Journal of Mathematical Inequalities
Volume 17, Number 4 (2023), 1481–1509
doi:10.7153/jmi-2023-17-98

GENERALIZATION OF TWO–POINT OSTROWSKI'S INEQUALITY

MOHAMMAD W. ALOMARI, NAZIA IRSHAD,
ASIF R. KHAN AND MUHAMMAD AWAIS SHAIKH

(*Communicated by J. Pečarić*)

Abstract. The paper presents a novel approach to generalize the two-point weighted Ostrowski's formula for Riemann-Stieltjes integrals by utilizing a unique class of functions of bounded r-variation. The proposed approach yields several results that exhibit sharp and better bounds compared to existing established results by using parameters and weights. Additionally, the paper also captures many of the known results as special cases.

Main objective of present paper is to generalize two-point Ostrowski's inequality by using weights with parameter. In this way our results would be more generalized than existing literature on the topic. This would be two-fold generalization, one in terms of weight and the other in terms of parameter.

1. Introduction and preliminaries

1.1. Ostrowski inequality

A celebrated integral inequality involving mapping with bounded derivative known with the name of Ostrowski's inequality was introduced by Alexander Markovich Ostrowski in year 1938 [19] can be stated as:

PROPOSITION 1.1. *Let ρ be real-valued continuous mapping on $[j,k]$ and differentiable on (j,k) such that ρ' is bounded by some real constant K. Then*

$$\left|\rho(\theta)-\frac{\int_j^k \rho(\dagger)d\dagger}{k-j}\right| \leqslant \left[\frac{1}{4}+\frac{\left(\theta-\frac{j+k}{2}\right)^2}{(k-j)^2}\right]K(k-j)$$
$$=\left[\frac{(\theta-j)^2+(k-\theta)^2}{2(k-j)}\right]K. \tag{1.1}$$

REMARK 1.2. (1) Here constant $\frac{1}{4}$ in first inequality is the best possible in the sense that it cannot be replaced by smaller one.

Mathematics subject classification (2020): 26D15, 26D20, 26D99.
Keywords and phrases: Ostrowski's inequality, bounded p-variation, Lipschitz continuous functions.
The research of the Third Author is financially supported by the Dean's Science Research Grant 2021-2022, University of Karachi, with grant number, KURP Award Letter No. 161/2021-22.

(2) In latest versions K is usually replace by $\|\rho'\|_\infty = ess \sup_{\theta\in(j,k)} |\rho'(\theta)| < \infty$.

(3) Since ρ' is bounded so the result is also valid for functions of bounded variation.

(4) In this result some assumptions may be relaxed by using the condition of absolutely continuous functions.

(5) This result may be proved in variety of ways by using different techniques including Lagrange mean value theorem, Montgomery identity and direct calculation etc.

(6) One of the importance of Ostrowski's inequality is that it is helpful to estimate the bound of first inequality in Hermite-Hadamard's double inequalities.

(7) This inequality may be interpreted in the following manners:

(a) It gives estimation of deviation of functional values with bounded derivative from integral mean.

(b) It measure the estimate of approximating area under the curve by rectangle.

(8) The celebrated inequality has vast applications in numerical integration, probability theory and special mean(s) among many others.

(9) It has close connection with other celebrated inequalities including Čebyšev and Grüss inequalities.

1.2. Two-point Ostrowski's formula

The well known integral mean-value theorem (IMVT) states that for a continuous mapping ρ defined on $[j,k]$ $\exists\, \theta \in [j,k]$ such that

$$\rho(\theta) = \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger.$$

In terms of numerical integration, the LHS of (1.1) can be regarded as a general one-point quadrature formula

$$\int_j^k \rho(\dagger)d\dagger \approx (k-j)\rho(\theta) \qquad \forall\, \theta \in [j,k],$$

with sharp error estimate in RHS of (1.1). Now we move towards two-point formula, for continuous mapping ρ defined on $[j,k]$, the IMVT guarantees that $\exists\, \kappa_1, \kappa_2 \in [j,k]$, such that

$$\int_j^k \rho(\dagger)d\dagger = (\theta - j)\rho(\kappa_1) + (k-\theta)\rho(\kappa_2), \tag{1.2}$$

for all $\theta \in [j,k]$ (see [1]). This is the starting point of two-point Ostrowski formula.

Related to this result in [7] author considered the case for $\kappa_1 = j$ and $\kappa_2 = k$ using different approach and this result named as generalized trapezoid formula, it may be read as:

$$\int_j^k \rho(\dagger)d\dagger \approx (\theta - j)\rho(j) + (k-\theta)\rho(k), \qquad \forall\, \theta \in [j,k] \tag{1.3}$$

Guessab and Schmeisser in [14] stated an important two-point formula using approach in connection with Ostrowski's formula. They stated that a real mapping ρ defined on $[j,k]$, gives us:

$$\int_j^k \rho(\dagger)d\dagger \approx (k-j)\frac{\rho(\theta)+\rho(j+k-\theta)}{2}, \qquad \forall\, \theta \in \left[j, \frac{j+k}{2}\right] \tag{1.4}$$

The formula (1.4) was further studied in [2, 3, 4, 5, 6, 8, 9, 10, 16, 20].

In year 2017 [1], Alomari worked on (1.2) to further generalize the two-point Ostrowski's formula. To be more specific, in his work he stated general two-point Ostrowski's formula

$$\int_j^k \rho(\dagger)d\dagger \approx (\theta - j)\rho(y_0) + (k-\theta)\rho(y_1),$$

for all $j \leqslant y_0 \leqslant \theta \leqslant y_1 \leqslant k$. In same article he also obtained some sharp bounds for mappings of bounded r-variation and mappings satisfy Lipschitz condition with L_k-bounds $(1 \leqslant k \leqslant \infty)$. So, (1.1) becomes special case when $y_0 = \theta = y_1$, similarly (1.3) follows when $y_0 = j$ and $y_1 = k$, and (1.4) holds when $y_0 = h$, $y_1 = j+k-h$ and $\theta = (j+k)/2$, for each $h \in [j,(j+k)/2]$.

Our aim of present work is to further generalize two-point Ostrowski's inequality given by Alomari [1] by using weight and parameter. In this way we would get more results with better bounds and would recapture many established results by varying on weight and parameter.

1.3. Bounded variation

Here we recall a definition from [17]. Throughout this section (X,d) is a metric space and T is a totally ordered set.

DEFINITION 1.3. Let $\rho : T \to (X,d)$ be a mapping. The total variation of mapping ρ is quantity

$$\bigvee_T(\rho) = \sup_D \sum_{\dagger_k \in D} d(\rho(\dagger_k), \rho(\dagger_{k-1}))$$

where D ranges over all finite partitions of the interval T.

DEFINITION 1.4. Let $\rho : T \to (X,d)$. The mapping ρ is of bounded variation $(BV(T))$ on T if its total variation is finite, i. e.,

$$\rho \in BV[T] \iff \bigvee_T(\rho) < +\infty.$$

Specifically for real-valued continuous mappings defined on compact subset of $\mathbb{R}$ we give definition in following way:

DEFINITION 1.5. Total variation of a continuous real-valued mapping ρ, defined on $[j,k]$ is

$$\bigvee_j^k(\rho) = \sup_{P\in\mathscr{P}[j,k]} \sum_{i=0}^{n_P-1} |\rho(\theta_{i+1}) - \rho(\theta_i)|$$

where $P = \{\theta_0, \cdots, \theta_{n_P}\}$, be a partition of $[j,k]$ satisfying $\theta_i \leqslant \theta_{i+1}$ for $0 \leqslant i \leqslant n_P - 1$ and supremum is taken over $\mathscr{P}[j,k] = \{P|P \text{ is partition of } [j,k]\}$ of all partitions of $[j,k]$.

DEFINITION 1.6. Continue real-valued mapping ρ on $\mathbb{R}$ is of bounded variation ($BV[j,k]$) on $[j,k] \subset \mathbb{R}$ if its total variation is finite, i. e.,

$$\rho \in BV[j,k] \iff \bigvee_j^k(\rho) < +\infty.$$

Function of bounded variation, also known as BV mapping, is a real-valued mapping whose total variation is bounded (finite): the graph of a mapping having this property is well behaved in a precise sense. For a continuous mapping of a single variable, being of bounded variation means that the distance along the direction of the y-axis, neglecting the contribution of motion along x-axis, traveled by a point moving along the graph has a finite value.

Functions of bounded variation are precisely those with respect to which one may find Riemann-Stieltjes integrals of all continuous mappings.

Another characterization states that the mappings of bounded variation on a compact interval are exactly those ρ which can be written as a difference $g - h$, where both g and h are bounded monotone.

One of the most important aspects of mappings of bounded variation is that they form an algebra of discontinuous mappings whose first derivative exists almost everywhere: due to this fact, they can use to define generalized solutions of nonlinear problems involving mappings, ordinary and partial differential equations in mathematics, physics and engineering. Further, the ability of BV mappings to deal with discontinuities has made their use widespread in the applied sciences: solutions of problems in mechanics, physics, chemical kinetics are very often representable by mappings of bounded variation. The book [15] details a very ample set of mathematical physics applications of BV mappings.

Now we move towards generalization of this important concept of mappings of bounded variation.

1.4. α-Hölder continuity

DEFINITION 1.7. A mapping $\rho : X \to Y$ is called α-Holder continuous. If $\exists\, C > 0$, such that $\forall\, y, y_0 \in X$

$$d_Y(\rho(y), \rho(y_0)) \leqslant C[d_X(y, y_0)]^{\alpha}. \tag{1.5}$$

where α[1] $\in [0,\infty)$ and (X,d_X) and (Y,d_Y) be two metric spaces.

Here we discuss different cases by varying on values of α and C.

Case 1*: Bounded function.* If $\alpha = 0$ and $d_X(y,y_0) \neq 0\ \forall\, y,y_0 \in X$, then inequality (1.5) becomes $d_Y(\rho(y),\rho(y_0)) \leqslant \mathrm{C}\ \forall\, y,y_0 \in X$, i. e., ρ is bounded.

Case 2*: Lipshitz continuity.* If $\alpha = 1$, then inequality (1.5) becomes

$$d_Y(\rho(y),\rho(y_0)) \leqslant C d_X(y,y_0), \qquad \forall y,y_0 \in X \tag{1.6}$$

here ρ is said to be Lipshitz continuous and C is called Lipshitz constant.

In theory of differential equations Lipshitz continuity is the central condition of the Picard-Linderlöff theorem which guarantees the existence and uniqueness of solution of Initial Value Problem.

