-z^\beta} z^{\kappa-1} \mathrm{d}z + 2\int_1^{\infty} \frac{1-z^\alpha}{1-z^\beta} z^{\kappa-\gamma-1} \mathrm{d}z \\ &= 2\int_0^1 \frac{z^{\kappa-1} - z^{\alpha+\kappa-1}}{1-z^\beta} \mathrm{d}z + 2\int_0^1 \frac{z^{\beta+\gamma-\alpha-\kappa-1} - z^{\beta+\gamma-\kappa-1}}{1-z^\beta} \mathrm{d}z \\ &= 2\int_0^1 \frac{z^{\kappa-1} - z^{\beta+\gamma-\kappa-1} + z^{\beta+\gamma-\alpha-\kappa-1} - z^{\alpha+\kappa-1}}{1-z^\beta} \mathrm{d}z. \end{aligned} \tag{2.14}$$

Expand $\frac{1}{1-z^\beta}$ $(z \in (0,1))$ into a power series at point $z = 0$, and employ Lebesgue term-by-term integration theorem, then it can be proved that

$$\int_0^1 \frac{z^{\kappa-1} - z^{\beta+\gamma-\kappa-1}}{1-z^\beta} \mathrm{d}z = \sum_{i=0}^{\infty} \left(\frac{1}{i\beta+\kappa} - \frac{1}{i\beta+\beta+\gamma-\kappa} \right), \tag{2.15}$$

$$\int_0^1 \frac{z^{\beta+\gamma-\alpha-\kappa-1} - z^{\alpha+\kappa-1}}{1-z^\beta} \mathrm{d}z = \sum_{i=0}^{\infty} \left(\frac{1}{i\beta+\beta+\gamma-\alpha-\kappa} - \frac{1}{i\beta+\alpha+\kappa} \right). \tag{2.16}$$

Plugging (2.15) and (2.16) back into (2.14), we get (2.13) for $\tau = -1$.

If $\tau = 1$, then we have

$$\int_{-\infty}^{\infty} K(z)\,|z|^{\kappa-1}\,\mathrm{d}z = 2\int_0^1 \frac{z^{\kappa-1} + z^{\beta+\gamma-\kappa-1} + z^{\beta+\gamma-\alpha-\kappa-1} + z^{\alpha+\kappa-1}}{1+z^\beta} \mathrm{d}z. \tag{2.17}$$

Expanding $\frac{1}{1+z^{\beta}}$ $(z \in (0,1))$ into a power series at point $z=0$, and using Lebesgue term-by-term integration theorem, then we obtain

$$\int_0^1 \frac{z^{\kappa-1}+z^{\beta+\gamma-\kappa-1}}{1+z^{\beta}}\mathrm{d}z = \sum_{i=0}^{\infty}\left(\frac{(-1)^i}{i\beta+\kappa}+\frac{(-1)^i}{i\beta+\beta+\gamma-\kappa}\right), \tag{2.18}$$

$$\int_0^1 \frac{z^{\beta+\gamma+\alpha-\kappa-1}+z^{\alpha+\kappa-1}}{1+z^{\beta}}\mathrm{d}z = \sum_{i=0}^{\infty}\left(\frac{(-1)^i}{i\beta+\beta+\gamma-\alpha-\kappa}+\frac{(-1)^i}{i\beta+\alpha+\kappa}\right). \tag{2.19}$$

Applying (2.18) and (2.19) to (2.17), we arrive at (2.13) for $\tau=1$. Lemma 2.4 is proved. $\square$

LEMMA 2.5. *Assume that* $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, $\gamma \in \mathbb{R}^+ \cup \{0\}$, $\kappa \in (0,1)$, $\delta \in \Omega$, *and* $\theta \in (0,1] \cap \Omega$. *Let* α,β,κ *be such that* $0<\alpha<\beta$, $\alpha+\kappa<1$, *and* $K(z)$ *is defined by* (2.1). *Set*

$$\hat{\boldsymbol{a}} := \{\hat{a}_n\}_{n\in\mathbb{Z}^0} := \left\{|n|^{\kappa\theta-1-\frac{2\theta}{ql}}\right\}_{n\in\mathbb{Z}^0},$$

$$\hat{f}(x) := \begin{cases} |x|^{\kappa\delta-1+\frac{2\delta}{pl}} & x\in F \\ 0 & x\in\mathbb{R}\setminus F \end{cases},$$

where l *is a sufficiently large natural number, and* $F := \left\{x : |x|^{\frac{\delta}{|\delta|}} < 1\right\}$. *Then*

$$\begin{aligned} \hat{I} :&= \sum_{n\in\mathbb{Z}^0}\hat{a}_n\int_{-\infty}^{\infty}K\left(x^{\delta}n^{\theta}\right)\hat{f}(x)\mathrm{d}x = \int_{-\infty}^{\infty}\hat{f}(x)\sum_{n\in\mathbb{Z}^0}K\left(x^{\delta}n^{\theta}\right)\hat{a}_n\mathrm{d}x \\ &> \frac{l}{|\delta\theta|}\left(\int_{[-1,1]}K(z)\,|z|^{\kappa-1+\frac{2}{pl}}\,\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]}K(z)\,|z|^{\kappa-1-\frac{2}{ql}}\,\mathrm{d}z\right). \end{aligned} \tag{2.20}$$

Proof. Let $F^+ := \{x : x\in F\cap\mathbb{R}^+\}, F^- := \{x : x\in F\cap\mathbb{R}^-\}$. Then

$$\begin{aligned} \hat{I} &= \int_{x\in F^-}\hat{f}(x)\sum_{n\in\mathbb{Z}^+}\hat{a}_nK\left(x^{\delta}n^{\theta}\right)\mathrm{d}x + \int_{x\in F^-}\hat{f}(x)\sum_{n\in\mathbb{Z}^-}\hat{a}_nK\left(x^{\delta}n^{\theta}\right)\mathrm{d}x \\ &\quad + \int_{x\in F^+}\hat{f}(x)\sum_{n\in\mathbb{Z}^+}\hat{a}_nK\left(x^{\delta}n^{\theta}\right)\mathrm{d}x + \int_{x\in F^+}\hat{f}(x)\sum_{n\in\mathbb{Z}^-}\hat{a}_nK\left(x^{\delta}n^{\theta}\right)\mathrm{d}x \\ &:= I_1+I_2+I_3+I_4. \end{aligned}$$

If $x\in F^-$ and $n\in\mathbb{Z}^+$, then $x^{\delta}n^{\theta}<0$ owing to $\delta,\theta\in\Omega$. By Lemma 2.1, it can be proved that $H\left(x^{\delta}n^{\theta}\right)$ decreases with n $(n\in\mathbb{Z}^+)$. Furthermore, since $\theta\in(0,1]$, it can also be proved that $|n|^{\theta-1-\frac{2\theta}{ql}}$ decreases with n $(n\in\mathbb{Z}^+)$. It implies that

$$\hat{a}_nK\left(x^{\delta}n^{\theta}\right) = |x|^{\delta(1-\kappa)}H\left(x^{\delta}n^{\theta}\right)|n|^{\theta-1-\frac{2\theta}{ql}}$$

decreases with n $(n \in \mathbb{Z}^+)$ for a fixed x $(x \in F^-)$. it follows therefore that

$$I_1 > \int_{x\in F^-} |x|^{\kappa\delta-1+\frac{2\delta}{pl}} \int_1^{\infty} K\left(x^\delta y^\theta\right) |y|^{\kappa\theta-1-\frac{2\theta}{ql}} \,\mathrm{d}y\mathrm{d}x := Q_1.$$

