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MAJORIZATION TYPE INEQUALITIES VIA 4–CONVEX FUNCTIONS

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This paper is dedicated to Professor Sanjo Zlobec on the occasion of his 80th birthday.

(*Communicated by M. Krnić*)

Abstract. The main aim of this paper is to prove several majorization type inequalities using Green and 4-convex functions. First of all, we drive generalized majorization inequality for arbitrary n-tuples and real weights. Further, we explore the inequality for majorized tuples, weighted majorization theorems given by Fuchs, Dragomir and Maligranda et al. For deriving another generalized majorization inequality, we use a simple form of Jensen's inequality, and by similar fashion we apply classical earlier majorization theorems for further elaborations of generalized inequality. Several applications of information theory are discussed at the end of the article.

1. Introduction and preliminaries

Mathematical inequalities play an excellent role in almost every field of science. Nowadays, several mathematicians are taking a keen interest in introducing new inequalities or refining the earlier inequalities and giving their applications. In the literature, several inequalities have been proved for the important class of convex functions such as Jensen's inequality, the Jensen-Steffensen inequality, majorization and Slater's inequalities etc. Among these, one of the generalized and applicable inequalities for the convex function is the majorization inequality. Initially, this inequality has been proved for majorized tuples. If $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,\ y_2,\ldots,y_n)$ are two n-tuples such that $n\in\mathbb{N}$ and $n\geqslant 2$, then $\mathbf{y}$ is said to be majorized by $\mathbf{x}$ (in symbol $\mathbf{y}\prec\mathbf{x}$ or $\mathbf{x}\succ\mathbf{y}$) if the sum of q largest entries of $\mathbf{y}$ are not greater than the sum of q largest entries of $\mathbf{x}$ for $q=1,2,\ldots,n-1$ and

$$\sum_{j=1}^{n} x_j=\sum_{j=1}^{n} y_j. \tag{1.1}$$

In [14], the majorization inequality

$$\sum_{j=1}^{n} H(y_j)\leqslant\sum_{j=1}^{n} H(x_j) \tag{1.2}$$

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has been proved that: if $H : [d_1, d_2] \to \mathbb{R}$ is a convex function and $\mathbf{y} \prec \mathbf{x}$ with $x_j, y_j \in [d_1, d_2]$ for $j = 1, 2, \ldots, n$. In 1947, Fuchs [13] proved the following weighted version of majorization inequality:

$$\sum_{j=1}^{n} w_j H(y_j) \leqslant \sum_{j=1}^{n} w_j H(x_j) \tag{1.3}$$

if $H : [d_1, d_2] \to \mathbb{R}$ is a convex function, $w_j \in \mathbb{R}$, $x_j, y_j \in [d_1, d_2]$ and $\mathbf{x}$, $\mathbf{y}$ are decreasing n-tuples, and the following conditions hold:

$$\sum_{j=1}^{q} w_j y_j \leqslant \sum_{j=1}^{q} w_j x_j, \quad \text{for} \quad q = 1, 2, \ldots, n-1 \tag{1.4}$$

and

$$\sum_{j=1}^{n} w_j y_j = \sum_{j=1}^{n} w_j x_j. \tag{1.5}$$

Maligranda et al. [18] proved (1.3) by using the relaxed condition that only one tuple should be monotonic while using the strict condition on weights that $w_j \geqslant 0 \ \forall\, j = 1, 2, \ldots, n$.

In 2004, Dragomir [11] proved the weighted majorization inequality by utilizing a more strict condition of monotonicity of $\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ with positive weights but without using condition (1.4). He also discussed the case of increasing convex function, but using the relaxed condition $\sum\limits_{j=1}^{n} w_j y_j \geqslant \sum\limits_{j=1}^{n} w_j x_j$ instead of (1.5). In the proof of Dragomir's result, he applied Chebyshev's inequality. Later on, in 2007 Neizgoda [21] introduced the concept of separable sequences and presented a generalized Chebyshev's inequality for these sequences. Neizgoda [21] proved majorization inequality for separable sequences. Further, he also gave several applications for particular bases [20].

In the literature, several generalizations, extensions and refinements have been presented for majorization inequality. In [5] Khan et al. used Taylor formula, the Green convex function, n-convex functions and obtained several generalizations of majorization inequality. In particular, they obtained exponential and log-convexity for parameterized functionals associated with generalized inequalities. Similarly, some other identities such as the Lidstone, Fink and Montogonmy identities have been used and obtained several related results for majorization inequalities [2]. For more results related to majorization and its applications see [1, 3, 6–10, 15–17, 19, 22, 23].

We use the following Green functions defined on $[d_1, d_2] \times [d_1, d_2]$ to obtain our main results [4].

$$G_1(x,s) = \begin{cases} d_1 - s, & d_1 \leqslant s \leqslant x, \\ d_1 - x, & x \leqslant s \leqslant d_2. \end{cases} \tag{1.6}$$

$$G_2(x,s) = \begin{cases} x - d_2, & d_1 \leqslant s \leqslant x, \\ s - d_2, & x \leqslant s \leqslant d_2. \end{cases} \tag{1.7}$$

$$G_3(x,s) = \begin{cases} x - d_1, & d_1 \leqslant s \leqslant x, \\ s - d_1, & x \leqslant s \leqslant d_2. \end{cases} \tag{1.8}$$

$$G_4(x,s) = \begin{cases} d_2 - s, & d_1 \leqslant s \leqslant x, \\ d_2 - x, & x \leqslant s \leqslant d_2. \end{cases} \tag{1.9}$$

With respect to both the variables s and $x,$ these Green functions are convex and continuous.