Secondly, Lipshitz continuous mappings are absolutely continuous. On one hand absolutely continuous mappings are of bounded variation and on the other hand absolutely continuous mappings are differentiable almost everywhere and hence satisfy Fundamental Theorem of Calculus, i. e., $\int_j^k \rho'(\theta)d\theta = \rho(k) - \rho(j)$. Here mapping need not be differentiable but absolutely continuous only. Further, absolute continuity gives us uniform continuity and in turn it gives us continuity.

We can also establish a connection between mapping of bounded variation and Lipshitz continuous mappings as follows:

If $\rho : [j,k] \to X$ is a mapping of bounded variation and $\nu : X \to Y$ a Lipschitz continuous map, then $g \circ \rho$ is also a mapping of bounded variation and

$$\bigvee_c^d (\nu \circ \rho) \leqslant Lip(\nu) \bigvee_c^d (\rho),$$

where $Lip(\nu)$ denotes the Lipschitz constant of ν.

Here we have further subcases of Lipshitz continuity by considering inequality (1.6):

Subcase $2(a)$*:* If $0 < C \leqslant 1$, then the mapping is called *Contraction* mapping.

Contraction mapping helps us prove *Banach Fixed Point Theorem.* Banach Fixed Point Theorem has numerous applications for example to solve iterative System of Liner Algebraic Equations, in study of existence and uniqueness of solution(s) of ODE, in Theory of Integral Equations and in Dynamical Systems etc.

Subcase $2(b)$*:* If we consider that ρ is mapping from a metric space (X,d) to itself and satisfies following equation

$$d(\rho(y),\rho(y_0)) = Cd(y,y_0)$$

for all points $y,y_0 \in X$ and C is some positive real number, then the mapping ρ is called *Dilation*.

[1] Actual definition is with $\alpha \in (0,\infty)$ but we have extended this definition a little bit by including a point $\alpha = 0$ in it.

In Euclidean space, such a dilation is a similarity of the space. Dilations change the size but not the shape of an object or figure.

Every dilation of a Euclidean space that is not a congruence has a unique fixed point that is called the center of dilation. Some congruences have fixed points and others do not.

Subcase $2(c)$*:* If $C=1$ and equality holds in (1.5) then we get *Isometry*. In mathematics, an isometry is a distance-preserving transformation between metric spaces, usually assumed to be bijective.

Isometries are often used in constructions where one space is embedded in another space. For instance, the completion of a metric space X involves an isometry from X into X', a quotient set of the space of Cauchy sequences on X. The original space X is thus isometrically isomorphic to a subspace of a complete metric space, and it is usually identified with this subspace. Other embedding constructions show that every metric space is isometrically isomorphic to a closed subset of some normed vector space and that every complete metric space is isometrically isomorphic to a closed subset of some Banach space.

Case 3*: Constant function.* If $\alpha>1$, then clearly $\alpha-1>0$. In this case inequality (1.5) becomes

$$d_Y(\rho(y),\rho(y_0)) \leqslant C\,[d_X(y,y_0)]^{\alpha}\ \forall\, y,y_0.$$

Subcase $3(a)$*:* If $d_X(y,y_0)=0\ \ \forall\, y,y_0\in X$, then clearly $0\leqslant d_Y(\rho(y),\rho(y_0))\leqslant 0 \iff d_Y(\rho(y),\rho(y_0))=0\ \forall\, y,y_0\in X \iff \rho(y)=\rho(y_0)\ \ \forall\, y,y_0\in X$, and hence we conclude that the mapping ρ is constant.

Subcase $3(b)$*:* If $d_X(y,y_0)\neq 0\ \ \forall\, y,y_0\in X$, then

$$\begin{aligned}
&0\leqslant \frac{d_Y(\rho(y),\rho(y_0))}{d_X(y,y_0)}\leqslant C[d_X(y,y_0)]^{\alpha-1}\\
&\Rightarrow 0\leqslant \lim_{d_X(y,y_0)\to 0}\frac{d_Y(\rho(y),\rho(y_0))}{d_X(y,y_0)}\leqslant \lim_{d_X(y,y_0)\to 0} C[d_X(y,y_0)]^{\alpha-1}\\
&\Rightarrow 0\leqslant \lim_{d_X(y,y_0)\to 0}\frac{d_Y(\rho(y),\rho(y_0))}{d_X(y,y_0)}\leqslant 0\\
&\iff \lim_{d_X(y,y_0)\to 0}\frac{d_Y(\rho(y),\rho(y_0))}{d_X(y,y_0)}=0.
\end{aligned}$$

That is metric derivative of the mapping ρ is zero for all values of y and y_0. Finally in this subcase as well we conclude that the mapping ρ is constant.

1.5. Bounded r-variation

Here we recall generalized notion of mappings of bounded r-variation taken from [1]. In mathematical analysis, r-variation is a collection of seminorms on mappings from an ordered set to a metric space, indexed by a real number $1\leqslant r<\infty$. r-variation is a measure of the regularity or smoothness of a mapping. Specifically, DEFINITION 1.8. Let $\rho : T \to (X,d)$ be a mapping from a totally ordered set T to a metric space (X,d). Then r-variation of ρ is defined as

$$\|\rho\|_{r\text{-var}} = \left(\sup_{D} \sum_{\dagger_k \in D} d(\rho(\dagger_k), \rho(\dagger_{k-1}))^r\right)^{1/r}$$

where D ranges over all finite partitions of the interval T.

DEFINITION 1.9. $\rho : T \to (X,d)$ is called mapping of bounded r-variation if its r-variation is finite, i. e., $\|\rho\|_{r\text{-var}} < \infty$.

The r-variation of a mapping decreases with ρ. If ρ has finite r-variation and g is an α-Hölder continuous mapping, then $g \circ \rho$ has finite $\dfrac{r}{\alpha}$-variation. Further Wiener in [21] showed that the class of α-Hölder continuous mappings is a subset of the class of mappings of bounded r- variation. More precisely, if ρ has an α-Hölder property, then ρ is of bounded r-variation with $r = \frac{1}{\alpha}$. However, a continuous mapping of bounded r-variation need not satisfies a Hölder condition. For example, the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k\dagger)}{\dagger \log(k)}$ $(0 < \dagger \leqslant 1)$ converges uniformly to the sum g, which is absolutely continuous and hence is of bounded r-variation for all $1 \leqslant r < \infty$, however g does not satisfies α-Hölder property for all $\alpha \in (0,1]$, for more details the reader may refer to [12, 13].

The case when $r = 1$ gives total variation, and mappings with a finite 1-variation becomes mapping of bounded variation.

For real-valued continuous mappings defined on a compact subset of $\mathbb{R}$, we may define in this way:

DEFINITION 1.10. Total r-variation of a continuous real-valued mapping ρ, defined on interval $[j,k]$ is the quantity

$$\bigvee_j^k(\rho;r) = \sup \sum_{i=0}^{n_P-1} \left(|\rho(\theta_{i+1}) - \rho(\theta_i)|^r\right)^{\frac{1}{r}} \qquad 1 \leqslant r < \infty$$

where $P = \{\theta_0, \cdots, \theta_{n_P}\}$, be a partition of $[j,k]$ satisfying $\theta_i \leqslant \theta_{i+1}$ for $0 \leqslant i \leqslant n_P - 1$ and supremum is taken over $\mathscr{P}[j,k] = \{P | P \text{ is partition of } [j,k]\}$ of all partitions of $[j,k]$.

DEFINITION 1.11. A continue real-valued mapping ρ on $\mathbb{R}$ is of bounded r-variation ($BV_r[j,k]$) on $[j,k]$ if its total r-variation is finite, i. e.,

$$\rho \in BV_r([j,k]) \iff \bigvee_j^k(\rho;r) < +\infty.$$

Variation of order ∞ of ρ on $[j,k]$ may be define as:

DEFINITION 1.12. A continue real-valued mapping ρ on $[j,k]$ is said to be of bounded ∞-*variation* on $[j,k]$ if

$$\sum_{i=1}^{n} Osc\left(\rho;\left[\theta_{i-1}^{(n)},\theta_{i}^{(n)}\right]\right)=\sum_{i=1}^{n}(\sup-\inf)\rho(\dagger_i)<R,\ \ \dagger_i\in\left[\theta_{i-1}^{(n)},\theta_{i}^{(n)}\right],$$

for all partition of $[j,k]$, where, R is a positive real constant and

$$\bigvee_{j}^{k}(\rho,\infty)=\sup\left\{\sum(\rho):\rho\in\mathscr{P}[j,k]\right\}:=Osc(\rho;[j,k]),$$

is called oscillation of ρ on $[j,k]$. Equivalently, [11]:

$$\begin{aligned}\bigvee_{j}^{k}(\rho;\infty)&=\lim_{r\to\infty}\bigvee_{j}^{k}(\rho;r)=\sup_{\theta\in[j,k]}\{\rho(\theta)\}-\inf_{\theta\in[j,k]}\{\rho(\theta)\}\\&=Osc(\rho:[j,k]).\end{aligned}$$

Interestingly

$$Osc(\rho;[j,k])\leqslant\bigvee_{j}^{k}(\rho;r)\leqslant\bigvee_{j}^{k}(\rho;1).$$

Moreover, if $\rho(\theta)$ is differentiable on $[j,k]$, then

$$\bigvee_{j}^{k}(\rho;r)=\left(\int_{j}^{k}|\rho'(\dagger)|^{r}d\dagger\right)^{\frac{1}{r}}=\|\rho'\|_{r},\ \ 1\leqslant r<\infty.$$

Further, [19, p. 232] if $\rho\in BV[j,k]$, the Riemann-Stieltjes integral $\int_{j}^{k}\omega(\dagger)d\rho(\dagger)$ exist and the inequality

$$\left|\int_{j}^{k}\omega(\dagger)d\rho(\dagger)\right|\leqslant\|\omega\|_{\infty}.\bigvee_{j}^{k}(\rho)\tag{1.7}$$

holds and sharp, where $\|\omega\|_{\infty}=\sup_{\dagger\in[j,k]}|\omega(\dagger)|<\infty$.

Its generalization is given in following lemmas [1]:

LEMMA 1.13. *Fix $1\leqslant r<\infty$. Let $\rho,\nu:[j,k]\to\mathbb{R}$ be such that ρ is continuous on $[j,k]$ and $\nu\in BV_r[j,k]$. Then $\int_{j}^{k}\rho(\dagger)d\nu(\dagger)$ exists and*

$$\left|\int_{j}^{k}\omega(\dagger)du(\dagger)\right|\leqslant\|\omega\|_{\infty}\cdot Osc(u;[j,k])\leqslant\|\omega\|_{\infty}\cdot\bigvee_{j}^{k}(u;r),\tag{1.8}$$

holds. Here 1 in both the inequalities is best possible constant.

