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STABILITY OF A NEW ADDITIVE FUNCTIONAL INEQUALITY IN BANACH SPACES

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Abstract. We propose and solve a new type of additive functional inequality. We also obtain the Hyers-Ulam stability of such functional inequality in a complex Banach space by using two different approaches.

1. Introduction and preliminaries

In 1940, Ulam [29] mentioned a question regarding the stability of (group) homomorphisms which motivated the study of the stability problems of functional equations.

Every consequence of the Cauchy functional equation:

$$g(u+v) = g(u) + g(u)$$

is said to be an additive mapping. The properties of Cauchy functional equations are vigorous tools in almost every field of natural and social sciences. Hyers [12] obtained a partial answer to the question for additive mappings in Banach spaces. The stability of functional equations has been also known as Hyers-Ulam stability. It was later extended by Aoki [1] for additive mappings, and by Rassias [27] for linear mappings by concerning an unbounded Cauchy difference. Replacing the unbounded Cauchy difference by a general control function, Găvruta [10] also extended the Rassias theorem. Park [22, 23] recently defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of those inequalities in non-Archimedean Banach spaces and Banach spaces. The stability results of the Jensen functional equation:

$$g\left(\frac{u+v}{2}\right) = \frac{g(u)+g(v)}{2}$$

have been scrutinized by several authors, see [14, 17, 18], for example.

Applications of stability theory of functional equations for proving fixed point theorems and applications in nonlinear analysis were introduced by Isac and Rassias [13]

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in 1996. A large number of research articles concerning the stability problems of several functional equations for instance set-valued functional equation, Cauchy functional equation, Drygas functional equation, and various definitions of stability by using the fixed pointed technique have been widely studied in great details in [4, 5, 7, 24, 26, 28].

In this article, we let $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ denote the set of positive integers, of real numbers, of positive real numbers and of complex numbers. Also, we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

For $(\rho_1, \rho_2) \in \left\{(z, z') \in \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} : \sqrt{2}|z| + |z'| < 1\right\}$, the following additive (ρ_1, ρ_2)-functional inequality:

$$\begin{aligned}\left\| 2g\left(\frac{u+v}{2}\right) - g(u) - g(v) \right\| \leqslant \|\rho_1 (g(u+v) + g(u-v) - 2g(u))\| \\ + \|\rho_2 (g(u+v) - g(u) - g(v))\|\end{aligned}$$

was introduced and solved in 2017 by Yun and Shin [30]. Recently, Park [25] also proposed and solved the additive (ρ_1, ρ_2)-functional inequality as follows:

$$\begin{aligned}\|g(u+v) - g(u) - g(v)\| \leqslant \|\rho_1 (g(u+v) + g(u-v) - 2g(u))\| \\ + \left\| \rho_2 \left(2g\left(\frac{u+v}{2}\right) - g(u) - g(v)\right) \right\|.\end{aligned}$$

Moreover, the Hyers-Ulam stability solutions for the additive (ρ_1, ρ_2)-functional inequalities are proved in a complex Banach space. The stability problems of varied functional equations and functional inequalities have been studied considerably; for instance, see [2, 8, 9, 15, 16, 19, 21, 31].

We begin with a useful result in theory of fixed point.

THEOREM 1. [3, 6] *Let $(\mathscr{X}, d)$ be a complete generalized metric space, and let $u \in \mathscr{X}$. For a strict Lipschitz contraction $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ with the Lipschitz constant $\mu < 1$, either*

- *$d(\mathscr{T}^n u, \mathscr{T}^{n+1} u) = \infty$ for all $n \in \mathbb{N}_0$ or*
- *there exists $n_0 \in \mathbb{N}$ for which $d(\mathscr{T}^n u, \mathscr{T}^{n+1} u) < \infty$ for all $n \geqslant n_0$; $\{\mathscr{T}^n u\} \to v^*$ where v^* is a unique fixed point of $\mathscr{T}$ in $\mathscr{X}_{n_0} := \{v \in \mathscr{X} \mid d(\mathscr{T}^{n_0} u, v) < \infty\}$; $d(v, v^*) \leqslant \frac{1}{1-\mu} d(v, \mathscr{T} v)$ for all $v \in \mathscr{X}_{n_0}$.*

In this paper, we study the functional inequality (1). This inequality is called additive (t_1, t_2)-functional inequality. We prove the Hyers-Ulam stability of such functional inequality by applying the direct technique in Section 2. While in Section 3, the Hyers-Ulam stability of (1) using the fixed point technique is given.