The following lemma is also useful for obtaining our main results.

LEMMA 1.1. ([4]) *Let $H \in C^2[d_1, d_2]$. Then the following identities hold.*

$$H(x) = H(d_1) + (x - d_1)H'(d_2) + \int_{d_1}^{d_2} G_1(x, s)H''(s)ds, \tag{1.10}$$

$$H(x) = H(d_2) + (x - d_2)H'(d_1) + \int_{d_1}^{d_2} G_2(x, s)H''(s)ds, \tag{1.11}$$

$$H(x) = H(d_2) + (x - d_1)H'(d_1) - (d_2 - d_1)H'(d_2) + \int_{d_1}^{d_2} G_3(x, s)H''(s)ds, \tag{1.12}$$

$$H(x) = H(d_1) + (d_2 - d_1)H'(d_1) - (d_2 - x)H'(d_2) + \int_{d_1}^{d_2} G_4(x, s)H''(s)ds, \tag{1.13}$$

where G_i $(i = 1,2,3,4)$ are given in (1.6)–(1.9).

In the main results, we use 4-convex function, therefore we include definition of 4-convex function in the following part.

DEFINITION 1.2. ([5]) Consider, the arbitrary function $H : [d_1,d_2] \to \mathbb{R}$ and let $\zeta_0, \zeta_1, \cdots, \zeta_m$ be any distinct points from $[d_1,d_2]$. Then the m^{th} ordered divided difference of H at the selected points is defined recursively as:

$$[\zeta_i]H = H(\zeta_i), \quad i = 1,2,\ldots,m,$$
$$[\zeta_0, \zeta_1, \ldots, \zeta_m]H = \frac{[\zeta_1, \ldots, \zeta_m]H - [\zeta_0, \ldots, \zeta_{m-1}]H}{\zeta_m - \zeta_0}.$$

The following theorem provides a criteria for a function to be 4-convex.

THEOREM 1.3. ([2]) *Let $H : [d_1,d_2] \to \mathbb{R}$ be any function such that $H^{(m)}$ exists. Then H is m-convex if and only if $H^{(m)} \geqslant 0$ on $[d_1,d_2]$.*

In our second main result, the following simple form of Jensen's inequality will be used.

LEMMA 1.4. *Let $H : [d_1,d_2] \to \mathbb{R}$ be a convex function and $h_3^*(s)$ be a weight function such that $h_3^*(s) \geqslant 0$ and $\int_{d_1}^{d_2} h_3^*(s)\,ds > 0$. Then*

$$H\left(\frac{\int_{d_1}^{d_2} s h_3^*(s)ds}{\int_{d_1}^{d_2} h_3^*(s)ds}\right) \leqslant \frac{1}{\int_{d_1}^{d_2} h_3^*(s)\,ds}\int_{d_1}^{d_2} H(s)h_3^*(s)ds. \tag{1.14}$$

2. Main results

We begin to present our first main result.

THEOREM 2.1. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x} = (x_1,x_2,\dots,x_n)$, $\mathbf{y} = (y_1,y_2,\dots,y_n)$ be n-tuples such that $x_j, y_j \in [d_1,d_2]$, for $j = 1,2,\dots,n$. Also, let $w_j \in \mathbb{R}$ for $j = 1,2,\dots,n$ and G_i $(i = 1,2,3,4)$ be Green functions as defined in (1.6)–(1.9). If*

$$\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s) \geqslant 0 \quad \text{for} \quad i \in \{1,2,3,4\},\ s \in [d_1,d_2], \tag{2.15}$$

then

$$\begin{aligned}
&\sum_{j=1}^{n} w_j H(x_j) - \sum_{j=1}^{n} w_j H(y_j) \\
&\leqslant \left(H'(d_k) - \frac{H''(d_2)}{d_2-d_1}\left(\frac{d_k^2}{2} - d_1 d_k\right) - \frac{H''(d_1)}{d_2-d_1}\left(d_k d_2 - \frac{d_k^2}{2}\right)\right)\bar{w}_0 \\
&\quad + \frac{H''(d_2)-H''(d_1)}{6(d_2-d_1)}\bar{w}_2 + \frac{d_2 H''(d_1) - d_1 H''(d_2)}{2(d_2-d_1)}\bar{w}_1, \quad \text{for} \quad k = 1,2.
\end{aligned} \tag{2.16}$$

Where

$$\bar{w}_0 = \sum_{j=1}^{n} w_j (x_j - y_j), \tag{2.17}$$

$$\bar{w}_1 = \sum_{j=1}^{n} w_j \left(x_j^2 - y_j^2\right), \tag{2.18}$$

$$\bar{w}_2 = \sum_{j=1}^{n} w_j (x_j^3 - y_j^3). \tag{2.19}$$

If the inequality in (2.15) holds in the opposite direction, then the inequality in (2.16) holds in the opposite direction.