Here we have another result related to Lipshitz function with Lipshitz constant M:

$$\left|\int_j^k \omega(\dagger)du(\dagger)\right| \leqslant M\|\omega\|_1, \tag{1.9}$$

Its further generalization for L_r-space is as under:

LEMMA 1.14. *Let $1 \leqslant r < \infty$. Let $\omega, u : [j,k] \to \mathbb{R}$ be such that $\omega \in L_r[j,k]$ and u has a Lipschitz properly on $[j,k]$. Then the inequality*

$$\left|\int_j^k \omega(\dagger)du(\dagger)\right| \leqslant Lip_M u(\dagger)(k-j)^{1-\frac{1}{r}} \cdot \|\omega\|_r, \tag{1.10}$$

holds and constant 1 in RHS is best possible, where

$$\|\omega\|_r = \left(\int_j^k |\omega(t)|^r d\dagger\right)^{\frac{1}{r}}, \quad 1 \leqslant r < \infty. \tag{1.11}$$

REMARK 1.15. Clearly, when $r = 1$ in (1.10) we get (1.9).

Also we have with weak conditions:

LEMMA 1.16. *Let $1 \leqslant r < \infty$. Let $\omega, \nu : [j,k] \to \mathbb{R}$ be such that $\omega \in L_r[j,k]$ and ν is of bounded variation on $[j,k]$. Then the inequality*

$$\left|\int_j^k \omega(\dagger)dv(\dagger)\right| \leqslant \left(\bigvee_j^k (v)\right)^{1-\frac{1}{r}} \cdot \|v'\|_\infty^{\frac{1}{r}} \cdot \|\omega\|_r \quad a.\,e. \tag{1.12}$$

holds provided that integral $\int_j^k w(\dagger)d\dagger_0(\dagger)$ exist. Here constant 1 on RHS is the best possible.

REMARK 1.17. If ρ is M-Lipschitz then

$$Lip_M(\rho) = \sup_{(y,y_0)\in[j,k]} \left|\frac{\rho(y_0)-\rho(\theta)}{y_0-\theta}\right| < \infty$$

Now (1.12) becomes:

$$\left|\int_j^k \omega(\dagger)d\rho(\dagger)\right| \leqslant (Lip_M(\rho))^{\frac{1}{r}} \left(\bigvee_j^k (\rho)\right)^{1-\frac{1}{r}} \cdot \|\omega\|_r, \tag{1.13}$$

which is valid everywhere and constant 1 on RHS is the best possible. Moreover, (1.13) reduces to (1.7), as $r \to \infty$, and to (1.9), as $r \to 1$.

2. Bounds for functions of bounded r-variation $(1 \leqslant r < \infty)$

Let $\rho : [j,k] \to \mathbb{R}$. To approximate $\int_j^k \rho(\dagger)d\dagger$, we define

$$\int_j^k \omega(\dagger)\rho(\dagger)d\dagger = Q_\omega(\rho;\lambda,y_0,\theta,y_1) + E_\omega(\rho;\lambda,y_0,\theta,y_1), \tag{2.1}$$

where $Q_\omega(\rho;\lambda,y_0,\theta,y_1)$ is generalized quadrature formula involving weight and parameter defined as

$$\begin{aligned} Q_\omega(\rho;\lambda,y_0,\theta,y_1) = {} & \rho(y_0)\int_\alpha^{y_0}\omega(\dagger_0)d\dagger_0 - \rho(j)\int_\alpha^{j}\omega(\dagger_0)d\dagger_0 \\ & + \rho(y_1)\int_\theta^{y_1}\omega(\dagger_0)d\dagger_0 - \rho(y_0)\int_\theta^{y_0}\omega(\dagger_0)d\dagger_0 \\ & + \rho(k)\int_\beta^{k}\omega(\dagger_0)d\dagger_0 - \rho(y_1)\int_\beta^{y_1}\omega(\dagger_0)d\dagger_0, \end{aligned}$$

where $\alpha = j + \lambda\frac{k-j}{2}$, $\beta = k - \lambda\frac{k-j}{2}$, $\forall\, \theta \in [y_0,y_1] \subseteq [j,k]$ with $\lambda \in [0,1]$ and ω is a probability density mapping. Here error term is given as

$$E_\omega(\rho;\lambda,y_0,\theta,y_1) := \int_j^k K_\omega(\dagger;\lambda,y_0,\theta,y_1)d\rho(\dagger), \tag{2.2}$$

where

$$K_\omega(\dagger;\lambda,y_0,\theta,y_1) = \begin{cases} \displaystyle\int_\alpha^\dagger \omega(\dagger_0)d\dagger_0, & \text{if} \quad j \leqslant \dagger \leqslant y_0, \\ \displaystyle\int_\theta^\dagger \omega(\dagger_0)d\dagger_0, & \text{if} \quad y_0 < \dagger < y_1, \\ \displaystyle\int_\beta^\dagger \omega(\dagger_0)d\dagger_0, & \text{if} \quad y_1 \leqslant \dagger \leqslant k. \end{cases}$$

We find bounds of $E_\omega(\rho;\lambda,y_0,\theta,y_1)$, as follows:

THEOREM 2.1. *Let $\rho : [j,k] \to \mathbb{R}$ be of bounded r-variation $(1 \leqslant r < \infty)$ on $[j,k]$. Then following inequality holds*

$$\begin{aligned} & \Bigg| \rho(y_0)\int_\alpha^{y_0}\omega(\dagger_0)d\dagger_0 - \rho(j)\int_\alpha^{j}\omega(\dagger_0)d\dagger_0 + \rho(y_1)\int_\theta^{y_1}\omega(\dagger_0)d\dagger_0 \\ & -\rho(y_0)\int_\theta^{y_0}\omega(\dagger_0)d\dagger_0 + \rho(k)\int_\beta^{k}\omega(\dagger_0)d\dagger_0 - \rho(y_1)\int_\beta^{y_1}\omega(\dagger_0)d\dagger_0 \\ & - \int_j^k \omega(\dagger)\rho(\dagger)d\dagger \Bigg| \\ \leqslant {} & \max\left\{ \sup_{\dagger\in[j,y_0]}\int_\alpha^\dagger \omega(\dagger_0)d\dagger_0,\ \sup_{\dagger\in[y_0,y_1]}\int_\theta^\dagger \omega(\dagger_0)d\dagger_0,\ \sup_{\dagger\in[y_1,k]}\int_\beta^\dagger \omega(\dagger_0)d\dagger_0 \right\} \\ & \times \bigvee_j^k(\rho;r), \end{aligned} \tag{2.3}$$

where $\alpha = j+\lambda\frac{k-j}{2}$, $\beta = k-\lambda\frac{k-j}{2}$, $\forall\,\theta\in[y_0,y_1]\subseteq[j,k]$ *with* $\lambda\in[0,1]$ *and* ω *is a probability density mapping.*

Proof. Using integration-by-part, we get

$$E_\omega(\rho;\lambda,y_0,\theta,y_1) = \int_j^k K_\omega(\dagger;\lambda,y_0,\theta,y_1)d\rho(\dagger) = Q_\omega(\rho;\lambda,y_0,\theta,y_1) - \int_j^k \omega(\dagger)\rho(\dagger)d\dagger.$$

Clearly, from the definition of the function $K_\omega(\dagger;y_0,\theta,y_1)$, we obtain

$$\begin{aligned}
&\left|\int_j^k K_\omega(\dagger;\lambda,y_0,\theta,y_1)d\rho(\dagger)\right| \\
&\leqslant \left|\int_j^{y_0}\int_\alpha^\dagger \omega(\dagger_0)d\dagger_0 d\rho(\dagger)\right| + \left|\int_{y_0}^{y_1}\int_\theta^\dagger \omega(\dagger_0)d\dagger_0 d\rho(\dagger)\right| + \left|\int_{y_1}^{k}\int_\beta^\dagger \omega(\dagger_0)d\dagger_0 d\rho(\dagger)\right| \\
&\leqslant \sup_{\dagger\in[j,y_0]}\int_\alpha^\dagger \omega(\dagger_0)d\dagger_0 \bigvee_j^{y_0}(\rho;r) + \sup_{\dagger\in[y_0,y_1]}\int_\theta^\dagger \omega(\dagger_0)d\dagger_0 \bigvee_{y_0}^{y_1}(\rho;r) \\
&\quad + \sup_{\dagger\in[y_1,k]}\int_\beta^\dagger \omega(\dagger_0)d\dagger_0 \bigvee_{y_1}^{k}(\rho;r) \\
&\leqslant \max\left\{\sup_{\dagger\in[j,y_0]}\int_\alpha^\dagger \omega(\dagger_0)d\dagger_0,\ \sup_{\dagger\in[y_0,y_1]}\int_\theta^\dagger \omega(\dagger_0)d\dagger_0,\ \sup_{\dagger\in[y_1,k]}\int_\beta^\dagger \omega(\dagger_0)d\dagger_0\right\}\bigvee_j^k(\rho;r).
\end{aligned}$$

This is our required inequality. □

If we put $\omega(\dagger) = \frac{1}{k-j}$ in (2.3), then:

COROLLARY 2.2. *Let* $\rho:[j,k]\to\mathbb{R}$ *be a mapping of bounded* r*-variation* $(1\leqslant r<\infty)$ *and* $\lambda\in[0,1]$. *Then we have*

$$\begin{aligned}
&\bigg|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\frac{\rho(j)+\rho(k)}{2} \\
&\quad+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)-\int_j^k\rho(\dagger)d\dagger\bigg| \\
&\leqslant \max\left\{y_0-j-\lambda\frac{k-j}{2},\frac{y_1-y_0}{2}+\left|\theta-\frac{y_0+y_1}{2}\right|,k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\} \\
&\quad\times\bigvee_j^k(\rho;r) \\
&\leqslant \max\left\{y_0-j-\lambda\frac{k-j}{2},\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|,k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\} \\
&\quad\times\bigvee_j^k(\rho;r)
\end{aligned}$$

where for $\lambda=0$ *we obtain* 1 *as the sharp constant while corresponding to value of* $\lambda=1$ *we get sharp constant* $1/2$.