Similar discussion yields

$$I_2 > \int_{x\in F^-} |x|^{\kappa\delta-1+\frac{2\delta}{pl}} \int_{-\infty}^{-1} K\left(x^\delta y^\theta\right) |y|^{\kappa\theta-1-\frac{2\theta}{ql}} \,\mathrm{d}y\mathrm{d}x := Q_2,$$
$$I_3 > \int_{x\in F^+} |x|^{\kappa\delta-1+\frac{2\delta}{pl}} \int_1^{\infty} K\left(x^\delta y^\theta\right) |y|^{\kappa\theta-1-\frac{2\theta}{ql}} \,\mathrm{d}y\mathrm{d}x := Q_3,$$
$$I_4 > \int_{x\in F^+} |x|^{\kappa\delta-1+\frac{2\delta}{pl}} \int_{-\infty}^{-1} K\left(x^\delta y^\theta\right) |y|^{\kappa\theta-1-\frac{2\theta}{ql}} \,\mathrm{d}y\mathrm{d}x := Q_4.$$

Consider the case where $\delta < 0$, that is, $\delta \in \Omega \cap \mathbb{R}^-$, then $F^- = F \cap \mathbb{R}^- = (-\infty, -1)$. Letting $x^\delta y^\theta = z$, and observing that $x^{-\frac{\delta}{\theta}} = -|x|^{-\frac{\delta}{\theta}}$ $(x<0)$ and $z^{\frac{1}{\theta}-1} = |z|^{\frac{1}{\theta}-1}$ $(z<0)$, we have

$$\begin{aligned} Q_1 &= \int_{-\infty}^{-1} |x|^{\kappa\delta-1+\frac{2\delta}{pl}} \int_1^{\infty} K\left(x^\delta y^\theta\right) |y|^{\kappa\theta-1-\frac{2\theta}{ql}} \,\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{\theta}\int_{-\infty}^{-1} |x|^{-1+\frac{2\delta}{l}} \int_{-\infty}^{x^\delta} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z\mathrm{d}x \\ &= \frac{1}{\theta}\int_{-\infty}^{-1} |x|^{-1+\frac{2\delta}{l}} \int_{-\infty}^{-1} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z\mathrm{d}x \\ &\quad + \frac{1}{\theta}\int_{-\infty}^{-1} |x|^{-1+\frac{2\delta}{l}} \int_{-1}^{x^\delta} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z\mathrm{d}x \\ &= \frac{l}{2\,|\delta\theta|}\int_{-\infty}^{-1} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z \\ &\quad + \frac{1}{\theta}\int_{-\infty}^{-1} |x|^{-1+\frac{2\delta}{l}} \int_{-1}^{x^\delta} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z\mathrm{d}x. \end{aligned} \tag{2.21}$$

Applying Fubini's theorem to (2.21), we have

$$\begin{aligned} Q_1 =& \frac{l}{2\,|\delta\theta|}\int_{-\infty}^{-1} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z + \frac{1}{\theta}\int_{-1}^{0} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \int_{-\infty}^{z^{1/\delta}} |x|^{-1+\frac{2\delta}{l}} \,\mathrm{d}x\mathrm{d}z \\ =& \frac{l}{2\,|\delta\theta|}\left(\int_{-\infty}^{-1} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z + \int_{-1}^{0} K(z)\,|z|^{\kappa-1+\frac{2}{pl}} \,\mathrm{d}z\right). \end{aligned}$$

Similarly, it can be obtained that $Q_4 = Q_1$, and

$$Q_2 = Q_3 = \frac{l}{2\,|\delta\theta|}\left(\int_1^{\infty} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z + \int_0^1 K(z)\,|z|^{\kappa-1+\frac{2}{pl}} \,\mathrm{d}z\right).$$

It follows therefore that

$$\begin{aligned} \hat{I} &> Q_1 + Q_2 + Q_3 + Q_4 \\ &= \frac{l}{|\delta\theta|}\left(\int_{[-1,1]} K(z)\,|z|^{\kappa-1+\frac{2}{pl}} \,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\kappa-1-\frac{2}{ql}} \,\mathrm{d}z\right). \end{aligned}$$

Thus, (2.20) is proved for $\delta < 0$. Similarly, it can also be proved that (2.20) holds true for $\delta > 0$. $\square$

REMARK 2.6. Assume that $\tau \in \{1,-1\}$, $\alpha,\beta \in \Theta$, $\gamma \in \mathbb{R}^+ \cup \{0\}$, $\kappa \in (0,1)$, $\delta \in \Omega$, and $\theta \in (0,1] \cap \Omega$. Let $0 < \alpha < \beta$ for $\tau = -1$. Let $0 \leqslant \alpha < \beta$ and $\alpha + \kappa < 1$ for $\tau = 1$. Let $K(z)$ be defined by (2.6), and $\hat{\boldsymbol{a}}$, $\hat{f}(x)$ be defined by Lemma 2.5. Then it can also be proved (2.20) holds true from the proof of Lemma 2.5.

LEMMA 2.7. *Let $z_1, z_2 > 0$, $z_1 + z_2 = z$, and $\psi(x) = \cot x$. Then*

$$\psi\left(\frac{z_1\pi}{z}\right) = \frac{z}{\pi}\sum_{i=0}^{\infty}\left(\frac{1}{zi+z_1} - \frac{1}{zi+z_2}\right). \tag{2.22}$$

Proof. Observing that $\psi(x) = \cot x$ $(0 < x < \pi)$ can be written as a partial fraction expansion [11] as follows:

$$\psi(x) = \frac{1}{x} + \sum_{i=1}^{\infty}\left(\frac{1}{x+i\pi} + \frac{1}{x-i\pi}\right),$$

and setting $x = \frac{z_1\pi}{z}$, we have

$$\begin{aligned}
\psi\left(\frac{z_1\pi}{z}\right) &= \frac{z}{\pi}\left[\frac{1}{z_1} + \sum_{i=1}^{\infty}\left(\frac{1}{zi+z_1} + \frac{1}{z_1 - zi}\right)\right] \\
&= \frac{z}{\pi}\lim_{n\to\infty}\left(\sum_{i=0}^{n}\frac{1}{zi+z_1} + \sum_{i=1}^{n}\frac{1}{z_1 - zi}\right) \\
&= \frac{z}{\pi}\lim_{n\to\infty}\left(\sum_{i=0}^{n}\frac{1}{zi+z_1} - \sum_{i=0}^{n-1}\frac{1}{zi+z_2}\right) \\
&= \frac{z}{\pi}\lim_{n\to\infty}\left[\sum_{i=0}^{n}\left(\frac{1}{zi+z_1} - \frac{1}{zi+z_2}\right) + \frac{1}{zn+z_2}\right] \\
&= \frac{z}{\pi}\sum_{i=0}^{\infty}\left(\frac{1}{zi+z_1} - \frac{1}{zi+z_2}\right).
\end{aligned}$$