For all over this article, let $\mathscr{X}$ and $\mathscr{Y}$ be a (real or complex) normed space and a complex Banach space, respectively. We let $(t_1, t_2) \in \left\{(z, z') \in \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} : |z| + |z'| < \frac{1}{\sqrt{2}}\right\}$. Also, we denote the class of mapping $\{f : \mathscr{X} \to \mathscr{Y} : f(0) = 0\}$ by $\mathscr{M}_0(\mathscr{X}, \mathscr{Y})$, the class of mapping $f : \mathscr{X} \to \mathscr{Y}$ which is additive by $\mathscr{A}(\mathscr{X}, \mathscr{Y})$, and $\mathscr{A}_0(\mathscr{X}, \mathscr{Y}) := \mathscr{M}_0(\mathscr{X}, \mathscr{Y}) \cap \mathscr{A}(\mathscr{X}, \mathscr{Y})$.

2. Stability results: direct technique

In this section, the stability outcomes of the additive (t_1, t_2)-functional inequality (1) are solved and investigated by using the direct technique. We start with the following lemma:

LEMMA 1. *If a mapping $g : \mathscr{X} \to \mathscr{Y}$ satisfies the following functional inequality:*

$$\|g(u+v) - g(u) - g(v)\| \leqslant \|t_1 (g(u+v) + g(u-v) - 2g(u))\| + \left\|t_2 \left(2g\left(\frac{u+v}{2}\right) + g(u-v) - 2g(u)\right)\right\| \tag{1}$$

for all $u, v \in \mathscr{X}$, then $g \in \mathscr{A}(\mathscr{X}, \mathscr{Y})$.

Proof. Putting $u = v = 0$ into (1), we gain $(1 - |t_2|)\|g(0)\| \leqslant 0$ and so $g(0) = 0$, since $|t_2| < 1$. Next, by taking $v = u$ in (1) and from $|t_1| < 1$, we obtain

$$2g(u) = g(2u) \tag{2}$$

for all $u \in \mathscr{X}$. From (1), it follows

$$\|g(u+v) - g(u) - g(v)\| \leqslant (|t_1| + |t_2|) \|g(u+v) + g(u-v) - 2g(u)\| \tag{3}$$

for all $u, v \in \mathscr{X}$. Setting $\bar{u} = u + v$ and $\bar{v} = u - v$ in (1), we gain

$$\left\|g(\bar{u}) - g\left(\frac{\bar{u}+\bar{v}}{2}\right) - g\left(\frac{\bar{u}-\bar{v}}{2}\right)\right\| \leqslant (|t_1| + |t_2|) \left\|g(\bar{u}) + g(\bar{v}) - 2g\left(\frac{\bar{u}+\bar{v}}{2}\right)\right\|.$$

This implies by the equation (2) that

$$\frac{1}{2} \|g(\bar{u}+\bar{v}) + g(\bar{u}-\bar{v}) - 2g(\bar{u})\| \leqslant (|t_1| + |t_2|) \|g(\bar{u}+\bar{v}) - g(\bar{u}) - g(\bar{v})\| \tag{4}$$

for all $\bar{u}, \bar{v} \in \mathscr{X}$. Using (3) and (4), we obtain

$$\frac{1}{2}\|g(u+v) - g(u) - g(v)\| \leqslant (|t_1| + |t_2|)^2 \|g(u+v) - g(u) - g(v)\|$$

for all $u, v \in \mathscr{X}$. This follows from $|t_1| + |t_2| < \frac{1}{\sqrt{2}}$ that $g(u+v) = g(u) + g(v)$ for all $u, v \in \mathscr{X}$, that is, $g \in \mathscr{A}(\mathscr{X}, \mathscr{Y})$. $\square$

This time, we provide the main results.