Proof. The first inequality easily follows by substituting $\omega(\dagger)=\dfrac{1}{k-j}$ in Theorem 2.1. The second inequality follows since $\frac{1}{2}\leqslant\frac{1}{2^{\frac{1}{r}}}$, for all $r\in[1,\infty)$ and hence result is proved.

Now we do work for sharpness of the inequalities.

For an arbitrary real constant $C>0$ let second inequality in (2.3) holds such as

$$\begin{aligned}&\Bigg|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\\&-\int_j^k\rho(\dagger)d\dagger\Bigg|\leqslant C\cdot\max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|\right],\right.\\&\left.k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}\cdot\bigvee_j^k(\rho;\dagger).\end{aligned}\tag{2.4}$$

Now we discuss different cases to find best bound(s).

Case 1*:* If

$$\begin{aligned}&\max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|\right],k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}\\&=y_0-j-\lambda\frac{k-j}{2},\end{aligned}$$

then consequently, (2.4) reduces to

$$\begin{aligned}&\Bigg|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\\&-\int_j^k\rho(\dagger)d\dagger\Bigg|\leqslant C\left(y_0-j-\lambda\frac{k-j}{2}\right)\cdot\bigvee_j^k(\rho;\dagger)\end{aligned}\tag{2.5}$$

Choose $\rho:[j,k]\ \to\mathbb{R}$ given by

$$\rho(\dagger)=\begin{cases}0, & \text{if}\quad \dagger\in[j,k),\\ 1, & \text{if}\quad \dagger=k.\end{cases}$$

Clear, $\displaystyle\int_j^k\rho(\dagger)d\dagger=0$ and $\displaystyle\bigvee_j^k(\rho,\dagger)=1$. For $y_0=\theta=y_1=k$, we get $\frac{(k-j)(4-3\lambda)}{2}$ $\leqslant C(k-j)(2-\lambda)$, and hence $C\geqslant\frac{4-3\lambda}{2(2-\lambda)}$. The first case in first inequality is similar.

Case 2*:* If

$$\begin{aligned}&\max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|\right],k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}\\&=k-\lambda\frac{k-j}{2}-y_1,\end{aligned}$$

then consequently, (2.4) reduces to

$$\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\right.$$
$$\left.-\int_j^k\rho(\dagger)d\dagger\right|\leqslant C\left(k-\lambda\frac{k-j}{2}-y_1\right)\cdot\bigvee_j^k(\rho;\dagger) \tag{2.6}$$

Choose $\rho:[j,k]\to\mathbb{R}$ given by

$$\rho(\dagger)=\begin{cases}0, & \text{if} \quad \dagger\in(j,k]\\ 1, & \text{if} \quad \dagger=j.\end{cases}$$

Clear, $\int_j^k\rho(\dagger)d\dagger=0$ and $\bigvee_j^k(\rho,\dagger)=1$. For $y_0=\theta=y_1=j$, we get $\frac{(k-j)(4-3\lambda)}{2}\leqslant C(k-j)(2-\lambda)$, which implies that $C\geqslant\frac{4-3\lambda}{2(2-\lambda)}$. The second case in the first inequality goes similar.

Case 3*:* If

$$\max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|\right],k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}$$
$$=\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|,$$

then consequently, (2.4) reduces to

$$\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\right.$$
$$\left.-\int_j^k\rho(\dagger)d\dagger\right|\leqslant C\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \tag{2.7}$$

Choose $\rho:[j,k]\to\mathbb{R}$ given by

$$\rho(\dagger)=\begin{cases}0, & \text{if} \quad \dagger\in(j,k)\\ 1, & \text{if} \quad \dagger=j,k.\end{cases}$$

Clear, $\int_j^k\rho(\dagger)d\dagger=0$ and $\bigvee_j^k(\rho,r)=2^{\frac{1}{r}}$. For $\theta=\frac{y_0+y_1}{2}$, $y_0=j$ and $y_1=k$, we get $\frac{(k-j)(2-\lambda)}{2}\leqslant C(k-j)$ which implies that $C\geqslant\frac{2-\lambda}{2}$. This case produces a bit different constant.

Now we consider the last case for second inequality.

Case 4*:* If

$$\max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\theta-\frac{y_0+y_1}{2}\right|\right],k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}$$
$$=\lambda\frac{k-j}{2},$$

then consequently, (2.4) reduces to

$$\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\right.$$
$$\left.-\int_j^k\rho(\dagger)d\dagger\right|\leqslant C\lambda\frac{k-j}{2}\cdot\bigvee_j^k(\rho;\dagger)$$

Choose $\rho:[j,k]\to\mathbb{R}$ given by

$$\rho(\dagger)=\begin{cases}0, & \text{if} \quad \dagger\in[j,k)\\ 1, & \text{if} \quad \dagger=k.\end{cases}$$

Clearly, $\int_j^k\rho(\dagger)d\dagger=0$ and $\bigvee_j^k(\rho;\dagger)=1$. For $\theta=\frac{j+k}{2}$, $y_0=j$, $y_1=k$ we get

$$\frac{(k-j)(2-\lambda)}{4}\leqslant C\lambda\frac{k-j}{2},$$

which implies that $C\geqslant\frac{2-\lambda}{2\lambda}$.

Finally we consider our last case for our first inequality.

Case 5*:* If

$$\max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2}+\left|\theta-\frac{y_0+y_1}{2}\right|\right],k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}$$
$$=\frac{y_1-y_0}{2}+\left|\theta-\frac{y_0+y_1}{2}\right|,$$

then consequently, (2.4) reduces to

$$\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\right.$$
$$\left.-\int_j^k\rho(\dagger)d\dagger\right|\leqslant C\left[\frac{y_1-y_0}{2}+\left|\theta-\frac{y_0+y_1}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger).$$

Choose $\rho:[j,k]\to\mathbb{R}$ given by

$$\rho(\dagger)=\begin{cases}0, & \text{if} \quad \dagger\in[j,k)\\ 1, & \text{if} \quad \dagger=k.\end{cases}$$

Clearly, $\int_j^k\rho(\dagger)d\dagger=0$ and $\bigvee_j^k(\rho;\dagger)=1$. For $\theta=\frac{j+k}{2}$, $y_0=j$, $y_1=k$ we get

$$\frac{(k-j)(2-\lambda)}{4}\leqslant C\frac{(k-j)}{2},$$

which implies that $C \geqslant \frac{(2-\lambda)}{2}$.

Now we have obtained three different values namely, $C \geqslant \frac{4-3\lambda}{2(2-\lambda)}$, $C \geqslant \frac{2-\lambda}{2}$ and $C \geqslant \frac{2-\lambda}{2\lambda}$,

To find the unique best possible constant in the first and second inequality we equate the three different values in order to find value(s) of λ.

$$\frac{4-3\lambda}{2(2-\lambda)} = \frac{2-\lambda}{2}$$

which gives us two different values of λ, $\lambda = 0$ and $\lambda = 1$.

$$\frac{4-3\lambda}{2(2-\lambda)} = \frac{2-\lambda}{2\lambda}$$

which gives us repeated values of λ, $\lambda = 1$.

$$\frac{2-\lambda}{2\lambda} = \frac{2-\lambda}{2}$$

which gives us only value of λ that is $\lambda = 1$.

Corresponding to $\lambda = 0$, in all cases we get $C \geqslant 1$ and hence 1 is our best possible constant. Corresponding to $\lambda = 1$, in all cases we get $C \geqslant \frac{1}{2}$ and hence $\frac{1}{2}$ is our best possible constant.

It should be noted that for $\lambda = 0$, our Case 4 would not be discussed because λ is in multiple and we cannot do further work in this case. In this way Case 4 would be vanish for $\lambda = 0$ it would only work for $\lambda = 1$. $\square$

REMARK 2.3. Previous corollary may be summarize in following manner: Under the assumptions of Corollary 2.2, we obtain here two inequalities for two different values of λ with sharp constants as follows:

$$\begin{aligned}
&\left|(\theta - j)\rho(y_0) + (k-\theta)\rho(y_1) - \int_j^k \rho(\dagger)d\dagger\right| \\
&\leqslant \max\left\{y_0 - j, \frac{y_1 - y_0}{2} + \left|\theta - \frac{y_0+y_1}{2}\right|, k - y_1\right\} \bigvee_j^k(\rho;r) \\
&\leqslant \max\left\{y_0 - j, \frac{y_1 - y_0}{2^{\frac{1}{r}}} + \left|\theta - \frac{y_0+y_1}{2}\right|, k - y_1\right\} \bigvee_j^k(\rho;r)
\end{aligned}$$

here in last two inequalities 1 is sharp constant.

$$\begin{aligned}
&\left|\left(\theta - j - \frac{k-j}{2}\right)\rho(y_0) + \frac{k-j}{2}\frac{\rho(j)+\rho(k)}{2}\right. \\
&\left. + \left(k - \frac{k-j}{2} - \theta\right)\rho(y_1) - \int_j^k \rho(\dagger)d\dagger\right| \\
&\leqslant \max\left\{y_0 - j - \frac{k-j}{2}, \frac{y_1-y_0}{2} + \left|\theta - \frac{y_0+y_1}{2}\right|, k - \frac{k-j}{2} - y_1, \frac{k-j}{2}\right\} \bigvee_j^k(\rho;r)
\end{aligned}$$

$$\leqslant \max\left\{y_0 - j - \frac{k-j}{2}, \frac{y_1-y_0}{2^{\frac{1}{r}}} + \left|\theta - \frac{y_0+y_1}{2}\right|, k - \frac{k-j}{2} - y_1, \frac{j-k}{2}\right\} \bigvee_j^k(\rho;r)$$

here in last two inequalities $\frac{1}{2}$ is sharp constant.

COROLLARY 2.4. *Let $\rho : [j,k] \to \mathbb{R}$ be of infinite variation on $[j,k]$. Then inequality*

$$\begin{aligned}
&\left|\left(\theta - j - \lambda\frac{k-j}{2}\right)\rho(y_0) + \lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right) + \left(k - \lambda\frac{k-j}{2} - \theta\right)\rho(y_1)\right.\\
&\left.-\int_j^k \rho(\dagger)d\dagger\right|\\
\leqslant &\max\left\{y_0 - j - \lambda\frac{k-j}{2}, \left[\frac{y_1-y_0}{2} + \left|\theta - \frac{y_0+y_1}{2}\right|\right], k - \lambda\frac{k-j}{2} - y_1, \lambda\frac{k-j}{2}\right\}\\
&\times Osc(\rho,[j,k])
\end{aligned}$$

holds for all $j \leqslant y_0 \leqslant \theta \leqslant y_1 \leqslant k$, where for $\lambda = 0$ we obtain 1 as the sharp constant while corresponding to value of $\lambda = 1$ we get sharp constant $1/2$.