We arrive at (2.22), and Lemma 2.7 is proved. $\square$

LEMMA 2.8. *Let $z_1, z_2 > 0$, $z_1 + z_2 = z$, and $\phi(x) = \csc x$. Then*

$$\phi\left(\frac{z_1\pi}{z}\right) = \frac{z}{\pi}\sum_{i=0}^{\infty}(-1)^i\left(\frac{1}{zi+z_1} + \frac{1}{zi+z_2}\right). \tag{2.23}$$

Proof. Write $\phi(x) = \csc x$ $(0 < x < \pi)$ in the form of partial fraction expansion [11] as follows:

$$\phi(x) = \frac{1}{x} + \sum_{i=1}^{\infty}(-1)^i\left(\frac{1}{x+i\pi} + \frac{1}{x-i\pi}\right).$$

Setting $x = \frac{z_1\pi}{z}$, we have

$$\begin{aligned}
\phi\left(\frac{z_1\pi}{z}\right) &= \frac{z}{\pi}\left[\frac{1}{z_1} + \sum_{i=1}^{\infty}(-1)^i\left(\frac{1}{zi+z_1} + \frac{1}{z_1 - zi}\right)\right] \\
&= \frac{z}{\pi}\lim_{n\to\infty}\left(\sum_{i=0}^{n}\frac{(-1)^i}{zi+z_1} + \sum_{i=1}^{n}\frac{(-1)^i}{z_1 - zi}\right) \\
&= \frac{z}{\pi}\lim_{n\to\infty}\left(\sum_{i=0}^{n}\frac{(-1)^i}{zi+z_1} + \sum_{i=0}^{n-1}\frac{(-1)^i}{zi + z_2}\right) \\
&= \frac{z}{\pi}\lim_{n\to\infty}\left[\sum_{i=0}^{n}(-1)^i\left(\frac{1}{zi+z_1} + \frac{1}{zi+z_2}\right) - \frac{(-1)^n}{zn+z_2}\right] \\
&= \frac{z}{\pi}\sum_{i=0}^{\infty}(-1)^i\left(\frac{1}{zi+z_1} + \frac{1}{zi+z_2}\right). \qquad (2.24)
\end{aligned}$$

Relation (2.23) follows by (2.24) obviously, and Lemma 2.8 is proved. □

3. Main results

THEOREM 3.1. *Assume that* $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, $\gamma \in \mathbb{R}^+ \cup \{0\}$, $\kappa \in (0,1)$, $\delta \in \Omega$, *and* $\theta \in (0,1] \cap \Omega$. *Let* α,β,κ *be such that* $0 < \alpha < \beta$, *and* $\alpha + \kappa < \min\{1, \beta + \gamma\}$. *Suppose that* $\mu(x) = |x|^{p(1-\kappa\delta)-1}$, $\nu_n = |n|^{q(1-\kappa\theta)-1}$. *Let* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $K(z)$ *and* $C(\alpha,\beta,\gamma,\kappa)$ *be defined by* (2.1) *and* (2.7), *respectively. Then*

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x &= \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n \mathrm{d}x \\
&< |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}} C(\alpha,\beta,\gamma,\kappa)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \qquad (3.1)
\end{aligned}$$

where the constant factor $|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}} C(\alpha,\beta,\gamma,\kappa)$ *in* (3.1) *is the best possible.*

Proof. Let $\hat{K}\left(x^\delta y^\theta\right) := K\left(x^\delta n^\theta\right)$, $g(y) := a_n$, and $h(y) := |n|$ when $y \in [n, n+1)$ $(n \in \mathbb{Z}^-)$. Let $\hat{K}\left(x^\delta y^\theta\right) := K\left(x^\delta n^\theta\right)$, $g(y) := a_n$, and $h(y) := n$ when $y \in [n-1, n)$ $(n \in \mathbb{Z}^+)$. By Hölder's inequality, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n \mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{K}\left(x^\delta y^\theta\right) f(x) g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\hat{K}\left(x^\delta y^\theta\right)\right]^{1/p} [h(y)]^{(\kappa\theta-1)/p} |x|^{(1-\kappa\delta)/q} f(x) \\
&\quad \times \left[\hat{K}\left(x^\delta y^\theta\right)\right]^{1/q} |x|^{(\kappa\delta-1)/q} [h(y)]^{(1-\kappa\theta)/p} g(y)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{K}\left(x^{\delta}y^{\theta}\right)[h(y)]^{\kappa\theta-1}|x|^{p(1-\kappa\delta)/q} f^{p}(x)\mathrm{d}y\mathrm{d}x \right\}^{1/p} \\
&\quad \times \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{K}\left(x^{\delta}y^{\theta}\right)|x|^{\kappa\delta-1}[h(y)]^{q(1-\kappa\theta)/p} g^{q}(y)\mathrm{d}x\mathrm{d}y \right\}^{1/q} \\
&= \left[\int_{-\infty}^{\infty} \omega_1(x)|x|^{p(1-\kappa\delta)/q} f^{p}(x)\mathrm{d}x \right]^{1/p} \left[\sum_{n\in\mathbb{Z}^0} \omega_2(n)|n|^{q(1-\kappa\theta)/p} a_n^{q} \right]^{1/q}, \qquad (3.2)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1(x) &= \sum_{n\in\mathbb{Z}^0} K\left(x^{\delta}n^{\theta}\right)|n|^{\kappa\theta-1}, \\
\omega_2(n) &= \int_{-\infty}^{\infty} K\left(x^{\delta}n^{\theta}\right)|x|^{\kappa\delta-1}\mathrm{d}x.
\end{aligned}
$$

Observing that $\theta \in (0,1]$, it is easy to show that $|n|^{\theta-1}$ decreases monotonically with n $(n \in \mathbb{Z}^{+})$ and increases monotonically with n $(n \in \mathbb{Z}^{-})$. Furthermore, in view of $\delta, \theta \in \Omega$, it follows from Lemma 2.1 that $H\left(x^{\delta}n^{\theta}\right)$ decreases monotonically with $n(n \in \mathbb{Z}^{+})$ and increases monotonically with $n(n \in \mathbb{Z}^{-})$, whether $x > 0$ or $x < 0$. Hence,

$$
K\left(x^{\delta}n^{\theta}\right)|n|^{\kappa\theta-1} = |x|^{\delta-\kappa\delta} H\left(x^{\delta}n^{\theta}\right)|n|^{\theta-1}
$$

decreases monotonically with $n(n \in \mathbb{Z}^{+})$ and increases monotonically with n $(n \in \mathbb{Z}^{-})$ for a fixed x. It follows therefore that

$$
\omega_1(x) = \sum_{n\in\mathbb{Z}^0} K\left(x^{\delta}n^{\theta}\right)|n|^{\kappa\theta-1} < \int_{-\infty}^{\infty} K\left(x^{\delta}y^{\theta}\right)|y|^{\kappa\theta-1}\mathrm{d}y.
$$

Setting $x^{\delta}y^{\theta} = z$, and supposing that $x < 0$, we have $x^{-\frac{\delta}{\theta}} = -|x|^{-\frac{\delta}{\theta}}$ and $z^{\frac{1}{\theta}-1} = |z|^{\frac{1}{\theta}-1}$ owing to $\delta, \theta \in \Omega$. It follows that

$$
\omega_1(x) < \int_{-\infty}^{\infty} K\left(x^{\delta}y^{\theta}\right)|y|^{\kappa\theta-1}\mathrm{d}y = \frac{|x|^{-\kappa\delta}}{\theta}\int_{-\infty}^{\infty} K(z)|z|^{\kappa-1}\mathrm{d}z. \qquad (3.3)
$$

If $x > 0$, then (3.3) can also be proved. Furthermore, setting $x^{\delta}n^{\theta} = z$, it follows that

$$
\omega_2(n) = \int_{-\infty}^{\infty} K\left(x^{\delta}n^{\theta}\right)|x|^{\kappa\delta-1}\mathrm{d}x = \frac{|n|^{-\kappa\theta}}{|\delta|}\int_{-\infty}^{\infty} K(z)|z|^{\kappa-1}\mathrm{d}z. \qquad (3.4)
$$

Apply (3.3) and (3.4) to (3.2), and use (2.8), then we arrive at (3.1).