If H is a 4-concave function, then (2.16) holds in the opposite direction.

Proof. Using (1.10) and (1.13) in $\sum_{j=1}^{n} w_j H(x_j) - \sum_{j=1}^{n} w_j H(y_j)$, we get

$$\begin{aligned}
&\sum_{j=1}^{n} w_j H(x_j) - \sum_{j=1}^{n} w_j H(y_j) \\
&= H'(d_2) \sum_{j=1}^{n} w_j (x_j - y_j) \\
&\quad + \int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s) \right) H''(s)ds.
\end{aligned} \tag{2.20}$$

Since H is a 4-convex function, so H'' is convex. Using definition of convexity, we have

$$H''(s) \leqslant \left(\frac{s-d_1}{d_2-d_1}\right) H''(d_2) + \frac{d_2-s}{d_2-d_1} H''(d_1). \tag{2.21}$$

Therefore, using (2.15) and (2.21) in the right-hand side of (2.20), we get

$$\begin{aligned}
&\int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s)\right) H''(s)ds \\
&\leqslant \frac{H''(d_1)}{d_2-d_1} \int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s)\right)(d_2-s)ds \\
&\quad + \frac{H''(d_2)}{d_2-d_1} \int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s)\right)(s-d_1)ds.
\end{aligned} \tag{2.22}$$

If $H(s) = \frac{s^2 d_2}{2} - \frac{s^3}{6}$, then $H'(s) = s d_2 - \frac{s^2}{2}$, $H''(s) = d_2 - s$ and using these functions in (2.20), we obtain

$$\begin{aligned}
&\int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s)\right)(d_2-s)ds \\
&= \sum_{j=1}^{n} w_j \left(\frac{x_j^2 d_2}{2} - \frac{x_j^3}{6}\right) - \sum_{j=1}^{n} w_j \left(\frac{y_j^2 d_2}{2} - \frac{y_j^3}{6}\right) - \frac{d_2^2}{2} \sum_{j=1}^{n} w_j (x_j - y_j).
\end{aligned} \tag{2.23}$$

Similarly, if $H(s) = \frac{s^3}{6} - \frac{s^2 d_1}{2}$, then $H'(s) = \frac{s^2}{2} - s d_1$, $H''(s) = s - d_1$ and using these functions in (2.20), we obtain

$$\begin{aligned}
&\int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s)\right)(s-d_1)ds \\
&= \sum_{j=1}^{n} w_j \left(\frac{x_j^3}{6} - \frac{x_j^2 d_1}{2}\right) - \sum_{j=1}^{n} w_j \left(\frac{y_j^3}{6} - \frac{y_j^2 d_1}{2}\right) - \left(\frac{d_2^2}{2} - d_1 d_2\right) \sum_{j=1}^{n} w_j (x_j - y_j).
\end{aligned} \tag{2.24}$$

Now using (2.23) and (2.24) in (2.22), we get

$$\begin{aligned}
&\int_{d_1}^{d_2} \left(\sum_{j=1}^{n} w_j G_i(x_j,s) - \sum_{j=1}^{n} w_j G_i(y_j,s)\right) H''(s)ds \\
&\leqslant \frac{H''(d_1)}{d_2-d_1} \left(\sum_{j=1}^{n} w_j \left(\frac{x_j^2 d_2}{2} - \frac{x_j^3}{6}\right) - \sum_{j=1}^{n} w_j \left(\frac{y_j^2 d_2}{2} - \frac{y_j^3}{6}\right)\right) \\
&\quad + \frac{H''(d_2)}{d_2-d_1} \left(\sum_{j=1}^{n} w_j \left(\frac{x_j^3}{6} - \frac{x_j^2 d_1}{2}\right) - \sum_{j=1}^{n} w_j \left(\frac{y_j^3}{6} - \frac{y_j^2 d_1}{2}\right)\right) \\
&\quad - \frac{d_2^2}{2} \frac{H''(d_1)}{d_2-d_1} \sum_{j=1}^{n} w_j (x_j - y_j) - \frac{H''(d_2)}{d_2-d_1} \left(\frac{d_2^2}{2} - d_1 d_2\right) \sum_{j=1}^{n} w_j (x_j - y_j).
\end{aligned} \tag{2.25}$$

Using (2.25) in (2.20), we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n} w_j H(x_j) - \sum_{j=1}^{n} w_j H(y_j) \\
&\leqslant H'(d_2) \sum_{j=1}^{n} w_j (x_j - y_j) \\
&\quad + \frac{H''(d_1)}{d_2 - d_1} \left(\sum_{j=1}^{n} w_j \left(\frac{x_j^2 d_2}{2} - \frac{x_j^3}{6} \right) - \sum_{j=1}^{n} w_j \left(\frac{y_j^2 d_2}{2} - \frac{y_j^3}{6} \right) \right) \\
&\quad + \frac{H''(d_2)}{d_2 - d_1} \left(\sum_{j=1}^{n} w_j \left(\frac{x_j^3}{6} - \frac{x_j^2 d_1}{2} \right) - \sum_{j=1}^{n} w_j \left(\frac{y_j^3}{6} - \frac{y_j^2 d_1}{2} \right) \right) \\
&\quad - \frac{H''(d_2)}{(d_2 - d_1)} \left(\frac{d_2^2}{2} - d_1 d_2 \right) \sum_{j=1}^{n} w_j (x_j - y_j) \\
&\quad - \frac{d_2^2}{2} \frac{H''(d_1)}{(d_2 - d_1)} \sum_{j=1}^{n} w_j (x_j - y_j),
\end{aligned} \tag{2.26}
$$

which is equivalent to (2.16), for $k = 2$.