Proof. The result is an immediate consequence of Theorem 2.1 and Lemma 1.13. □

COROLLARY 2.5. *Let all the assumptions of Corollary 2.2 be valid. Then the following inequalities hold:*

(1) *The Ostrowski type inequality:*

$$\begin{aligned}
&\left|(k-j)(1-\lambda)\rho(\theta) - \lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right) - \int_j^k \rho(\dagger)d\dagger\right|\\
&\leqslant \max\left(\theta - j - \lambda\frac{k-j}{2}, k - \lambda\frac{k-j}{2} - \theta, \lambda\frac{k-j}{2}\right)\cdot\bigvee_j^k(\rho;\dagger) \qquad (2.8)
\end{aligned}$$

for all $\theta \in [j,k]$.

(2) *The generalized two-point inequality:*

$$\begin{aligned}
&\left|(k-j)(1-\lambda)\left(\frac{\rho(y_0)+\rho(y_1)}{2}\right) + \lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right) - \int_j^k \rho(\dagger)d\dagger\right|\\
\leqslant &\max\left\{y_0 - j - \lambda\frac{k-j}{2}, \left[\frac{y_1-y_0}{2} + \left|\frac{j+k}{2} - \frac{y_0+y_1}{2}\right|\right], k - \lambda\frac{k-j}{2} - y_1, \lambda\frac{k-j}{2}\right\}\\
&\times\bigvee_j^k(\rho;\dagger)
\end{aligned}$$

$$\leqslant \max\left\{y_0-j-\lambda\frac{k-j}{2},\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\frac{j+k}{2}-\frac{y_0+y_1}{2}\right|\right],k-\lambda\frac{k-j}{2}-y_1,\lambda\frac{k-j}{2}\right\}$$
$$\times\bigvee_j^k(\rho;\dagger) \tag{2.9}$$

for all $j\leqslant y_0\leqslant\frac{j+k}{2}\leqslant y_1\leqslant k$. *Both inequalities are sharp corresponding to values of* $\lambda=0$ *and* $\lambda=1$ *as discussed earlier. In special case, if one chooses* $y_0=j$ *and* $y_1=k$*, then we refer to the trapezoid type inequality*

$$\begin{aligned}&\left|(k-j)\left(1-\frac{\lambda}{2}\right)\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|\\&\leqslant\max\left\{\frac{k-j}{2},\lambda\frac{k-j}{2}\right\}\bigvee_j^k(\rho;\dagger)\\&\leqslant\max\left\{\frac{k-j}{2^{\frac{1}{r}}},\lambda\frac{k-j}{2}\right\}\bigvee_j^k(\rho;\dagger).\end{aligned}\tag{2.10}$$

(3) *The companion of Ostrowski type inequality*

$$\left|(k-j)(1-\lambda)\left(\frac{\rho(y_0)+\rho(y_1)}{2}\right)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|$$
$$\leqslant\max\left\{\lambda-j-\lambda\frac{k-j}{2},\frac{j+k}{2},\lambda\frac{k-j}{2}\right\}\bigvee_j^k(\rho;\dagger) \tag{2.11}$$

for all $\theta\in[j,\frac{j+k}{2}]$. *Previous remarks about* λ *are still valid here.*

(4) *The Cerone-Dragomir type inequality:*

$$\begin{aligned}&\left|(\theta-j)\rho(j)+(k-\theta)\rho(k)-\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|\\&\leqslant\max\left\{\frac{k-j}{2}+\left|\theta-\frac{j+k}{2}\right|,\lambda\frac{k-j}{2}\right\}\cdot\bigvee_j^k(\rho;\dagger)\\&\leqslant\max\left\{\frac{k-j}{2^{\frac{1}{r}}}+\left|\theta-\frac{j+k}{2}\right|,\lambda\frac{k-j}{2}\right\}\cdot\bigvee_j^k(\rho;\dagger)\end{aligned}\tag{2.12}$$

for all $\theta\in[j,k]$. *One can choose* $\lambda=0$ *and* $\lambda=1$ *to find corresponding best possible constant.*

(5) *The midpoint-trapezoid Ostrowski's type inequality:*

$$\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho\left(\frac{j+\theta}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho\left(\frac{\theta+k}{2}\right)\right.$$
$$\left.+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|$$

$$\leqslant \frac{\max}{2}\left\{\theta-j-\lambda(k-j),\frac{k-j}{2}+\left|\theta-\frac{j+k}{2}\right|,(k-\theta)-\lambda(k-j),\lambda(k-j)\right\}$$
$$\times\bigvee_j^k(\rho;\dagger)$$
$$\leqslant \frac{\max}{2}\left\{\theta-j-\lambda(k-j),\frac{k-j}{2^{\frac{1}{r}}}+\left|\theta-\frac{j+k}{2}\right|,(k-\theta)-\lambda(k-j),\lambda(k-j)\right\}$$
$$\times\bigvee_j^k(\rho;\dagger) \tag{2.13}$$

for all $\theta \in [j,k]$. *To calculate the corresponding best possible constant values one can use* $\lambda=0$ *and* $\lambda=1$ *respectively.*

Proof. (1) The Ostrowski type inequality: setting $y_0=\theta=y_1$, in (2.3), then we get inequality

$$\left|(k-j)(1-\lambda)\rho(\theta)-\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|$$
$$\leqslant \max\left(\theta-j-\lambda\frac{k-j}{2},k-\lambda\frac{k-j}{2}-\theta,\lambda\frac{k-j}{2}\right)\cdot\bigvee_j^k(\rho;\dagger)$$

and this proves (2.8).

(2) The generalized two-point inequality: Setting $\theta=\frac{j+k}{2}$, in (2.3), then we get the result (1.2).

(3) The companion of Ostrowski type inequality: Setting $\theta=\frac{j+k}{2}$, $y_0=\lambda$ and $y_1=j+k-\lambda$ in (2.3).

$$\left|(k-j)(1-\lambda)\left(\frac{\rho(y_0)+\rho(y_1)}{2}\right)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|$$
$$\leqslant \max\left\{\lambda-j-\lambda\frac{k-j}{2},\frac{j+k}{2},\lambda\frac{k-j}{2}\right\}\bigvee_j^k(\rho;\dagger).$$

(4) The Cerone-Dragomir type inequality: Setting $y_0=c$ and $y_1=k$, in (2.3), then we get the desired result (2.12).

(5) The midpoint-trapezoid Ostrowski's type inequality: Setting $y_0=\frac{j+\theta}{2}$ and $y_1=\frac{\theta+k}{2}$, we get

$$\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho\left(\frac{j+\theta}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho\left(\frac{\theta+k}{2}\right)\right.$$
$$\left.+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k\rho(\dagger)d\dagger\right|$$

$$\leqslant \frac{\max}{2}\left\{\theta - j - \lambda(k-j), \frac{k-j}{2} + \left|\theta - \frac{j+k}{2}\right|, k-\theta-\lambda(k-j), \lambda(k-j)\right\}$$
$$\times \bigvee_j^k(\rho;\dagger)$$
$$\leqslant \frac{\max}{2}\left\{\theta - j - \lambda(k-j), \frac{k-j}{2^{\frac{1}{r}}} + \left|\theta - \frac{j+k}{2}\right|, k-\theta-\lambda(k-j), \lambda(k-j)\right\}$$
$$\times \bigvee_j^k(\rho;\dagger) \tag{2.14}$$

Now, since

$$M := \left[\frac{k-j}{2} + \left|\theta - \frac{j+k}{2}\right|\right] = \max_{\theta\in[j,k]}\{\theta - j - \lambda(k-j), k-\theta-\lambda(k-j)\}$$

therefore, $M \geqslant (\theta - j - \lambda(k-j))$ and $M \geqslant (k-\theta-\lambda(k-j))$. Thus,

$$\max\left\{\theta - j - \lambda(k-j), \frac{k-j}{2} + \left|\theta - \frac{j+k}{2}\right|, k-\theta-\lambda(k-j)\right\} = M,$$

i. e.

$$\left|\left(\theta - j - \lambda\frac{k-j}{2}\right)\rho\left(\frac{j+\theta}{2}\right) + \left(k - \lambda\frac{k-j}{2} - \theta\right)\rho\left(\frac{\theta+k}{2}\right)\right.$$
$$\left. + \lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right) - \int_j^k \rho(\dagger)d\dagger\right|$$
$$\leqslant \frac{1}{2}\left[\frac{(k-j)(1-2\lambda)}{2} + \left|\theta - \frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger)$$
$$\leqslant \frac{1}{2}\left[\frac{(k-j)(1-2\lambda)}{2^{\frac{1}{r}}} + \left|\theta - \frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \tag{2.15}$$

and the last inequality holds since $\frac{1}{2} \leqslant \frac{1}{2^{\frac{1}{r}}}$, for all $\dagger \in [1,\infty)$ and this proves the inequality (2.13).

The sharpness of each inequality follows from (2.3). Hence, the proof is completely established. □

If we put $\lambda = 0$ in Corollary 2.5, then we get the following results which could be found in [1].