In what follows, we will prove that the constant factor $|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)$ in (3.1) is the best possible. In fact, if there exists a constant T which satisfies

$$
0 < T \leqslant |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa), \qquad (3.5)
$$

and

$$\sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^{\delta}n^{\theta}\right) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} K\left(x^{\delta}n^{\theta}\right) a_n \mathrm{d}x$$
$$< T\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{3.6}$$

Let a_n and $f(x)$ in (3.6) be replaced by $\hat{a}_n$ and $\hat{f}(x)$ defined in Lemma 2.5, respectively, then we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^0} \hat{a}_n \int_{-\infty}^{\infty} K\left(x^{\delta}n^{\theta}\right) \hat{f}(x)\mathrm{d}x &= \int_{-\infty}^{\infty} \hat{f}(x) \sum_{n\in\mathbb{Z}^0} K\left(x^{\delta}n^{\theta}\right) \hat{a}_n \mathrm{d}x \\ &< T\|\hat{f}\|_{p,\mu}\|\hat{\boldsymbol{a}}\|_{q,\nu} \\ &= T\left(2\int_{F^+} x^{\frac{2\delta}{l}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(2+2\sum_{n=2}^{\infty} n^{\frac{-2\theta}{l}-1}\right)^{\frac{1}{q}} \\ &< T\left(2\int_{F^+} x^{\frac{2\delta}{l}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(2+2\int_{1}^{\infty} x^{-\frac{2\theta}{l}-1}\mathrm{d}x\right)^{\frac{1}{q}} \\ &= T\left(\frac{l}{|\delta|}\right)^{\frac{1}{p}}\left(2+\frac{l}{\theta}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.7}$$

Combining (2.20) and (3.7), we have

$$\begin{aligned}&\int_{[-1,1]} K(z)\,|z|^{\kappa-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\kappa-1-\frac{2}{ql}}\,\mathrm{d}z \\ &< T\,|\delta\theta|\left(\frac{1}{|\delta|}\right)^{\frac{1}{p}}\left(\frac{2}{l}+\frac{1}{\theta}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.8}$$

Applying Fatou's lemma to (3.8), and using (2.8), it follows that

$$\begin{aligned}C(\alpha,\beta,\gamma,\kappa) &= \int_{-\infty}^{\infty} K(z)\,|z|^{\kappa-1}\,\mathrm{d}z \\ &= \int_{[-1,1]} \underline{\lim}_{l\to\infty} K(z)\,|z|^{\kappa-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} \underline{\lim}_{l\to\infty} K(z)\,|z|^{\kappa-1-\frac{2}{ql}}\,\mathrm{d}z \\ &\leqslant \underline{\lim}_{l\to\infty}\left(\int_{[-1,1]} K(z)\,|z|^{\kappa-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\kappa-1-\frac{2}{ql}}\,\mathrm{d}z\right) \\ &\leqslant \underline{\lim}_{l\to\infty} T\,|\delta\theta|\left(\frac{1}{|\delta|}\right)^{\frac{1}{p}}\left(\frac{2}{l}+\frac{1}{\theta}\right)^{\frac{1}{q}} = T\,|\delta|^{\frac{1}{q}}\,\theta^{\frac{1}{p}}.\end{aligned}$$

It implies

$$T \geqslant |\delta|^{-\frac{1}{q}}\,\theta^{-\frac{1}{p}} C(\alpha,\beta,\gamma,\kappa). \tag{3.9}$$

combine (3.5) and (3.9), then we obtain $T = |\delta|^{-\frac{1}{q}}\,\theta^{-\frac{1}{p}} C(\alpha,\beta,\gamma,\kappa)$, and therefore the constant factor $|\delta|^{-\frac{1}{q}}\,\theta^{-\frac{1}{p}} C(\alpha,\beta,\gamma,\kappa)$ in (3.1) is the best possible. □

THEOREM 3.2. *Assume that* $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, $\gamma \in \mathbb{R}^+ \cup \{0\}$, $\kappa \in (0,1)$, $\delta \in \Omega$, *and* $\theta \in (0,1] \cap \Omega$. *Let* α,β,κ *be such that* $\alpha+\kappa < \min\{1,\beta+\gamma\}$. *Suppose that* $\mu(x) = |x|^{p(1-\kappa\delta)-1}$, $\nu_n = |n|^{q(1-\kappa\theta)-1}$. *Let* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $K(z)$ *and* $C(\alpha,\beta,\gamma,\kappa)$ *be defined by* (2.1) *and* (2.7), *respectively. Then*

$$\sum_{n\in\mathbb{Z}^0} |n|^{p\kappa\theta-1}\left(\int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x\right)^p < \left[|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\right]^p \|f\|_{p,\mu}^p, \tag{3.10}$$

$$\int_{-\infty}^{\infty} |x|^{q\kappa\delta-1}\left(\sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n\right)^q \mathrm{d}x < \left[|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{3.11}$$

where the constant factors $\left[|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\right]^p$ *and* $\left[|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\right]^q$ *are the best possible, and* (3.1), (3.10) *and* (3.11) *are equivalent.*

Proof. Letting $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{Z}}$,

$$x_n := |n|^{p\kappa\theta-1}\left(\int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x\right)^{p-1},$$

and using (3.1), we have

$$\begin{aligned} J_1 &:= \sum_{n\in\mathbb{Z}^0} |n|^{p\kappa\theta-1}\left(\int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x\right)^p \\ &= \sum_{n\in\mathbb{Z}^0} x_n \int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x \\ &< |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\|f\|_{p,\mu}\|\boldsymbol{x}\|_{q,\nu} \\ &= |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\|f\|_{p,\mu}J_1^{1/q}. \end{aligned} \tag{3.12}$$

Inequaltiy (3.12) implies (3.10) obviously. Additionally, setting

$$F(x) := |x|^{q\kappa\delta-1}\left(\sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n\right)^{q-1},$$

and using (3.1), we have

$$\begin{aligned} J_2 &:= \int_{-\infty}^{\infty} |x|^{q\kappa\delta-1}\left(\sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n\right)^q \mathrm{d}x \\ &= \int_{-\infty}^{\infty} F(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n \mathrm{d}x \\ &< |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\|F\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu} \\ &= |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}C(\alpha,\beta,\gamma,\kappa)\|\boldsymbol{a}\|_{q,\nu}J_2^{1/p}. \end{aligned} \tag{3.13}$$