Similarly, the required inequality for $k = 1$, can be obtained using the identities (1.11) and (1.12). □

The following theorem is the integral version of the above theorem.

THEOREM 2.2. *Let $H \in C^2[d_1, d_2]$ be a 4-convex function, let $h_1, h_2 : [b_1, b_2] \to [d_1, d_2]$, $g : [b_1, b_2] \to \mathbb{R}$ be three integrable functions, let G_i $(i = 1, 2, 3, 4,)$ be Green functions as defined in (1.6)–(1.9) and*

$$
\int_{b_1}^{b_2} g(y)\, G_i(h_1, s) dy - \int_{b_1}^{b_2} g(y)\, G_i(h_2, s) dy \geqslant 0 \quad \textit{for} \quad i \in \{1, 2, 3, 4\}. \tag{2.27}
$$

Then the following inequality holds:

$$
\begin{aligned}
&\int_{b_1}^{b_2} g(y) H(h_1(y)) dy - \int_{b_1}^{b_2} g(y) H(h_2(y)) dy \\
&\leqslant \left(H'(d_k) - \frac{H''(d_2)}{d_2 - d_1} \left(\frac{d_k^2}{2} - d_1 d_k \right) - \frac{H''(d_1)}{d_2 - d_1} \left(d_k d_2 - \frac{d_k^2}{2} \right) \right) \tilde{w}_0 \\
&\quad + \frac{H''(d_2) - H''(d_1)}{6(d_2 - d_1)} \tilde{w}_2 + \frac{d_2 H''(d_1) - d_1 H''(d_2)}{2(d_2 - d_1)} \tilde{w}_1, \quad \textit{for} \quad k = 1, 2.
\end{aligned} \tag{2.28}
$$

Where

$$
\tilde{w}_0 = \int_{b_1}^{b_2} g(y)(h_1(y) - h_2(y)) y, \tag{2.29}
$$

$$
\tilde{w}_1 = \int_{b_1}^{b_2} g(y)(h_1^2(y) - h_2^2(y)) dy, \tag{2.30}
$$

$$
\tilde{w}_2 = \int_{b_1}^{b_2} g(y)(h_1^2(y) - h_2^2(y)) dy. \tag{2.31}
$$

If the inequality (2.27) *holds in the reverse direction, then the inequality* (2.28) *holds in the opposite direction.*

The inequality in (2.28) *holds in the reverse direction if H is a 4-concave function.*

We use majorized tuples in the following result to construct bounds for the difference obtained from majorization inequality.

COROLLARY 2.3. *Let $H\in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ be two n-tuples such that $\mathbf{x}\succ\mathbf{y}$ and $x_j,y_j\in[d_1,d_2]$, for $j=1,2,3,\ldots,n$. Then*

$$\sum_{j=1}^{n} H(x_j)-\sum_{j=1}^{n} H(y_j)\leqslant \frac{H''(d_2)-H''(d_1)}{6(d_2-d_1)}\sum_{j=1}^{n}(x_j^3-y_j^3) + \frac{d_2H''(d_1)-d_1H''(d_2)}{2(d_2-d_1)}\sum_{j=1}^{n}\left(x_j^2-y_j^2\right). \tag{2.32}$$

Proof. Since $\mathbf{x}\succ\mathbf{y}$ and G_i is convex for each $i\in\{1,2,3,4\}$, so by majorization theorem [15], we have

$$\sum_{j=1}^{n} G_i(x_j,s)-\sum_{j=1}^{n} G_i\,(y_j,s)\geqslant 0.$$

Therefore, for $w_j=1$, the inequality (2.15) holds. Also, by majorization condition (1.5), the equality

$$H'(d_k)\sum_{j=1}^{n} w_j\,(x_j-y_j)=0$$

holds, for $k=1,2$. Hence using Theorem 2.1, we obtain (2.32). □

In the following result, we use Fuchs majorization inequality for the derivation of majorization type inequality for the 4-convex function.