COROLLARY 2.6. *Let* $\rho : [j,k] \to \mathbb{R}$ *be of bounded* r*-variation* $(1 \leqslant r < \infty)$. *Then following inequalities hold:*

(1) *The Ostrowski's inequality:*

$$\left|(k-j)\rho(\theta)-\int_j^k \rho(\dagger)d\dagger\right| \leqslant \left[\frac{k-j}{2}+\left|\theta-\frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger)$$
$$\leqslant \left[\frac{k-j}{2^{\frac{1}{r}}}+\left|\theta-\frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \tag{2.16}$$

for all $\theta\in[j,k]$. *The constants* 2^{-1} *and* $2^{-\frac{1}{r}}$ *are the best possible. For instant, choosing* $\theta=\frac{j+k}{2}$, *we get the Midpoint inequality:*

$$\left|(k-j)\rho\left(\frac{j+k}{2}\right)-\int_j^k \rho(\dagger)d\dagger\right| \leqslant \frac{k-j}{2}\cdot\bigvee_j^k(\rho;\dagger)\leqslant\frac{k-j}{2^{\frac{1}{r}}}\cdot\bigvee_j^k(\rho;\dagger)$$

(2) *The generalized two-point inequality:*

$$\left|(k-j)\left(\frac{\rho(y_0)+\rho(y_1)}{2}\right)-\int_j^k \rho(\dagger)d\dagger\right|$$
$$\leqslant \max\left\{y_0-j,\left[\frac{y_1-y_0}{2}+\left|\frac{j+k}{2}-\frac{y_0+y_1}{2}\right|\right],k-y_1\right\}\bigvee_j^k(\rho;\dagger)$$
$$\leqslant \max\left\{y_0-j,\left[\frac{y_1-y_0}{2^{\frac{1}{r}}}+\left|\frac{j+k}{2}-\frac{y_0+y_1}{2}\right|\right],k-y_1\right\}\bigvee_j^k(\rho;\dagger),$$

for all $j\leqslant y_0\leqslant\frac{j+k}{2}\leqslant y_1\leqslant k$. *Both inequalities are sharp. In special case, if one chooses* $y_0=j$ *and* $y_1=k$, *then we refer to the trapezoid inequality*

$$\left|(k-j)\left(\frac{\rho(j)+\rho(k)}{2}\right)-\int_j^k \rho(\dagger)d\dagger\right| \leqslant \frac{k-j}{2}\bigvee_j^k(\rho;\dagger)\leqslant\frac{k-j}{2^{\frac{1}{r}}}\bigvee_j^k(\rho;\dagger)$$

(3) *The companion of Ostrowski type inequality*

$$\left|(k-j)\frac{\rho(j+k-\theta)+\rho(\theta)}{2}-\int_j^k \rho(\dagger)d\dagger\right|$$
$$\leqslant \left[\frac{k-j}{4}+\frac{\theta}{2}-\left|\theta-\frac{3j+k}{4}\right|\right]\bigvee_j^k(\rho;r)$$
$$\leqslant \left[\frac{k-j}{4^{\frac{1}{p}}}+\frac{\theta}{2}-\left|\theta-\frac{3j+k}{4}\right|\right]\bigvee_j^k(\rho;r) \tag{2.17}$$

for all $\theta\in[j,\frac{j+k}{2}]$, *The constant* 4^{-1} *and* $4^{\frac{-1}{r}}$ *are the best possible.*

(4) *The Cerone-Dragomir type inequality:*

$$\begin{aligned}
&\left|(\theta-j)\rho(j)+(k-\theta)\rho(k)-\int_j^k \rho(\dagger)d\dagger\right| \\
&\leqslant \left[\frac{k-j}{2}+\left|\theta-\frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \\
&\leqslant \left[\frac{k-j}{2^{\frac{1}{r}}}+\left|\theta-\frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \qquad (2.18)
\end{aligned}$$

for all $\theta\in[j,k]$*, The constant* 2^{-1} *and* $2^{\frac{-1}{r}}$ *are the best possible for all* $r\geqslant 1$.

(5) *The midpoint-trapezoid Ostrowski's type inequality:*

$$\begin{aligned}
&\left|(\theta-j)\rho\left(\frac{j+\theta}{2}\right)+(k-\theta)\rho\left(\frac{\theta+k}{2}\right)-\int_j^k \rho(\dagger)d\dagger\right| \\
&\leqslant \frac{1}{2}\left[\frac{k-j}{2}+\left|\theta-\frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \\
&\leqslant \frac{1}{2}\left[\frac{k-j}{2^{\frac{1}{r}}}+\left|\theta-\frac{j+k}{2}\right|\right]\cdot\bigvee_j^k(\rho;\dagger) \qquad (2.19)
\end{aligned}$$

for all $\theta\ \in[j,k]$*, The constant* $\frac{1}{2}$ *in the first inequality, and the constants* $\frac{1}{2},\frac{1}{2^{\frac{1}{r}}}$ *in the second inequality are the best possible.*

COROLLARY 2.7. *Let* I *be an interval of real numbers, such that* $j,k\in I^0$: *the interior of* I *where* $j<k$. *Let* $\rho:[j,k]\to\mathbb{R}$ *be of bounded* r*-variation* $(1\leqslant r<\infty)$ *on I. If* ρ *is differentiable on* $[j,k]$*, then inequality*

$$\begin{aligned}
&\left|\left(\theta-j-\lambda\frac{k-j}{2}\right)\rho(y_0)+\lambda\frac{k-j}{2}\left(\frac{\rho(j)+\rho(k)}{2}\right)+\left(k-\lambda\frac{k-j}{2}-\theta\right)\rho(y_1)\right. \\
&\left.-\int_j^k \rho(\dagger)d\dagger\right| \\
&\leqslant \max\left\{y_0-j-\lambda\frac{k-j}{2},\frac{y_1-y_0}{2}+\left|\theta-\frac{y_0+y_1}{2}\right|,k-\lambda\frac{k-j}{2}-y_1\right\}\|\rho'\|_r,
\end{aligned}$$

holds for all $j\leqslant y_0\leqslant\theta\leqslant y_1\leqslant k$ *with* $\lambda\in[0,1]$.

Proof. The result is an immediate consequence of Corollary 2.2 and Lemma 1.13; since ρ is differentiable on $[j,k]$, then we have

$$\bigvee_j^k(\rho;r)=\left(\int_j^k|\rho'(\dagger)|^r d\dagger\right)^{\frac{1}{r}}=\|\rho'\|_r,\quad 1\leqslant r<\infty,$$

which proof the inequality. □

If we put $\lambda = 0$, then we get the result of Corollary 3 of [1] which may be stated as:

COROLLARY 2.8. *Let I be an interval of real numbers, such that $j,k \in I^0$: the interior of I where $j<k$. Let $\rho : [j,k] \to \mathbb{R}$ be of bounded r-variation $(1 \leqslant r < \infty)$ on I. If ρ is differentiable on $[j,k]$, then following inequality holds*

$$\left|(\theta - j)\rho(y_0) + (k-\theta)\rho(y_1) - \int_j^k \rho(s)ds\right|$$
$$\max\left\{(y_0 - j), \left(\frac{y_1 - y_0}{2} + \left|\theta - \frac{y_0+y_1}{2}\right|, (k-y_1)\right)\right\} \|\rho'\|_p.$$

3. $\mathbf{L_p}$-bounds for Lipshitz functions $(1 \leqslant p < \infty)$

In this section we obtain L_p-bounds for our proposed generalized Two-point Ostrowski's inequality using the lemma in Section 2.

THEOREM 3.1. *Let $\rho : [j,k] \to \mathbb{R}$ be of M-Lipschitz on $[j,k]$ and $\lambda \in [0,1]$. Then we have*

$$\begin{aligned}
&\Bigg|\rho(y_0)\int_\alpha^{y_0} \omega(\dagger_0)d\dagger_0 - \rho(j)\int_\alpha^{j} \omega(\dagger_0)d\dagger_0 + \rho(y_1)\int_\theta^{y_1} \omega(\dagger_0)d\dagger_0 \\
&-\rho(y_0)\int_\theta^{y_0} \omega(\dagger_0)d\dagger_0 + \rho(k)\int_\beta^{k} \omega(\dagger_0)d\dagger_0 - \rho(y_1)\int_\beta^{y_1} \omega(\dagger_0)d\dagger_0 - \int_j^k \omega(\dagger)\rho(\dagger)d\dagger\Bigg| \\
&\leqslant \frac{Lip_M(\rho)}{(p+1)^{\frac{1}{p}}} \max\{(y_0-j)^{1-\frac{1}{p}}, (y_1-y_0)^{1-\frac{1}{p}}, (k-y_1)^{1-\frac{1}{p}}\} \\
&\left[\left(\left(\int_j^{y_0}\left(\int_\alpha^{\dagger} \omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\right) + \left(\left(\int_{y_0}^{y_1}\left(\int_\theta^{\dagger} \omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\right)\right. \\
&\left.+\left(\left(\int_{y_1}^{k}\left(\int_\beta^{\dagger} \omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\right)\right]
\end{aligned} \tag{3.1}$$

where $\alpha = j + \lambda\frac{k-j}{2}$, $\beta = k - \lambda\frac{k-j}{2}$, $\forall\, \theta \in [y_0,y_1] \subseteq [j,k]$ with $\lambda \in [0,1]$ and ω is a probability density function.

Proof. Employing the triangle inequality on the identity (2.2) and then using Lemma 1.14, we get

$$\begin{aligned}
&\left|\int_j^k K(\dagger; y_0, \theta, y_1)d\rho(\dagger)\right| \\
&\leqslant \left|\int_j^{y_0}\int_\alpha^{\dagger} \omega(\dagger_0)d\dagger_0 d\rho(\dagger)\right| + \left|\int_{y_0}^{y_1}\int_\theta^{\dagger} \omega(\dagger_0)d\dagger_0 d\rho(\dagger)\right| + \left|\int_{y_1}^{k}\int_\beta^{\dagger} \omega(\dagger_0)d\dagger_0 d\rho(\dagger)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant Lip_M(\rho)\left(\int_j^{y_0}\left(\int_\alpha^{\dagger}\omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}(y_0-j)^{1-\frac{1}{p}}\\
&\quad+Lip_M(\rho)(y_1-y_0)^{1-\frac{1}{p}}\left(\int_{y_0}^{y_1}\left(\int_\theta^{\dagger}\omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\\
&\quad+Lip_M(\rho)(k-y_1)^{1-\frac{1}{p}}\left(\int_{y_1}^{k}\left(\int_\beta^{\dagger}\omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\\
&\leqslant Lip_M(\rho)\max\{(y_0-j)^{1-\frac{1}{p}},(y_1-y_0)^{1-\frac{1}{p}},(k-y_1)^{1-\frac{1}{p}}\}\\
&\quad\left[\left(\int_j^{y_0}\left(\int_\alpha^{\dagger}\omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\right.\\
&\quad\left.+\left(\int_{y_0}^{y_1}\left(\int_\theta^{\dagger}\omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}+\left(\int_{y_1}^{k}\left(\int_\beta^{\dagger}\omega(\dagger_0)d\dagger_0\right)^p d\dagger\right)^{\frac{1}{p}}\right]. \quad \square
\end{aligned}
$$

REMARK 3.2. Another bound can be obtained by applying (1.13) instead of (1.8).

If we put $\omega(\dagger)=\frac{1}{k-j}$ in (3.1), then we get the following result.