THEOREM 2. *Let a fixed function $\varphi : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_0^+$ satisfy*

$$\Phi(u,v) := \sum_{j=1}^{\infty} 2^j \varphi\left(2^{-j}u, 2^{-j}v\right) < \infty \tag{5}$$

for all $u,v \in \mathscr{X}$. *If* $g \in \mathscr{M}_0(\mathscr{X},\mathscr{Y})$ *and*

$$\begin{aligned}\|g(u+v)-g(u)-g(v)\| \leqslant{} & \|t_1\left(g(u+v)+g(u-v)-2g(u)\right)\| \\ & +\left\|t_2\left(2g\left(\frac{u+v}{2}\right)+g(u-v)-2g(u)\right)\right\|+\varphi(u,v) \end{aligned} \tag{6}$$

for all $u,v \in \mathscr{X}$*, then there is a unique mapping* $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ *such that*

$$\|g(u)-G(u)\| \leqslant \frac{\Phi(u,u)}{2(1-|t_1|)} \tag{7}$$

for all $u \in \mathscr{X}$.

Proof. By taking $v=u$ in (6), for all $u \in \mathscr{X}$, we gain

$$(1-|t_1|)\|g(2u)-2g(u)\| \leqslant \varphi(u,u) \tag{8}$$

and so

$$\left\|g(u)-2g\left(\frac{u}{2}\right)\right\| \leqslant \frac{1}{1-|t_1|}\varphi\left(\frac{u}{2},\frac{u}{2}\right).$$

Then, for all $m,l \in \mathbb{N}_0$ with $m>l$ and all $u \in \mathscr{X}$,

$$\begin{aligned}\left\|2^l g\left(2^{-l}u\right)-2^m g\left(2^{-m}u\right)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|2^j g\left(2^{-j}u\right)-2^{j+1}g\left(2^{-(j+1)}u\right)\right\| \\ &\leqslant \frac{1}{2(1-|t_1|)}\sum_{j=l+1}^{m}2^j\varphi\left(2^{-j}u,2^{-j}u\right). \end{aligned} \tag{9}$$

Thus, $\{2^n g(2^{-n}u)\}$ is a Cauchy sequence for all $u \in \mathscr{X}$ and hence a convergent sequence due to the completeness of $\mathscr{Y}$. Define $G:\mathscr{X}\to\mathscr{Y}$ by

$$G(u) := \lim_{n\to\infty} 2^n g\left(2^{-n}u\right)$$

for all $u \in \mathscr{X}$. Next, select $l=0$ and let $m\to\infty$ in (9). Then, we have that the mapping G satisfies (7). That follows from (5) and (6) as

$$\begin{aligned}&\|G(u+v)-G(u)-G(v)\| \\ &\quad = \lim_{n\to\infty} 2^n\left\|g\left(2^{-n}(u+v)\right)-g\left(2^{-n}u\right)-g\left(2^{-n}v\right)\right\| \\ &\quad \leqslant |t_1|\lim_{n\to\infty}2^n\left\|g\left(2^{-n}(u+v)\right)+g\left(2^{-n}(u-v)\right)-2g\left(2^{-n}u\right)\right\| \\ &\qquad +|t_2|\lim_{n\to\infty}2^n\left\|2g\left(2^{-(n+1)}(u+v)\right)+g\left(2^{-n}(u-v)\right)-2g\left(2^{-n}u\right)\right\| \\ &\qquad +\lim_{n\to\infty}2^n\varphi\left(2^{-n}u,2^{-n}v\right) \\ &\quad = \|t_1\left(G(u+v)+G(u-v)-2G(u)\right)\|+\left\|t_2\left(2G\left(\frac{u+v}{2}\right)+G(u-v)-2G(u)\right)\right\| \end{aligned}$$

for all $u,v \in \mathscr{X}$. By the definition of G with Lemma 1, we obtain that $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$. Finally, let another mapping $F \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ satisfy (7). Then,