COROLLARY 2.4. *Let $H\in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ be two decreasing n-tuples such that $x_j,y_j\in[d_1,d_2]$, for $j=1,2,3,\ldots,n$. Also, let $w_j\in\mathbb{R}$ for $j=1,2,\ldots,n$ with*

$$\sum_{j=1}^{q} w_jy_j\leqslant\sum_{j=1}^{q} w_jx_j \quad \textit{for} \quad q=1,2,\ldots,n-1,$$

$$\sum_{j=1}^{n} w_jy_j=\sum_{j=1}^{n} w_jx_j.$$

Then the following inequality holds:

$$\sum_{j=1}^{n} w_jH(x_j)-\sum_{j=1}^{n} w_jH(y_j)\leqslant\frac{H''(d_2)-H''(d_1)}{6(d_2-d_1)}\bar{w}_2+\frac{d_2H''(d_1)-d_1H''(d_2)}{2(d_2-d_1)}\bar{w}_1, \tag{2.33}$$

where $\bar{w_1}$ and $\bar{w_2}$ are defined in (2.18) *and* (2.19) *respectively.*

Proof. The idea of the proof is similar to the proof of Corollary 2.3 but instead of majorization theorem, using Fuchs majorization theorem [13]. □

The following corollary is the integral version of the preceding corollary.

COROLLARY 2.5. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function, let $h_1,h_2:[b_1,b_2] \to [d_1,d_2]$ be two decreasing functions, $g:[b_1,b_2] \to \mathbb{R}$ be any integrable function and*

$$\int_{b_1}^{\vartheta} h_2(y)g(y)dy \leqslant \int_{b_1}^{\vartheta} g(y)h_1(y)dy, \quad \textit{for} \quad \vartheta \in [b_1,b_2],$$

$$\int_{b_1}^{b_2} g(y)h_2(y)dy = \int_{b_1}^{b_2} g(y)h_1(y)dy.$$

Then the following inequality holds:

$$\begin{aligned} &\int_{b_1}^{b_2} g(y)H(h_1(y))dy - \int_{b_1}^{b_2} g(y)H(h_2(y))dy \\ &\leqslant \frac{H''(d_2)-H''(d_1)}{6(d_2-d_1)}\tilde{w}_2 + \frac{d_2H''(d_1)-d_1H''(d_2)}{2(d_2-d_1)}\tilde{w}_1, \end{aligned} \tag{2.34}$$

where $\tilde{w_1}$ and $\tilde{w_2}$ are defined in (2.30) and (2.31) respectively.

The following generalized majorization inequality has been obtained by applications of the Dragomir majorization result.

COROLLARY 2.6. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ be two real n-tuples such that $x_j,y_j \in [d_1,d_2]$, for $j=1,2,3,\ldots,n$. Also, let $\mathbf{w}=(w_1,w_2,\ldots,w_n)$ be non-negative real n-tuple with $W=\sum_{j=1}^{n} w_j > 0$. If $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}$ are monotonic in the same sense and $\sum_{j=1}^{n} w_jx_j = \sum_{j=1}^{n} w_jy_j$, then the inequality in (2.33) holds.*

Proof. The proof follows the same steps as the proof of Corollary 2.3, but use the Dragomir majorization theorem [11] rather than the majorization theorem. □

The integral form of the preceding corollary is given below.

COROLLARY 2.7. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and $h_1,h_2:[b_1,b_2] \to [d_1,d_2]$ be two integrable functions, $g:[b_1,b_2] \to \mathbb{R}$ be a non-negative integrable function with $\int_{b_1}^{b_2} g(y)dy > 0$. If h_2 and h_1-h_2 are monotonic in the same sense and $\int_{b_1}^{b_2} h_1(y)g(y)dy = \int_{b_1}^{b_2} h_2(y)g(y)dy$, then the inequality in (2.34) holds.*

The following generalized discrete version of majorization inequality has been obtained by application of the result of Maligranda et al. given in [18].

COROLLARY 2.8. *Let $H\in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x}=(x_1,x_2,\dots,x_n)$, $\mathbf{y}=(y_1,y_2,\dots,y_n)$ be two n-tuples such that $x_j,y_j\in[d_1,d_2]$ and $w_j\geqslant 0$ for $j=1,2,\dots,n$.*

(i) If $y_1\geqslant y_2\geqslant\dots\geqslant y_n$, then the inequality in (2.33) holds.

(ii) If $x_1\leqslant x_2\leqslant\dots\leqslant x_n$, then the inequality in (2.33) holds in opposite direction.

Proof. The proof of this corollary is similar to the proof of Corollary 2.3. □

The integral version of the above corollary is as follows.

COROLLARY 2.9. *Let $H\in C^2[d_1,d_2]$ be a 4-convex function, $h_1,h_2:[b_1,b_2]\to[d_1,d_2]$ be two integrable functions and $g:[b_1,b_2]\to\mathbb{R}$ be a non-negative integrable function.*

(i) If h_1 is an increasing function, then the inequality in (2.34) holds.

(ii) If h_2 is a decreasing function, then inequality in (2.34) holds in opposite direction.

Now, we are going to give our second main theorem.