COROLLARY 3.3. *Let $\rho:[j,k]\to\mathbb{R}$ be of M-Lipschitz on $[j,k]$, then*

$$
\begin{aligned}
&\left|(\theta-\alpha)\rho(y_0)+(\beta-\theta)\rho(y_1)-\int_j^k\rho(s)ds\right|\\
&\leqslant\frac{Lip_M(\rho)}{(p+1)^{\frac{1}{p}}}\cdot\max\{(y_0-\alpha)^{1-\frac{1}{p}},(y_1-y_0)^{1-\frac{1}{p}},(\beta-y_1)^{1-\frac{1}{p}}\}\\
&\quad\times[(y_0-\alpha)^{\frac{p+1}{p}}+\{(y_1-\theta)^{p+1}+(\theta-y_0)^{p+1^{\frac{1}{p}}}\}+(\beta-y_1)^{\frac{p+1}{p}}],
\end{aligned}
$$

for all $j\leqslant y_0\leqslant\theta\leqslant y_1\leqslant k$ and $\forall p\geqslant 1$, where $\alpha=j+\lambda\frac{k-j}{2}$, $\beta=k-\lambda\frac{k-j}{2}$ with $\lambda\in[0,1]$.

REMARK 3.4. If we put $\lambda=0$ then we get following results as special case of previous corollary which can be found in [1]. We can also state similar results for $\lambda=1$ as well.

COROLLARY 3.5. *Let $\rho:[j,k]\to\mathbb{R}$ be of M-Lipschitz on $[j,k]$, then*

$$
\begin{aligned}
&\left|(\theta-j)\rho(y_0)+(k-\theta)\rho(y_1)-\int_j^k\rho(s)ds\right|\\
&\leqslant\frac{Lip_M(\rho)}{(p+1)^{\frac{1}{p}}}\cdot\max\{(y_0-j)^{1-\frac{1}{p}},(y_1-y_0)^{1-\frac{1}{p}},(k-y_1)^{1-\frac{1}{p}}\}\\
&\quad\times[(y_0-j)^{\frac{p+1}{p}}+\{(y_1-\theta)^{p+1}+(\theta-y_0)^{p+1}\}^{\frac{1}{p}}+(k-y_1)^{\frac{p+1}{p}}],
\end{aligned}
$$

for all $j\leqslant y_0\leqslant\theta\leqslant y_1\leqslant k$ and $\forall p\geqslant 1$.

Further consequences of Corollary 3.5 are as under (see also [1]):

COROLLARY 3.6. *Let all assumptions of Corollary 3.5 be valid. Then the following inequalities holds:*

(1) The Ostrowski inequality:

$$\begin{aligned}&\left|(k-j)\rho(\theta)-\int_j^k \rho(s)ds\right| \\ &\leqslant Lip_M(\rho)\cdot\left[\frac{k-j}{2}+\left|\theta-\frac{j+k}{2}\right|\right]^{1-\frac{1}{p}}\cdot\frac{(\theta-j)^{\frac{p+1}{p}}+(k-\theta)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}}\end{aligned}$$

$\forall\ \theta\in[j,k]$. *For instance, choosing* $\theta=\frac{j+k}{2}$, *we get the Midpoint inequality:*

$$\left|(k-j)\rho\left(\frac{j+k}{2}\right)-\int_j^k \rho(s)ds\right|\leqslant Lip_M(\rho)\frac{(k-j)^2}{2(p+1)/p},$$

(2) *The generalized two-point inequality: For all* $j\leqslant y_0\leqslant\frac{j+k}{2}\leqslant y_1\leqslant k$, *we have*

$$\begin{aligned}&\left|(k-j)\frac{\rho(y_0)+\rho(y_1)}{2}-\int_j^k \rho(s)ds\right| \\ &\leqslant\frac{Lip_M(\rho)}{(p+1)\frac{1}{p}}\max\left\{(y_0-j)^{1-\frac{1}{p}},(y_1-y_0)^{1-\frac{1}{p}},(k-y_1)^{1-\frac{1}{p}}\right\} \\ &\quad\times\left[(y_0-j)^{\frac{p+1}{p}}+\left\{\left(y_1-\frac{j+k}{2}\right)^{p+1}+\left(\frac{j+k}{2}-y_0\right)^{p+1}\right\}^{\frac{1}{p}}+(k-y_1)^{\frac{p+1}{p}}\right].\end{aligned}$$

In special case, if we choose $y_0=j$ *and* $y_1=k$, *then we get the trapezoid inequality*

$$\left|(k-j)\cdot\frac{\rho(j)+\rho(k)}{2}-\int_j^k \rho(s)ds\right|\leqslant Lip_M(\rho)\cdot\frac{(k-j)^2}{2(p+1)\frac{1}{p}}.$$

(3) *The companion of Ostrowski inequality: For all* $h\in[j,\frac{j+k}{2}]$, *we have*

$$\begin{aligned}&\left|(k-j)\cdot\frac{\rho(h)+\rho(j+k-h)}{2}-\int_j^k \rho(s)ds\right| \\ &\leqslant\frac{Lip_M(\rho)}{(p+1)^{\frac{1}{p}}}\cdot\left[\frac{k-j}{4}+\left|h-\frac{3j+k}{4}\right|\right]^{1-\frac{1}{p}}\left[2(h-j)^{\frac{p+1}{p}}+2^{\frac{1}{p}}\left(\frac{j+k}{2}-h\right)^{\frac{p+1}{p}}\right].\end{aligned}$$

(4) *The Cerone–Dragomir inequality: For all* $\theta\in[j,k]$, *we have*

$$\begin{aligned}&\left|(\theta-j)\rho(j)+(k-\theta)\rho(k)-\int_j^k \rho(s)ds\right| \\ &\leqslant Lip_M(\rho)\cdot(k-j)^{1-\frac{1}{p}}\frac{\left((k-\theta)^{p+1}+(\theta-j)^{p+1}\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}.\end{aligned}$$

(5) *The midpint–trapezoid–Ostrowski's inequality: For all* $\theta \in [j,k]$, *we have*

$$\left|(\theta - j)\cdot\rho\left(\frac{j+\theta}{2}\right) + (k-\theta)\rho\left(\frac{j+\theta}{2}\right) - \int_j^k \rho(s)ds\right|$$
$$\leqslant \frac{Lip_M(\rho)}{(p+1)^{\frac{1}{p}}}\cdot\left(\frac{k-j}{2}\right)^{1-\frac{1}{p}}$$
$$\times\left[\left(\frac{\theta-j}{2}\right)^{\frac{p+1}{p}} + \left\{\left(\frac{k-\theta}{2}\right)^{p+1} + \left(\frac{\theta-j}{2}\right)^{p+1}\right\}^{\frac{1}{p}} + \left(\frac{j-\theta}{2}\right)^{\frac{p+1}{p}}\right].$$

4. Bounds in L_∞-norm

Here we state some results which involves L_∞-norm.

THEOREM 4.1. *Let* I *be an interval of real numbers, such that* $j,k \in I^0$; *the interior of* I *where* $j<k$. *Let* $\rho : I \to \mathbb{R}$ *be a differentiable mapping whose first derivative is bounded on* $[j,k]; \sup_{\dagger\in[j,k]}|\rho(\dagger)| = \|\rho'\|_{\infty,[j,k]} < \infty$, *then*

$$\left|\rho(y_0)\int_\alpha^{y_0}\omega(\dagger_0)d\dagger_0 - \rho(j)\int_\alpha^{j}\omega(\dagger_0)d\dagger_0 + \rho(y_1)\int_\theta^{y_1}\omega(\dagger_0)d\dagger_0\right.$$
$$\left.-\rho(y_0)\int_\theta^{y_0}\omega(\dagger_0)d\dagger_0 + \rho(k)\int_\beta^{k}\omega(\dagger_0)d\dagger_0 - \rho(y_1)\int_\beta^{y_1}\omega(\dagger_0)d\dagger_0 - \int_j^k\omega(\dagger)\rho(\dagger)d\dagger\right|$$
$$\leqslant\left[\int_j^{y_0}\left|\int_\alpha^{\dagger}\omega(\dagger)d\dagger_0\right|d\dagger + \int_{y_0}^{\theta}\left|\int_\theta^{\dagger}\omega(\dagger_0)d\dagger_0\right|d\dagger + \int_\theta^{y_1}\left|\int_\theta^{\dagger}\omega(\dagger_0)d\dagger_0\right|d\dagger\right.$$
$$\left.+\int_{y_1}^{k}\left|\int_\beta^{\dagger}\omega(\dagger_0)d\dagger_0\right|d\dagger\right]\|\rho'\|_{\infty,[j,k]}$$

where $\alpha = j+\lambda\frac{k-j}{2}$, $\beta = k-\lambda\frac{k-j}{2}$, $\forall\,\theta\in[y_0,y_1]\subseteq[j,k]$ *with* $\lambda\in[0,1]$ *and* ω *is a probability density function.*

Proof. Employing the triangle inequality on the identity (2.2), since ρ' is bounded on $[j,k]$, we have

$$\left|\int_j^k K_w(\dagger;\lambda,y_0,\theta,y_1)\rho'(\dagger)d\dagger\right|$$
$$\leqslant\left|\int_j^{y_0}\left(\int_\alpha^{\dagger}\omega(\dagger_0)d\dagger_0\right)\rho'(\dagger)d\dagger\right| + \left|\int_{y_0}^{y_1}\left(\int_\theta^{\dagger}\omega(\dagger_0)d\dagger_0\right)\rho'(\dagger)d\dagger\right|$$
$$+\left|\int_{y_1}^{k}\left(\int_\beta^{\dagger}\omega(\dagger_0)d\dagger_0\right)\rho'(\dagger)d\dagger\right|$$
$$\leqslant\int_j^{y_0}\left|\int_\alpha^{\dagger}\omega(\dagger_0)d\dagger_0\right||\rho'(\dagger)|d\dagger + \int_{y_0}^{y_1}\left|\int_\theta^{\dagger}\omega(\dagger_0)d\dagger_0\right||\rho'(\dagger)|d\dagger$$
$$+\int_{y_1}^{k}\left|\int_\beta^{\dagger}\omega(\dagger_0)d\dagger_0\right||\rho'(\dagger)|d\dagger$$

$$\leqslant \|\rho'\|_{\infty,[j,y_0]} \int_j^{y_0} \left| \int_\alpha^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger + \|\rho'\|_{\infty,[y_0,\theta]} \int_{y_0}^{\theta} \left| \int_\theta^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger$$
$$+ \|\rho'\|_{\infty,[\theta,y_1]} \int_\theta^{y_1} \left| \int_\theta^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger + \|\rho'\|_{\infty,[y_1,k]} \int_{y_1}^{k} \left| \int_\beta^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger$$
$$\leqslant \left[\int_j^{y_0} \left| \int_\alpha^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger + \int_{y_0}^{\theta} \left| \int_\theta^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger + \int_\theta^{y_1} \left| \int_\theta^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger \right.$$
$$\left. + \int_{y_1}^{k} \left| \int_\beta^\dagger \omega(\dagger_0) d\dagger_0 \right| d\dagger \right] \|\rho'\|_{\infty,[j,k]}. \quad \square$$

If we put $\omega(\dagger) = \frac{1}{k-j}$ in (3.1), then we get the upcoming result.