$$\begin{aligned}\|G(u)-F(u)\| &= \left\|2^pG\left(2^{-p}u\right)-2^pF\left(2^{-p}u\right)\right\| \\ &\leqslant \left\|2^pG\left(2^{-p}u\right)-2^pg\left(2^{-p}u\right)\right\|+\left\|2^pF\left(2^{-p}u\right)-2^pg\left(2^{-p}u\right)\right\| \\ &\leqslant \frac{2^p}{(1-|t_1|)}\Phi\left(2^{-p}u,2^{-p}u\right)\end{aligned}$$

for all $u \in \mathscr{X}$. Then, $\|G(u)-F(u)\| \to 0$ when $p \to \infty$ and this confirms the uniqueness of G. □

COROLLARY 1. *For $s,\vartheta \in \mathbb{R}_0^+$ with $s>1$, let $g \in \mathscr{M}_0(\mathscr{X},\mathscr{Y})$ and*

$$\begin{aligned}\|g(u+v)-g(u)-g(v)\| \leqslant &\ \|t_1\left(g(u+v)+g(u-v)-2g(u)\right)\| \\ &+\left\|t_2\left(2g\left(\frac{u+v}{2}\right)+g(u-v)-2g(u)\right)\right\| \\ &+\vartheta\left(\|u\|^s+\|v\|^s\right)\end{aligned} \tag{10}$$

for all $u,v \in \mathscr{X}$. Then, there is a unique mapping $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ such that

$$\|g(u)-G(u)\| \leqslant \frac{2\vartheta\|u\|^s}{(1-|t_1|)(2^s-2)}$$

for all $u \in \mathscr{X}$.

Proof. Let $\varphi(u,v)=\vartheta(\|u\|^s+\|v\|^s)$ for all $u,v \in \mathscr{X}$, we immediately obtain the result. □

The structure of the proof of the next result is analogous to the proof of Theorem 2. We include some details for convenience of the readers.

THEOREM 3. *Let a fixed function $\varphi : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_0^+$ satisfy*

$$\Psi(u,v) := \sum_{j=0}^{\infty} 2^{-j}\varphi\left(2^ju,2^jv\right) < \infty$$

for all $u,v \in \mathscr{X}$, and let $g \in \mathscr{M}_0(\mathscr{X},\mathscr{Y})$ satisfy (6). Then, there is a unique mapping $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ such that

$$\|g(u)-G(u)\| \leqslant \frac{\Psi(u,u)}{2(1-|t_1|)} \tag{11}$$

for all $u \in \mathscr{X}$.

Proof. It follows from (8) that

$$\left\|g(u)-\frac{1}{2}g(2u)\right\| \leqslant \frac{\varphi(u,u)}{2(1-|t_1|)}$$

for all $u \in \mathscr{X}$. Then, for $m, l \in \mathbb{N}_0$ with $m > l$,

$$\left\|2^{-l}g(2^l u) - 2^{-m}g(2^m u)\right\| \leqslant \sum_{j=l}^{m-1} \left\|2^{-j}g\left(2^j u\right) - 2^{-(j+1)}g\left(2^{j+1}u\right)\right\|$$
$$\leqslant \frac{1}{1-|t_1|} \sum_{j=l}^{m-1} 2^{-(j+1)}\varphi(2^j u, 2^j u) \tag{12}$$

for all $u \in \mathscr{X}$. Then, the completeness of $\mathscr{Y}$ implies that $\{2^{-n}g(2^n u)\}$ is convergent for all $u \in \mathscr{X}$. Next, we let a mapping $G : \mathscr{X} \to \mathscr{Y}$ by

$$G(u) := \lim_{n\to\infty} 2^{-n}g(2^n u)$$

for all $u \in \mathscr{X}$. Select $l = 0$ and let $m \to \infty$ in (12). Then, we have that the mapping G satisfies (11). The rest of the proof is analogous to that of the former. □

Finally, let $\varphi(u, v) = \vartheta(\|u\|^s + \|v\|^s)$ for all $u, v \in \mathscr{X}$. Then, we yield the following corollary.