THEOREM 2.10. *Let $H\in C^2[d_1,d_2]$ be a 4-convex function and $x_j,y_j\in[d_1,d_2]$, $w_j\in\mathbb{R}$ for $j=1,2,\dots,n$. Also, let G_i $(i=1,2,3,4)$ be Green functions as defined in (1.6)–(1.9) and (2.15) holds. Then*

$$\sum_{j=1}^{n}H(x_j)w_j-\sum_{j=1}^{n}H(y_j)w_j\geqslant H'(d_k)\bar{w}_0+\left(\frac{\bar{w}_1}{2}-d_k\bar{w}_0\right)H''\left(\frac{\frac{\bar{w}_2}{6}-\frac{d_k^2\bar{w}_0}{2}}{\frac{\bar{w}_1}{2}-d_k\bar{w}_0}\right),\ \textit{for}\ k=1,2. \tag{2.35}$$

Where $\bar{w_0}$, $\bar{w_1}$ and $\bar{w_2}$ are defined in (2.17), (2.18) and (2.19) respectively.

If (2.15) holds in the opposite direction, then (2.35) holds in the opposite direction.

If H is a 4-concave function, then (2.35) holds in the opposite direction.

Proof. Using (1.14) with $h_3^*(s)$ replaced by $F_i(s)=\sum_{j=1}^{n}w_jG_i(x_j,s)-\sum_{j=1}^{n}w_jG_i(y_j,s)$ and H replaced by H'', we get

$$\int_{d_1}^{d_2}F_i(s)dsH''\left(\frac{1}{\int_{d_1}^{d_2}F_i(s)ds}\int_{d_1}^{d_2}sF_i(s)ds\right)\leqslant\int_{d_1}^{d_2}F_i(s)H''(s)ds. \tag{2.36}$$

Using (2.20) in (2.36), we get

$$\begin{aligned}&\int_{d_1}^{d_2}F_i(s)dsH''\left(\frac{1}{\int_{d_1}^{d_2}F_i(s)\,ds}\int_{d_1}^{d_2}sF_i(s)\,ds\right)\\&\leqslant\sum_{j=1}^{n}H(x_j)w_j-\sum_{j=1}^{n}H(y_j)w_j-H'(d_2)\sum_{j=1}^{n}(x_j-y_j)w_j.\end{aligned} \tag{2.37}$$

Now, if $H(s)=\frac{s^2}{2}$, then $H^{'}(s)=s$ and $H^{''}(s)=1$ and using these functions in (2.20), we get

$$\int_{d_1}^{d_2} F_i(s)ds=\frac{1}{2}\left(\sum_{j=1}^{n} w_j x_j^2-\sum_{j=1}^{n} w_j y_j^2\right)-d_2\sum_{j=1}^{n} w_j\,(x_j-y_j). \tag{2.38}$$

Now, if $H(s)=\frac{s^3}{6}$, then $H'(s)=\frac{s^2}{2}$ and $H^{''}(s)=s$ and using these functions in (2.20), we get

$$\int_{d_1}^{d_2} sF_i(s)ds=\frac{1}{6}\left(\sum_{j=1}^{n} w_j x_j^3-\sum_{j=1}^{n} w_j y_j^3\right)-\frac{d_2^2}{2}\sum_{j=1}^{n} w_j\,(x_j-y_j). \tag{2.39}$$

Using (2.39) and (2.38) in (2.37), we get

$$\left(\frac{1}{2}\bar{w}_1-d_2\sum_{j=1}^{n} w_j\,(x_j-y_j)\right)H^{''}\left(\frac{\frac{1}{6}\bar{w}_2-\frac{d_2^2}{2}\sum_{j=1}^{n} w_j(x_j-y_j)}{\frac{1}{2}\bar{w}_1-d_2\sum_{j=1}^{n} w_j(x_j-y_j)}\right)$$
$$\leqslant \sum_{j=1}^{n} H(x_j)w_j-\sum_{j=1}^{n} H(y_j)w_j-H'(d_2)\sum_{j=1}^{n} w_j\,(x_j-y_j), \tag{2.40}$$

which is equivalent to (2.35) for $k=2$. Similarly we can prove for the case $k=1$. □

The following is an integral form of the aforementioned theorem:

THEOREM 2.11. *Let $H\in C^2[d_1,d_2]$ be a 4-convex function, let $h_1,h_2:[b_1,b_2]\to[d_1,d_2]$, $g:[b_1,b_2]\to\mathbb{R}$ be three integrable functions, let G_i $(i=1,2,3,4,)$ be Green functions as defined in (1.6)–(1.9) and (2.27) holds. Then the following inequality holds:*

$$\int_{b_1}^{b_2} H(h_1(y))g(y)\,dy-\int_{b_1}^{b_2} H(h_2(y))g(y)dy$$
$$\geqslant H'(d_k)\tilde{w}_0+\left(\frac{\tilde{w}_1}{2}-d_k\tilde{w}_0\right)H^{''}\left(\frac{\frac{\tilde{w}_2}{6}-\frac{d_k^2\tilde{w}_0}{2}}{\frac{\tilde{w}_1}{2}-d_k\tilde{w}_0}\right),\quad \textit{for}\quad k=1,2. \tag{2.41}$$

Where $\tilde{w}_0$, $\tilde{w}_1$ and $\tilde{w}_2$ are defined in (2.17), (2.18) and (2.19) respectively.

If the inequality (2.27) holds in the opposite way, then the inequality (2.41) holds in the opposite direction.

The inequality in (2.41) holds in the opposite direction if H is a 4-concave function.

In the following result, we use majorized tuples and derive bounds for the difference obtained from majorization inequality.