COROLLARY 4.2. *Under the assumptions of Theorem 4.1 with* $\omega(\dagger) = \dfrac{1}{k-j}$ *we have*

$$\left| (\theta-\alpha)\rho(y_0) + \lambda \frac{k-j}{2} \frac{\rho(j)+\rho(k)}{2} + (\beta-\theta)\rho(y_1) - \int_j^k \rho(s)ds \right|$$
$$\leqslant \left[\left(\frac{(\theta-\alpha)^2 + (\beta-\theta)^2}{4} \right) + \left(y_0 - \frac{\alpha+\theta}{2} \right)^2 + \left(y_1 - \frac{\beta+\theta}{2} \right) \right] \cdot \|\rho'\|_{\infty,[j,k]},$$

for all $j \leqslant y_0 \leqslant \theta \leqslant y_1 \leqslant k$.

COROLLARY 4.3. *Under the assumptions of Theorem 4.1 with* $\omega(\dagger) = \dfrac{1}{k-j}$ *and* $\lambda = 0$ *we have*

$$\left| (\theta-j)\rho(y_0) + (k-\theta)\rho(y_1) - \int_j^k \rho(s)ds \right|$$
$$\leqslant \left[\left(\frac{(\theta-j)^2 + (k-\theta)^2}{4} \right) + \left(y_0 - \frac{j+\theta}{2} \right)^2 + \left(y_1 - \frac{k+\theta}{2} \right) \right] \cdot \|\rho'\|_{\infty,[j,k]},$$

for all $j \leqslant y_0 \leqslant \theta \leqslant y_1 \leqslant k$. *The constant* $\frac{1}{4}$ *is the best possible.*

REMARK 4.4. Previous result was stated for $\lambda = 0$ which gives us results of article [1] as its special case.

Further results can be obtained by using different values of λ, specifically $\lambda = 1$ would give us some interesting results.

COROLLARY 4.5. *Let* ρ *be as in Theorem 4.1. Then, the following inequalities hold:*

(1) The Ostrowski inequality: For all $\theta \in [j,k]$, *we have*

$$\left| (k-j)\rho(\theta) - \int_j^k \rho(s)ds \right| \leqslant \left[\frac{(k-j)^2}{4} + \left(\theta - \frac{j+k}{2} \right) \right] \cdot \|\rho'\|_{\infty,[j,k]}.$$

(2) The generalized two-point inequality: for all $j \leqslant y_0 \leqslant \frac{j+k}{2} \leqslant y_1 \leqslant k$, *we have*

$$\left|(k-j)\frac{\rho(y_0)+\rho(y_1)}{2} - \int_j^k \rho(s)ds\right| \leqslant \left[\frac{(k-j)^2}{8} + \left(y_0 - \frac{3j+k}{2}\right)^2 + \left(\theta - \frac{a+3b}{2}\right)^2\right] \cdot \|\rho'\|_{\infty,[j,k]}.$$

(3) The companion of Ostrowski inequality: For all $h \in \left[j, \frac{j+k}{2}\right]$, *we have*

$$\left|(k-j)\frac{\rho(h)+\rho(j+k-h)}{2} - \int_j^k \rho(s)ds\right| \leqslant \left[\frac{(k-j)^2}{8} + 2\left(\frac{3j+k}{4} - h\right)^2\right] \times \|\rho'\|_{\infty,[j,k]}.$$

(4) The Cerone-Dragomir inequality: For all $\theta \in [j,k]$, *we have*

$$\left|(\theta-j)\rho(j) + (k-\theta)\rho(k) - \int_j^k \rho(s)ds\right| \leqslant \left[\frac{(k-j)^2}{4} + \left(\theta - \frac{j+k}{2}\right)^2\right] \cdot \|\rho'\|_{\infty,[j,k]}.$$

(5) The midpoint-trapezoid-Ostrowski's inequality: For all $\theta \in [j,k]$, *we have*

$$\left|(\theta-j)\rho\left(\frac{j+\theta}{2}\right) + (k-\theta)\rho\left(\frac{k+\theta}{2}\right) - \int_j^k \rho(s)ds\right| \leqslant \frac{1}{2}\left[\frac{(k-j)^2}{4} + 2\left(\theta - \frac{j+k}{2}\right)^2\right] \cdot \|\rho'\|_{\infty,[j,k]}.$$

The constants $\frac{1}{4}, \frac{1}{8}$, *and in the last inequality the both constants* $\frac{1}{2}, \frac{1}{4}$ *are all best possible.*

Following remarks are valid for our results as well as can be found in [1].

REMARK 4.6. In the representations $(3.1)-(3.3)$, if one assumes that ρ' is convex, r-convex, quasi-convex, s-convex, P-convex, or Q-convex; we can obtain other new bounds involving convexity.

REMARK 4.7. By following the same approach in establishing Theorem 2.1 in [14], we can obtain bound of $E(\rho; y_0, \theta, y_1)$ involving mappings possess Hölder property of order $\alpha \in (0,1]$. An extension to higher order derivatives with symmetric and/or translation assumptions can be sated by following the same approach of Theorem 2.6 in [14]. We leave the details of this part of the remark for further discussion in future studies.

Remark 4.8. To approximate $\int_j^k \rho(s)ds$, the symmetric formula (2.15) was presented in [14]. However, our formula (3.1) is presented for symmetric and non-symmetric points y_0 and y_1, which they can be chosen arbitrarily in $[j,k]$.

Remark 4.9. In [9], Dragomir introduced a new type of Ostrowski inequality based on Pompiue's mean-value theorem. To obtain the corresponding two point of Dragomir type; we may use the same approach considered in [9] taking into account the representations $(3.1)-(3.3)$.

REFERENCES

[1] M. W. Alomari, *Two–point Ostrowski's inequality*, Results in Math., **72** (2017), 1499–1523.

[2] M. W. Alomari, *A sharp companion of Ostrowski's inequality for the Riemann Stieltjes integral and applications*, Ann. Univ. Paedagog. Crac. Stud. Math., **15** (2016), 69–78.

[3] M. W. Alomari, *A companion of Ostrowski's inequality for the Riemann-Stieltjes integral, where f is of bounded variation and u is of r-H-Hölder type and applications*, Appl. Math. Comput., **219** (2013), 4792–4799.

[4] M. W. Alomari, *New sharp inequalities of Ostrowski and generalized trapezoid type for the Riemann–Stieltjes integrals and applications*, Ukrainian Math. J., **65** (7) (2013), 995–1018.

[5] M. W. Alomari, *A companion of the generalized trapezoid inequality and applications*, J. Math. Appl., **36** (2013), 5–15.

[6] M. W. Alomari, *A companion of Dragomir's generalization of Ostrowski's inequality and applications to numerical integration*, Ukrainian Math. J., **64** (4) (2012), 491–510.

[7] P. Cerone, S. S. Dragomir, C. E. M. Pearce, *Ageneralized trapezoid inequality for functions of bounded variation*, Turk. J. Math., **24** (2000), 147–163.

[8] S. S. Dragomir, *A companion of J. Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl., **5** (2014), no. 1, 89–97.

[9] S. S. Dragomir, *Some companions of Ostrowski's inequality for absolutely continuous functions and applications*, Bull. Korean Math. Soc., **42** (2) (2005), 213–230.

[10] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull. Austral. Math. Soc., **60** (3) (1999), 495–508.

[11] R. M. Dudley, *Frechet differentiability, p-variation and uniform Donsker classes*, Ann. Probab., **20** (4) (1992), 1968–1982.

[12] B. I. Golubov, *On criteria for the continuity of functions of bounded p-variation*, Sibirskii Mat. Zhurnal, **13** (5) (1972), 693–702.

[13] B. I. Golubov, *On functions of bounded p-variation*, Math. USSR Izvest., **2** (4) (1968), 799–819.

[14] A. Guessab, G. Schmeisser, *Sharp integral inequalities of the Hermite-Hadamard type*, J. approx. Theory, **15** (2) (2002), 260–288.

[15] Sergei Ivanovich Hudjaev, Aizik Isaakovich Vol'pert, *Analysis in classes of discontinuous functions and equations of mathematical physics, Mechanics: analysis, 8*, Dordrecht-Boston-Lancaster: Martinus Nijhoff Publishers (1985).

[16] S. Kovac, J. Pečarić , A. Vukelić, *A generalization of general two-point formulawith applications in numerical integration*, Nonlinear Anal.: Theory, Methods and Appl., **68** (8) (2008), 2445–2463.

[17] D. Lesnic, *Characterization of the functions with bounded variation*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics – ICTAMI 2003, Alba Iulia.

[18] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Calassical and new inequalities in analysis*, in: Mathematics and its application (East European Series), vol. 61, Kluwer, Dordrecht (1993).

[19] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihren integralmittelwert*, Comment. Math. Helv., **10** (1937), 226–227.

[20] J. Pečarić, I. Perić, A. Vukelić, *Sharp integral inequalities based on general Euler two-point formulae*, ANZIAM J. **46** (2005), 555–574.

[21] N. WIENER, *The quadratic variation of a function and its Fourier coefficients*, Mass. J. Math., **3** (1924), 72–94.

[22] F. ZAFAR, *Some Generalizations of Ostrowski Inequalities and their Applications to Numerical Integration and Special Means*, Unpublished doctoral dissertation, Bahauddin Zakariya University, Multan, Pakistan, 2009.

(Received July 11, 2021)

Mohammad W. Alomari
Department of Mathematics
Faculty of Science and Information Technology, Jadara University
P. O. Box 733, Irbid, P. C. 21110, Jordan
e-mail: malomari@jadara.edu.jo

Nazia Irshad
Department of Mathematics
Dawood University of Engineering and Technology
Karachi, Pakistan
e-mail: nazia.irshad@duet.edu.pk

Asif R. Khan
Department of Mathematics
University of Karachi
University Road, Karachi-75270, Pakistan
e-mail: asifrk@uok.edu.pk

Muhammad Awais Shaikh
Department of Mathematics
University of Karachi
University Road, Karachi-75270, Pakistan
e-mail: m.awaisshaikh2014@gmail.com