It follows from (3.13) that (3.11) holds true. Conversely, we can get (3.1) if inequality (3.10) or (3.11) is valid. In fact, assume that (3.10) holds, then it follows from Hölder's inequality that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x \\
&= \sum_{n\in\mathbb{Z}^0} \left(|n|^{\kappa\delta-1/p}\int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x\right)\left(a_n|n|^{-\kappa\delta+1/p}\right) \\
&\leqslant J_1^{1/p}\left(\sum_{n\in\mathbb{Z}^0} a_n^q |n|^{q(1-\kappa\delta)-1}\right)^{1/q} \\
&= J_1^{1/p}\|\boldsymbol{a}\|_{q,\nu}.
\end{aligned} \tag{3.14}$$

Apply inequality (3.10) to (3.14), then we obtain (3.1). In addition, suppose that (3.11) is valid, then it can also be proved that (3.1) holds true. Therefore, inequalities (3.1), (3.10) and (3.11) are equivalent, and from the equivalence of the three inequalities, it is obvious that the constant factors in (3.10) and (3.11) are the best possible. Theorem 3.2 is proved. □

Lastly, we will present the following theorem without giving detailed proof. In fact, by using Lemma 2.2, Lemma 2.4 and Remark 2.6, and referring to the proof of Theorem 3.1 and Theorem 3.2, we can easily establish the following theorem.

THEOREM 3.3. *Assume that* $\tau\in\{1,-1\}$, $\alpha,\beta\in\Theta$, $\gamma\in\mathbb{R}^+\cup\{0\}$, $\kappa\in(0,1)$, $\delta\in\Omega$, *and* $\theta\in(0,1]\cap\Omega$. *Let* $0<\alpha<\beta$ *and* $\alpha+\kappa<\beta+\gamma$ *for* $\tau=-1$. *Let* $0\leqslant\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta+\gamma\}$ *for* $\tau=1$. *Suppose that* $\mu(x)=|x|^{p(1-\kappa\delta)-1}$, *and* $\nu_n=|n|^{q(1-\kappa\theta)-1}$. *Let* $f(x),a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Let* $K(z)$ *and* $c(\alpha,\beta,\gamma,\kappa)$ *be defined by* (2.6) *and* (2.12), *respectively. Then*

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n\mathrm{d}x < |\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}c(\alpha,\beta,\gamma,\kappa)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{3.15}$$

$$\sum_{n\in\mathbb{Z}^0}|n|^{p\kappa\theta-1}\left(\int_{-\infty}^{\infty} K\left(x^\delta n^\theta\right) f(x)\mathrm{d}x\right)^p < \left[|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}c(\alpha,\beta,\gamma,\kappa)\right]^p\|f\|_{p,\mu}^p, \tag{3.16}$$

$$\int_{-\infty}^{\infty}|x|^{q\kappa\delta-1}\left(\sum_{n\in\mathbb{Z}^0} K\left(x^\delta n^\theta\right) a_n\right)^q\mathrm{d}x < \left[|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}c(\alpha,\beta,\gamma,\kappa)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{3.17}$$

where the constant $|\delta|^{-\frac{1}{q}}\theta^{-\frac{1}{p}}c(\alpha,\beta,\gamma,\kappa)$ *in* (3.15), (3.16) *and* (3.17) *is the best possible.*

4. Some corollaries

Suppose that $\gamma = 0,\ \delta = \theta = 1$ in Theorem 3.1, and use Lemma 2.7, then we obtain the following Hilbert-type inequality involving a non-homogeneous kernel.

COROLLARY 4.1. *Suppose that* $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, *and* $\kappa \in (0,1)$. *Let* α,β,κ *be such that* $0<\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\psi(x) = \cot x$, *and* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{1+\tau(xn)^\alpha}{1+\tau(xn)^\beta} a_n \mathrm{d}x < \frac{\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\alpha+\beta+\kappa)\pi}{2\beta}\right)\right] \times \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.1}$$

where $\mu(x) = |x|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\beta = (2j+1)\alpha$, $\kappa = j\alpha$ $(j \in \mathbb{N}^+)$, we have $0<\alpha<\frac{1}{j+1}$ $(\alpha\in\Omega)$, and (4.1) is transformed into the following inequality:

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{i=0}^{2j}(-\tau)^i (xn)^{i\alpha}} \mathrm{d}x < \frac{2\pi}{(2j+1)\alpha}\psi\left(\frac{j\pi}{4j+2}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.2}$$

where $\mu(x) = |x|^{p(1-j\alpha)-1}$, $\nu_n = |n|^{q(1-j\alpha)-1}$.

Let $j=1$ in (4.2), then $0<\alpha<\frac{1}{2}$ $(\alpha\in\Omega)$, and we have

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1-\tau(xn)^\alpha+(xn)^{2\alpha}} \mathrm{d}x < \frac{2\sqrt{3}\pi}{3\alpha}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.3}$$

where $\mu(x) = |x|^{p(1-\alpha)-1}$, $\nu_n = |n|^{q(1-\alpha)-1}$.

Suppose that $\gamma = 0$, $\delta = -1$, $\theta = 1$ in Theorem 3.1, and replace $f(x)x^{\beta-\alpha}$ by $f(x)$, then we have the following Hilbert-type inequality involving a homogeneous kernel.

COROLLARY 4.2. *Suppose that* $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, *and* $\kappa \in (0,1)$. *Let* α,β,κ *be such that* $0<\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\psi(x) = \cot x$, *and* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{x^\alpha+\tau n^\alpha}{x^\beta+\tau n^\beta} a_n \mathrm{d}x < \frac{\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\alpha+\beta+\kappa)\pi}{2\beta}\right)\right] \times \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.4}$$

where $\mu(x) = |x|^{p(1+\alpha-\beta+\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\beta = (2j+1)\alpha$, $\kappa = j\alpha$ $(j \in \mathbb{N}^+)$, we have $0<\alpha<\frac{1}{j+1}$ $(\alpha\in\Omega)$, and (4.4) reduces to

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{i=0}^{2j} x^{i\alpha}(-\tau n^\alpha)^{2j-i}} \mathrm{d}x < \frac{2\pi}{(2j+1)\alpha}\psi\left(\frac{j\pi}{4j+2}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.5}$$

where $\mu(x) = |x|^{p(1-j\alpha)-1}$, $\nu_n = |n|^{q(1-j\alpha)-1}$.

Let $j = 1$ and $\tau = -1$ in (4.5), we get (1.7) with $\alpha \in \Omega$. Let $j = 2$ in (4.5), then

$$\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}^0} \frac{a_n f(x)}{x^{4\alpha} - x^{3\alpha}n^{\alpha} + (xn)^{2\alpha} - x^{\alpha}n^{3\alpha} + n^{4\alpha}} \mathrm{d}x < \frac{2\pi}{5\alpha} \psi\left(\frac{\pi}{5}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu},$$

where $\mu(x) = |x|^{p(1-2\alpha)-1}$, $\nu_n = |n|^{q(1-2\alpha)-1}$.