COROLLARY 2.12. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and* $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ *be two n-tuples such that $\mathbf{x} \succ \mathbf{y}$ and $x_j,y_j \in [d_1,d_2]$, for $j=1,2,3,\ldots,n$. Then the following inequality holds:*

$$\sum_{j=1}^{n} H(x_j) - \sum_{j=1}^{n} H(y_j) \geqslant \frac{1}{2}\left(\sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} y_j^2\right) H''\left(\frac{\sum_{j=1}^{n} x_j^3 - \sum_{j=1}^{n} y_j^3}{3\left(\sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} y_j^2\right)}\right). \tag{2.42}$$

Proof. Since $\mathbf{x} \succ \mathbf{y}$ and G_i is convex for each $i \in \{1,2,3,4\}$, so by majorization theorem [15], we have

$$\sum_{j=1}^{n} G_i(x_j,s) - \sum_{j=1}^{n} G_i(y_j,s) \geqslant 0.$$

Therefore, for $w_j = 1$, the inequality (2.15) holds. Also, by majorization condition (1.1), the equality

$$\sum_{j=1}^{n} (x_j - y_j) = 0$$

holds. Hence applying Theorem 2.10, we obtain (2.42). □

The following generalized majorization inequality for 4-convex function has been obtained by applying of Fuchs majorization result [13].

COROLLARY 2.13. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and* $\mathbf{y}=(y_1,y_2,\ldots,y_n)$, $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ *be two decreasing n-tuples such that $x_j,y_j \in [d_1,d_2]$, for $j=1,2,3,\ldots,n$. Also, let $w_j \in \mathbb{R}$ for $j=1,2,\ldots,n$ and*

$$\sum_{j=1}^{q} w_j x_j \geqslant \sum_{j=1}^{q} w_j y_j \quad \textit{for} \quad q=1,2,3,\ldots,n-1,$$

$$\sum_{j=1}^{n} w_j x_j = \sum_{j=1}^{n} w_j y_j.$$

Then the following inequality holds:

$$\sum_{j=1}^{n} w_j H(x_j) - \sum_{j=1}^{n} w_j H(y_j) \geqslant \frac{\bar{w_1}}{2} H''\left(\frac{\bar{w_2}}{3\bar{w_1}}\right), \tag{2.43}$$

where $\bar{w_1}$ and $\bar{w_2}$ are defined in (2.18) and (2.19) respectively.

Proof. The proof is analogous to the proof of Corollary 2.12 but use Fuchs majorization theorem instead of the classical majorization theorem. □

The integral version of the preceding corollary is as follows.

COROLLARY 2.14. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function, let $h_1,h_2 : [b_1,b_2] \to [d_1,d_2]$ be two decreasing functions, $g : [b_1,b_2] \to \mathbb{R}$ be any integrable function and*

$$\int_{b_1}^{\vartheta} h_1(y)g(y)dy \geqslant \int_{b_1}^{\vartheta} g(y)h_2(y)dy \quad \text{for} \quad \vartheta \in [b_1,b_2],$$

$$\int_{b_1}^{b_2} g(y)h_1(y)dy = \int_{b_1}^{b_2} g(y)h_2(y)dy.$$

Then the following inequality holds:

$$\int_{b_1}^{b_2} g(y)H(h_1(y))dy - \int_{b_1}^{b_2} g(y)H(h_2(y))dy \geqslant \frac{\tilde{w_1}}{2} H''\left(\frac{\tilde{w_2}}{3\tilde{w_1}}\right), \tag{2.44}$$

where $\tilde{w_1}$ and $\tilde{w_2}$ are defined in (2.18) and (2.19) respectively.

The following generalized majorization inequality has been obtained by applying Dragomir's majorization result.

COROLLARY 2.15. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ be two real n-tuples such that $x_j,y_j \in [d_1,d_2]$, for $j=1,2,3,\ldots,n$. Also, let $\mathbf{w}=(w_1,w_2,\ldots,w_n)$ be non-negative real n-tuple such that $W=\sum_{j=1}^{n} w_j > 0$. If $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}$ are monotonic in the same sense and $\sum_{j=1}^{n} w_j x_j = \sum_{j=1}^{n} w_j y_j$, then the inequality in (2.43) holds.*

Proof. The proof is similar to the proof of Corollary 2.6. □

The following corollary is the integral version of the above corollary.

COROLLARY 2.16. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and $h_1,h_2 : [b_1,b_2] \to [d_1,d_2]$ be two integrable functions, $g : [b_1,b_2] \to \mathbb{R}$ be a non-negative integrable function with $\int_{b_1}^{b_2} g(y)dy > 0$. If h_2 and $h_1 - h_2$ are monotonic in the same sense and $\int_{b_1}^{b_2} h_1(y)g(y)dy = \int_{b_1}^{b_2} h_2(y)g(y)dy$, then the inequality in (2.44) holds.*

The following generalized discrete version of majorization inequality has been obtained by applications of Maligranda majorization result [18].

COROLLARY 2.17. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ be n-tuples such that $x_j,y_j \in [d_1,d_2]$ and $w_j > 0$ for $j=1,2,\ldots,n$.*

(i) If $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n$, then the inequality in (2.43) holds.