COROLLARY 2. *Let $s, \vartheta \in \mathbb{R}_0^+$ with $s < 1$. If $g \in \mathscr{M}_0(\mathscr{X}, \mathscr{Y})$ satisfies (10), then there is a unique mapping $G \in \mathscr{A}_0(\mathscr{X}, \mathscr{Y})$ such that*

$$\|g(u) - G(u)\| \leqslant \frac{2\vartheta\|u\|^s}{(1-|t_1|)(2-2^s)}$$

for all $u \in \mathscr{X}$.

3. Stability results: fixed point technique

In this part, we use the fixed point technique to prove the Hyers-Ulam stability of the additive (t_1, t_2)-functional inequality (1).

THEOREM 4. *Let $\varphi : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_0^+$ be a function such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\varphi\left(\frac{u}{2}, \frac{v}{2}\right) \leqslant \frac{L}{2}\varphi(u, v) \tag{13}$$

for all $u, v \in \mathscr{X}$. Then, for a mapping $g \in \mathscr{M}_0(\mathscr{X}, \mathscr{Y})$ satisfying (6), there is a unique mapping $G \in \mathscr{A}_0(\mathscr{X}, \mathscr{Y})$ such that

$$\|g(u) - G(u)\| \leqslant \frac{L\varphi(u, u)}{2(1-L)(1-|t_1|)} \tag{14}$$

for all $u \in \mathscr{X}$.

Proof. We equip the set $\mathscr{M}_0(\mathscr{X},\mathscr{Y})$ with the generalized metric defined by

$$d(h,k)=\inf\left\{\beta\in\mathbb{R}^+:\|h(u)-k(u)\|\leqslant\beta\varphi(u,u),\ \text{ for all } u\in\mathscr{X}\right\},$$

where $\inf\emptyset=\infty$ as typical. Then, $(\mathscr{M}_0(\mathscr{X},\mathscr{Y}),d)$ is complete, in [20]. Define a mapping $\mathscr{T}:\mathscr{M}_0(\mathscr{X},\mathscr{Y})\to\mathscr{M}_0(\mathscr{X},\mathscr{Y})$ by

$$\mathscr{T}h(u):=2h\left(\frac{u}{2}\right)$$

for all $u\in\mathscr{X}$. Let $h,k\in\mathscr{M}_0(\mathscr{X},\mathscr{Y})$ where $d(h,k)=\varepsilon$. Then,

$$\|h(u)-k(u)\|\leqslant\varepsilon\varphi(u,u)$$

for all $u\in\mathscr{X}$. Consequently,

$$\begin{aligned}\|\mathscr{T}h(u)-\mathscr{T}k(u)\|&=\left\|2h\left(\frac{u}{2}\right)-2k\left(\frac{u}{2}\right)\right\|\leqslant 2\varepsilon\varphi\left(\frac{u}{2},\frac{u}{2}\right)\\&\leqslant 2\varepsilon\frac{L}{2}\varphi(u,u)=L\varepsilon\varphi(u,u)\end{aligned}$$

for all $u\in\mathscr{X}$. Then $d(\mathscr{T}h,\mathscr{T}k)\leqslant L\varepsilon$ which means

$$d(\mathscr{T}h,\mathscr{T}k)\leqslant Ld(h,k)$$

for all $h,k\in\mathscr{M}_0(\mathscr{X},\mathscr{Y})$. It follows from (8) and (13) that

$$\left\|g(u)-2g\left(\frac{u}{2}\right)\right\|\leqslant\frac{1}{1-|t_1|}\varphi\left(\frac{u}{2},\frac{u}{2}\right)\leqslant\frac{L}{2(1-|t_1|)}\varphi(u,u)$$

for all $u\in\mathscr{X}$. Thus

$$d(g,\mathscr{T}g)\leqslant\frac{L}{2(1-|t_1|)}.$$

From Theorem 1, there exists $G:\mathscr{X}\to\mathscr{Y}$ satisfying as follows:

(1) G is a unique fixed point of $\mathscr{T}$, i.e.,

$$G(u)=2G\left(\frac{u}{2}\right)$$

for all $u\in\mathscr{X}$. Thus, there exists $\beta\in(0,\infty)$ satisfying

$$\|g(u)-G(u)\|\leqslant\beta\varphi(u,u)$$

for all $u\in\mathscr{X}$;

(2) $d(\mathscr{T}^l g,G)\to 0$ as $l\to\infty$, which implies that

$$\lim_{l\to\infty}2^l g\left(2^{-l}u\right)=G(u)$$

for all $u\in\mathscr{X}$;

(3) $d(g,G) \leqslant \frac{1}{1-L} d(g,\mathscr{T}g)$, which can be implied that

$$\|g(u) - G(u)\| \leqslant \frac{L\varphi(u,u)}{2(1-L)(1-|t_1|)}$$

for all $u \in \mathscr{X}$. By using the same technique as in the proof of Theorem 2, we can conclude that $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$. $\square$

COROLLARY 3. *Let $s, \vartheta \in \mathbb{R}_0^+$ with $s > 1$. If $g \in \mathscr{M}_0(\mathscr{X},\mathscr{Y})$ satisfies (10), then there is a unique mapping $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ such that*

$$\|g(u) - G(u)\| \leqslant \frac{2\vartheta\|u\|^s}{(1-|t_1|)(2^s - 2)}$$

for all $u \in \mathscr{X}$.

Proof. The proof follows from Theorem 4 by taking $L = 2^{1-s}$ and $\varphi(u,v) = \vartheta(\|u\|^s + \|v\|^s)$ for all $u,v \in \mathscr{X}$. $\square$

THEOREM 5. *Let $\varphi : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_0^+$ be a function such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\varphi(u,v) \leqslant 2L\varphi\left(\frac{u}{2}, \frac{v}{2}\right) \tag{15}$$

for all $u,v \in \mathscr{X}$. Then, for a mapping $g \in \mathscr{M}_0(\mathscr{X},\mathscr{Y})$ satisfying (6), there is a unique mapping $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ such that

$$\|g(u) - G(u)\| \leqslant \frac{\varphi(u,u)}{2(1-L)(1-|t_1|)} \tag{16}$$

for all $u \in \mathscr{X}$.

Proof. Regard the complete metric space $(\mathscr{M}_0(\mathscr{X},\mathscr{Y}),d)$ given as in the proof of Theorem 4. We consider a mapping $\mathscr{T} : \mathscr{M}_0(\mathscr{X},\mathscr{Y}) \to \mathscr{M}_0(\mathscr{X},\mathscr{Y})$ defined by

$$\mathscr{T}h(u) := \frac{1}{2}h(2u)$$

for all $u \in \mathscr{X}$. As follows from (8),

$$\left\|g(u) - \frac{1}{2}g(2u)\right\| \leqslant \frac{\varphi(u,u)}{2(1-|t_1|)}$$

for all $u \in \mathscr{X}$. Also, in the proof of Theorems 2 and 4, there exists a unique mapping $G \in \mathscr{A}_0(\mathscr{X},\mathscr{Y})$ satisfying (16). $\square$

Let $L = 2^{s-1}$ and $\varphi(u,v) = \vartheta(\|u\|^s + \|v\|^s)$ for all $u,v \in \mathscr{X}$. The following corollary is obtained.

COROLLARY 4. *Let* $s, \vartheta \in \mathbb{R}_0^+$ *with* $s < 1$, *and let* $g \in \mathscr{M}_0(\mathscr{X}, \mathscr{Y})$ *be a mapping satisfying* (10). *Then there exists a unique mapping* $G \in \mathscr{A}_0(\mathscr{X}, \mathscr{Y})$ *such that*

$$\|g(u) - G(u)\| \leqslant \frac{2\vartheta \|u\|^s}{(1-|t_1|)(2-2^s)}$$

for all $u \in \mathscr{X}$.

Conclusion

We have proposed the additive (t_1, t_2)-functional inequality (1) and have proved the Hyers-Ulam stability of the proposed functional inequality (1) in a complex Banach space.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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