Suppose that $\gamma = 2\beta$, $\delta = \theta = 1$ in Theorem 3.1, then we have $\alpha + \kappa < 3\beta$. If we suppose that $\kappa + \alpha < \beta$, then $\kappa < \beta < 2\beta$, and

$$\begin{aligned}
C(\alpha,\beta,\gamma,\kappa) &= 2\sum_{i=0}^{\infty} \left(\frac{1}{2i\beta + \kappa} - \frac{1}{2i\beta + 4\beta - \kappa} \right) \\
&\quad + 2\sum_{i=0}^{\infty} \left(\frac{1}{2i\beta + 3\beta - \alpha - \kappa} - \frac{1}{2i\beta + \alpha + \beta + \kappa} \right) \\
&= 2\sum_{i=0}^{\infty} \left(\frac{1}{2i\beta + \kappa} - \frac{1}{2i\beta + 2\beta - \kappa} \right) + \frac{2}{2\beta - \kappa} - \frac{2}{\beta - \alpha - \kappa} \\
&\quad + 2\sum_{i=0}^{\infty} \left(\frac{1}{2i\beta + \beta - \alpha - \kappa} - \frac{1}{2i\beta + \alpha + \beta + \kappa} \right) \\
&= \frac{\pi}{\beta} \left[\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\alpha+\beta+\kappa)\pi}{2\beta}\right) - c_0 \right],
\end{aligned} \tag{4.6}$$

where

$$c_0 = \frac{2\beta(\beta+\alpha)}{(2\beta-\kappa)(\beta-\alpha-\kappa)\pi}.$$

Therefore, we can establish the following corollary.

COROLLARY 4.3. *Suppose that $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, and $\kappa \in (0,1)$. Let α,β,κ be such that $0 < \alpha < \beta$ and $\alpha + \kappa < \min\{1,\beta\}$. Suppose that $\psi(x) = \cot x$, and $f(x), a_n > 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{1 + \tau(xn)^{\alpha}}{\left(1 + \tau(xn)^{\beta}\right) \max\left\{1, (xn)^{2\beta}\right\}} a_n \mathrm{d}x \\
&\quad < \frac{\pi}{\beta} \left[\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\alpha+\beta+\kappa)\pi}{2\beta}\right) - c_0 \right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}.
\end{aligned} \tag{4.7}$$

where $\mu(x) = |x|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\beta = (2j+1)\alpha$, $\kappa = j\alpha$ $(j \in \mathbb{N}^+)$ in (4.7), we have $0 < \alpha < \frac{1}{j+1}$ $(\alpha \in \Omega)$, and (4.7) is transformed into the following inequality:

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\max\left\{1, (xn)^{(4j+2)\alpha}\right\} \sum_{i=0}^{2j} (-\tau)^i (xn)^{i\alpha}} \mathrm{d}x \\
&\quad < \left[\frac{2\pi}{(2j+1)\alpha} \psi\left(\frac{j\pi}{4j+2}\right) - \frac{4(j+1)}{(3j+2)j\alpha} \right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu},
\end{aligned} \tag{4.8}$$

where $\mu(x) = |x|^{p(1-j\alpha)-1}$, $\nu_n = |n|^{q(1-j\alpha)-1}$.

Let $j=1$ in (4.8), then we get (1.8).

Furthermore, suppose that $\gamma = 2\beta$, $\delta = -1$, $\theta = 1$ in Theorem 3.1, and replace $f(x)x^{3\beta-\alpha}$ by $f(x)$, then we have the following corollary.

COROLLARY 4.4. *Suppose that* $\tau \in \{1,-1\}$, $\alpha,\beta \in \Omega$, *and* $\kappa \in (0,1)$. *Let* α,β,κ *be such that* $0<\alpha<\beta$, *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\psi(x) = \cot x$, *and* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{x^\alpha + \tau n^\alpha}{(x^\beta+\tau n^\beta)\max\{x^{2\beta},n^{2\beta}\}} a_n \mathrm{d}x < \frac{\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\alpha+\beta+\kappa)\pi}{2\beta}\right) - c_0\right] \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.9}$$

where $\mu(x) = |x|^{p(1+\alpha-3\beta+\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\beta = (2j+1)\alpha$, $\kappa = j\alpha$ $(j\in\mathbb{N}^+)$ in (4.9), we can obtain the homogeneous form of (4.8).

Suppose that $\tau = 1$, $\gamma = 0$, $\delta = \theta = 1$ in Theorem 3.3, and use Lemma 2.8, then we obtain the following Hilbert-type inequality involving a non-homogeneous kernel.

COROLLARY 4.5. *Suppose that* $\alpha,\beta \in \Theta$, $\kappa \in (0,1)$. *Let* $0 \leqslant \alpha < \beta$ *and* $\alpha + \kappa < \min\{1,\beta\}$. *Suppose that* $\phi(x) = \csc x$, *and* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{1+(xn)^\alpha}{1+(xn)^\beta} a_n \mathrm{d}x < \frac{2\pi}{\beta}\left[\phi\left(\frac{\kappa\pi}{\beta}\right) + \phi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)\right] \times \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{4.10}$$

where $\mu(x) = |x|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\alpha = 0$ in (4.10) we have $0<\kappa<\min\{1,\beta\}$ $(\beta\in\Theta)$, and (4.10) is transformed into the following inequality:

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1+(xn)^\beta} \mathrm{d}x < \frac{2\pi}{\beta}\phi\left(\frac{\kappa\pi}{\beta}\right) \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.11}$$

where $\mu(x) = |x|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\beta = (2j+1)\alpha$, $\kappa = j\alpha$ $(j\in\mathbb{N}^+)$ in (4.10), we have $0<\alpha<\frac{1}{j+1}$ $(\alpha\in\Theta)$, and (4.10) reduces to the following inequality.

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{i=0}^{2j}(-\tau)^i(xn)^{i\alpha}} \mathrm{d}x < \frac{4\pi}{(2j+1)\alpha}\phi\left(\frac{j\pi}{2j+1}\right) \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.12}$$

where $\mu(x) = |x|^{p(1-j\alpha)-1}$, $\nu_n = |n|^{q(1-j\alpha)-1}$.

Let $j=1$ in (4.12), then $0<\alpha<\frac{1}{2}$ $(\alpha\in\Theta)$, and we have

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{1-(xn)^{\alpha}+(xn)^{2\alpha}}\mathrm{d}x<\frac{8\sqrt{3}\pi}{9\alpha}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.13}$$

where $\mu(x)=|x|^{p(1-\alpha)-1}$, $\nu_n=|n|^{q(1-\alpha)-1}$.

It is of interest that although the form of the kernel function in inequality (4.13) is the same as that in (4.3) ($\tau=1$), the constant factors in (4.3) and (4.13) are completely different since α belongs to different sets.

Suppose that $\tau=1$, $\gamma=0$, $\delta=-1$, $\theta=1$ in Theorem 3.3. Replace $f(x)x^{\beta-\alpha}$ by $f(x)$, and use Lemma 2.8, then we have the following Hilbert-type inequality involving a homogeneous kernel.

COROLLARY 4.6. *Suppose that* $\alpha,\beta\in\Theta$, $\kappa\in(0,1)$. *Let* $0\leqslant\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\phi(x)=\csc x$, *and* $f(x),a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{x^{\alpha}+n^{\alpha}}{x^{\beta}+n^{\beta}}a_n\mathrm{d}x<\frac{2\pi}{\beta}\left[\phi\left(\frac{\kappa\pi}{\beta}\right)+\phi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)\right] \times\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.14}$$

where $\mu(x)=|x|^{p(1+\alpha-\beta+\kappa)-1}$, $\nu_n=|n|^{q(1-\kappa)-1}$.

Setting $\alpha=0$ in (4.14), we have $0<\kappa<\min\{1,\beta\}$ $(\beta\in\Theta)$, and (4.14) reduces to

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{x^{\beta}+n^{\beta}}\mathrm{d}x<\frac{2\pi}{\beta}\phi\left(\frac{\kappa\pi}{\beta}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.15}$$

where $\mu(x)=|x|^{p(1-\beta-\kappa)-1}$, $\nu_n=|n|^{q(1-\kappa)-1}$.

Setting $\beta=(2j+1)\alpha$, $\kappa=j\alpha$ $(j\in\mathbb{N}^{+})$ in (4.14), we have $0<\alpha<\frac{1}{j+1}$ $(\alpha\in\Theta)$, and (4.14) reduces to

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{\sum_{i=0}^{2j}x^{i\alpha}(-n^{\alpha})^{2j-i}}\mathrm{d}x<\frac{4\pi}{(2j+1)\alpha}\phi\left(\frac{j\pi}{2j+1}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.16}$$

where $\mu(x)=|x|^{p(1-j\alpha)-1}$, $\nu_n=|n|^{q(1-j\alpha)-1}$.

Suppose that $\tau=1$, $\gamma=2\beta$, $\delta=\theta=1$ in Theorem 3.3, and use Lemma 2.8, then

$$\begin{aligned}c(\alpha,\beta,\gamma,\kappa)&=2\sum_{i=0}^{\infty}\left(\frac{(-1)^i}{i\beta+\kappa}+\frac{(-1)^i}{i\beta+3\beta-\kappa}\right)\\&\quad+2\sum_{i=0}^{\infty}\left(\frac{(-1)^i}{i\beta+3\beta-\alpha-\kappa}+\frac{(-1)^i}{i\beta+\alpha+\kappa}\right)\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i=0}^{\infty}\left(\frac{(-1)^i}{i\beta+\kappa}+\frac{(-1)^i}{i\beta+\beta-\kappa}\right) \\
&\quad + 2\sum_{i=0}^{\infty}\left(\frac{(-1)^i}{i\beta+\beta-\alpha-\kappa}+\frac{(-1)^i}{i\beta+\alpha+\kappa}\right) \\
&\quad - \frac{2}{\beta-\kappa}+\frac{2}{2\beta-\kappa}-\frac{2}{\beta-\alpha-\kappa}+\frac{2}{2\beta-\alpha-\kappa} \\
&= \frac{2\pi}{\beta}\left[\phi\left(\frac{\kappa\pi}{\beta}\right)+\phi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)-c_1\right], \qquad (4.17)
\end{aligned}
$$

where

$$
c_1 = \frac{\beta}{\pi}\left(\frac{1}{\beta-\kappa}-\frac{1}{2\beta-\kappa}+\frac{1}{\beta-\alpha-\kappa}-\frac{1}{2\beta-\alpha-\kappa}\right).
$$

Therefore, we get the following corollary.

COROLLARY 4.7. *Suppose that* $\alpha,\beta\in\Theta$, $\kappa\in(0,1)$. *Let* $0\leqslant\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\phi(x)=\csc x$, *and* $f(x),a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Then*

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{1+(xn)^{\alpha}}{\left(1+(xn)^{\beta}\right)\max\left\{1,(xn)^{2\beta}\right\}}a_n\mathrm{d}x \\
&\qquad < \frac{2\pi}{\beta}\left[\phi\left(\frac{\kappa\pi}{\beta}\right)+\phi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)-c_1\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \qquad (4.18)
\end{aligned}
$$

where $\mu(x)=|x|^{p(1-\kappa)-1}$, $\nu_n=|n|^{q(1-\kappa)-1}$.

Setting $\alpha=0$ in (4.18), we have $0<\kappa<\min\{1,\beta\}$ $(\beta\in\Theta)$, and (4.18) is transformed into the following inequality:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{\left(1+(xn)^{\beta}\right)\max\left\{1,(xn)^{2\beta}\right\}}\mathrm{d}x \\
&\qquad < \left[\frac{2\pi}{\beta}\phi\left(\frac{\kappa\pi}{\beta}\right)-\frac{2\beta}{(\beta-\kappa)(2\beta-\kappa)}\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \qquad (4.19)
\end{aligned}
$$

where $\mu(x)=|x|^{p(1-\kappa)-1}$, $\nu_n=|n|^{q(1-\kappa)-1}$.

Additionally, we can establish the homogeneous form of (4.18).

COROLLARY 4.8. *Suppose that* $\alpha,\beta\in\Theta$, $\kappa\in(0,1)$. *Let* $0\leqslant\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\phi(x)=\csc x$, *and* $f(x),a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Then*

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{x^{\alpha}+n^{\alpha}}{(x^{\beta}+n^{\beta})\max\left\{x^{2\beta},n^{2\beta}\right\}}a_n\mathrm{d}x \\
&\qquad < \frac{2\pi}{\beta}\left[\phi\left(\frac{\kappa\pi}{\beta}\right)+\phi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)-c_1\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \qquad (4.20)
\end{aligned}
$$

where $\mu(x) = |x|^{p(1+\alpha-3\beta+\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Setting $\beta = (2j+1)\alpha$, $\kappa = j\alpha$ $(j \in \mathbb{N}^+)$ in (4.20), we have $0 < \alpha < \frac{1}{j+1}$ $(\alpha \in \Theta)$, and (4.20) reduces to

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\max\left\{x^{(4j+2)\alpha}, n^{(4j+2)\alpha}\right\} \sum_{i=0}^{2j} x^{i\alpha}(-n^{\alpha})^{2j-i}} \mathrm{d}x < \left[\frac{4\pi}{(2j+1)\alpha}\phi\left(\frac{j\pi}{2j+1}\right) - \frac{4(2j+1)(3j^2+3j+1)}{j(j+1)(3j+1)(3j+2)\alpha}\right] \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \quad (4.21)$$

where $\mu(x) = |x|^{p(1-(5j+2)\alpha)-1}$, $\nu_n = |n|^{q(1-j\alpha)-1}$.

Let $j = 1$ in (4.21), then (4.21) is transformed into the following inequality:

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{[x^{2\alpha} - (xn)^{\alpha} + n^{2\alpha}] \max\{x^{6\alpha}, n^{6\alpha}\}} \mathrm{d}x < \left(\frac{8\sqrt{3}\pi}{9\alpha} - \frac{21}{10\alpha}\right) \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \quad (4.22)$$

where $\mu(x) = |x|^{p(1-7\alpha)-1}$, $\nu_n = |n|^{q(1-\alpha)-1}$.

Suppose that $\tau = -1$, $\gamma = 0$, $\delta = \theta = 1$ in Theorem 3.3, and use Lemma 2.7, then we obtain the following corollary.

COROLLARY 4.9. *Suppose that* $\alpha, \beta \in \Theta$, $\kappa \in (0,1)$. *Let* $0 < \alpha < \beta$ *and* $\alpha + \kappa < \min\{1, \beta\}$. *Suppose that* $\psi(x) = \cot x$, *and* $f(x), a_n > 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{1-(xn)^{\alpha}}{1-(xn)^{\beta}} a_n \mathrm{d}x < \frac{2\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{\beta}\right) - \psi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)\right] \times \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \quad (4.23)$$

where $\mu(x) = |x|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$.

Replace α and β in (4.23) with β and 2β, respectively, then we have $0 < \kappa < \min\{1-\beta, \beta\}$ $(\beta \in \Theta \cap (0,1))$, and

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1+(xn)^{\beta}} \mathrm{d}x < \frac{\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\beta+\kappa)\pi}{2\beta}\right)\right] \times \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \quad (4.24)$$

where $\mu(x) = |x|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\kappa)-1}$. Observing that

$$\psi\left(\frac{\kappa\pi}{2\beta}\right) - \psi\left(\frac{(\beta+\kappa)\pi}{2\beta}\right) = 2\phi\left(\frac{\kappa\pi}{\beta}\right),$$

it follows that (4.11) and (4.24) are equivalent.

Setting $\beta=(2j+1)\alpha$, $\kappa=j\alpha$ $(j\in\mathbb{N}^{+})$ in (4.23), we have $0<\alpha<\frac{1}{j+1}$ $(\alpha\in\Theta)$, and (4.23) reduces to

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{\sum_{i=0}^{2j}(xn)^{i\alpha}}\mathrm{d}x<\frac{4\pi}{(2j+1)\alpha}\psi\left(\frac{j\pi}{2j+1}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.25}$$

where $\mu(x)=|x|^{p(1-j\alpha)-1}$, $\nu_n=|n|^{q(1-j\alpha)-1}$.

Suppose that $\tau=-1$, $\gamma=0$, $\delta=-1$, $\theta=1$ in Theorem 3.3. Replace $f(x)x^{\beta-\alpha}$ by $f(x)$, and use Lemma 2.7, then Theorem 3.3 is transformed into the following corollary.

COROLLARY 4.10. *Suppose that* $\alpha,\beta\in\Theta$, $\kappa\in(0,1)$. *Let* $0<\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\psi(x)=\cot x$, *and* $f(x),a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{x^\alpha-n^\alpha}{x^\beta-n^\beta}a_n\mathrm{d}x<\frac{2\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{\beta}\right)-\psi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)\right]\times\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{4.26}$$

where $\mu(x)=|x|^{p(1+\alpha-\beta+\kappa)-1}$, $\nu_n=|n|^{q(1-\kappa)-1}$.

Let $\beta=3\alpha$, $\kappa=\alpha$, we get (1.7) with $\alpha\in\Theta$.

Suppose that $\tau=-1$, $\gamma=\beta$, $\delta=\theta=1$ in Theorem 3.3, then we have $\alpha+\kappa<2\beta$. If we suppose that $\kappa+\alpha<\beta$, then

$$\begin{aligned}c(\alpha,\beta,\gamma,\kappa)&=2\sum_{i=0}^{\infty}\left(\frac{1}{i\beta+\kappa}-\frac{1}{i\beta+2\beta-\kappa}\right)\\&\quad+2\sum_{i=0}^{\infty}\left(\frac{1}{i\beta+2\beta-\alpha-\kappa}-\frac{1}{i\beta+\alpha+\kappa}\right)\\&=2\sum_{i=0}^{\infty}\left(\frac{1}{i\beta+\kappa}-\frac{1}{i\beta+\beta-\kappa}\right)+\frac{2}{\beta-\kappa}-\frac{2}{\beta-\alpha-\kappa}\\&\quad+2\sum_{i=0}^{\infty}\left(\frac{1}{i\beta+\beta-\alpha-\kappa}-\frac{1}{i\beta+\alpha+\kappa}\right)\\&=\frac{2\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{\beta}\right)-\psi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)-c_2\right],\end{aligned} \tag{4.27}$$

where

$$c_2=\frac{\alpha\beta}{(\beta-\kappa)(\beta-\alpha-\kappa)\pi}.$$

Therefore, we can establish the following corollary.

COROLLARY 4.11. *Suppose that* $\alpha,\beta \in \Theta,\ \kappa \in (0,1)$. *Let* $0<\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\phi(x)=\csc x$, *and* $f(x), a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{1-(xn)^{\alpha}}{\left(1-(xn)^{\beta}\right)\max\left\{1,(xn)^{\beta}\right\}}a_n\mathrm{d}x < \frac{2\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{\beta}\right)-\psi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)-c_2\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.28}$$

where $\mu(x)=|x|^{p(1-\kappa)-1},\ \nu_n=|n|^{q(1-\kappa)-1}$.

Replace α and β in (4.28) with β and 2β, respectively, then we have then we have $0<\kappa<\min\{1-\beta,\beta\}$ $(\beta\in\Theta\cap(0,1))$, and

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{\left(1+(xn)^{\beta}\right)\max\left\{1,(xn)^{2\beta}\right\}}\mathrm{d}x < \left[\frac{2\pi}{\beta}\phi\left(\frac{\kappa\pi}{\beta}\right)-\frac{2\beta}{(\beta-\kappa)(2\beta-\kappa)}\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.29}$$

where $\mu(x)=|x|^{p(1-\kappa)-1},\ \nu_n=|n|^{q(1-\kappa)-1}$.

Suppose that $\tau=-1,\ \gamma=\beta,\ \delta=-1, \theta=1$ in Theorem 3.3, then we obtain a Hilbert-type inequality with a homogeneous kernel.

COROLLARY 4.12. *Suppose that* $\alpha,\beta \in \Theta,\ \kappa \in (0,1)$. *Let* $0<\alpha<\beta$ *and* $\alpha+\kappa<\min\{1,\beta\}$. *Suppose that* $\phi(x)=\csc x$, *and* $f(x), a_n>0$ *with* $f(x)\in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. *Then*

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{x^{\alpha}-n^{\alpha}}{\left(x^{\beta}-n^{\beta}\right)\max\left\{x^{\beta},n^{\beta}\right\}}a_n\mathrm{d}x < \frac{2\pi}{\beta}\left[\psi\left(\frac{\kappa\pi}{\beta}\right)-\psi\left(\frac{(\alpha+\kappa)\pi}{\beta}\right)-c_2\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.30}$$

where $\mu(x)=|x|^{p(1+\alpha-2\beta+\kappa)-1},\ \nu_n=|n|^{q(1-\kappa)-1}$.

Setting $\beta=(2j+1)\alpha,\ \kappa=j\alpha\ (j\in\mathbb{N}^+)$ in (4.30), we have $0<\alpha<\frac{1}{j+1}\ (\alpha\in\Theta)$, and (4.30) is transformed into the following inequality:

$$\int_{-\infty}^{\infty} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{\max\left\{x^{(2j+1)\alpha},n^{(2j+1)\alpha}\right\}\sum_{i=0}^{2j}x^{i\alpha}n^{(2j-i)\alpha}}\mathrm{d}x < \left[\frac{4\pi}{(2j+1)\alpha}\psi\left(\frac{j\pi}{2j+1}\right)-\frac{2}{j(j+1)\alpha}\right]\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.31}$$

where $\mu(x)=|x|^{p(1-(3j+1)\alpha)-1},\ \nu_n=|n|^{q(1-j\alpha)-1}$.

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