(ii) If $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, then the inequality in (2.43) holds in the opposite direction.

Proof. The proof of this corollary is identical to that of the proof of Corollary 2.12. □

The integral form of the preceding corollary is given below.

COROLLARY 2.18. *Let $H \in C^2[d_1,d_2]$ be a 4-convex function, $h_1,h_2 : [b_1,b_2] \to [d_1,d_2]$ be two integrable functions and $g : [b_1,b_2] \to \mathbb{R}$ be a non-negative integrable function.*

(i) If h_1 is an increasing function, then the inequality in (2.44) holds.

(ii) If h_2 is a decreasing function, then the inequality in (2.44) is reversed.

3. Applications in information theory

DEFINITION 3.1. ([4]) (Csiszár divergence) Let $g : [d_1,d_2] \to \mathbb{R}$ be a function, $\mathbf{u}=(u_1,u_2,\ldots,u_n) \in \mathbb{R}^n$ and $\mathbf{w}=(w_1,w_2,\ldots,w_n)\in\mathbb{R}^n_+$ with $\frac{u_j}{w_j}\in[d_1,d_2]$ $(j=1,2,\ldots,n)$. Then the Csiszár divergence is defined as

$$\bar{D}_c(\mathbf{u},\mathbf{w}) = \sum_{j=1}^{n} w_j g\left(\frac{u_j}{w_j}\right).$$

THEOREM 3.2. *Let $g \in C^2[d_1,d_2]$ be a 4-convex function and $\mathbf{r}=(r_1,r_2,\ldots,r_n)$, $\mathbf{u}=(u_1,u_2,\ldots,u_n) \in \mathbb{R}^n$. Also, let $\mathbf{w}=(w_1,w_2,\ldots,w_n) \in \mathbb{R}^n_+$ such that $\frac{u_j}{w_j}, \frac{r_j}{w_j} \in [d_1,d_2]$ for $j=1,2,\ldots,n$ and G_i $(i=1,2,3,4)$ be Green functions as defined in (1.6)–(1.9). If*

$$\sum_{j=1}^{n} w_j G_i\left(\frac{r_j}{w_j},s\right) - \sum_{j=1}^{n} w_j G_i\left(\frac{u_j}{w_j},s\right) \geqslant 0 \quad \textit{for} \quad i \in \{1,2,3,4\}, \tag{3.45}$$

then

$$\begin{aligned}
&\bar{D}_c(\mathbf{r},\mathbf{w}) - \bar{D}_c(\mathbf{u},\mathbf{w}) \\
&\leqslant g'(d_k)\sum_{j=1}^{n}(r_j-u_j) + \frac{g''(d_2)-g''(d_1)}{6(d_2-d_1)}\left(\sum_{j=1}^{n}\left(\frac{r_j^3}{w_j^2}-\frac{u_j^3}{w_j^2}\right)\right) \\
&\quad - \frac{g''(d_2)}{(d_2-d_1)}\left(\frac{d_k^2}{2}-d_1d_k\right)\sum_{j=1}^{n}(r_j-u_j) \\
&\quad + \frac{d_2g''(d_1)-d_1g''(d_2)}{2(d_2-d_1)}\left(\sum_{j=1}^{n}\left(\frac{r_j^2}{w_j}-\frac{u_j^2}{w_j}\right)\right) \\
&\quad - \frac{g''(d_1)}{d_2-d_1}\left(d_kd_2-\frac{d_k^2}{2}\right)\sum_{j=1}^{n}(r_j-u_j), \quad \textit{for} \quad k=1,2.
\end{aligned} \tag{3.46}$$

Proof. Using (2.16) for $H=g$, $x_j=\frac{r_j}{w_j}$, $y_j=\frac{u_j}{w_j}$, we get (3.46). □

THEOREM 3.3. *Let* $g \in C^2[d_1,d_2]$ *be a* 4*-convex function and* $\mathbf{u} = (u_1,u_2,\ldots,u_n)$, $\mathbf{r} = (r_1,r_2,r_3,\ldots,r_n) \in \mathbb{R}^n$ *and* $\mathbf{w} = (w_1,w_2,w_3,\ldots,w_n) \in \mathbb{R}^n_+$ *such that* $\frac{u_j}{w_j}, \frac{r_j}{w_j} \in [d_1,d_2]$ *for* $j = 1,2,\ldots,n$. *If* (3.45) *holds, then*

$$\begin{aligned}&\bar{D}_c(\mathbf{r},\mathbf{w}) - \bar{D}_c(\mathbf{u},\mathbf{w})\\ &\geqslant g'(d_k)\sum_{j=1}^{n}(r_j-u_j) + \left(\frac{\hat{w}_1}{2} - d_k\sum_{j=1}^{n}(r_j-u_j)\right)g''\left(\frac{\frac{\hat{w}_2}{6} - \frac{d_k^2\sum_{j=1}^{n}(r_j-u_j)}{2}}{\frac{\hat{w}_1}{2} - d_k\sum_{j=1}^{n}(r_j-u_j)}\right),\\ &for \quad k = 1,2.\end{aligned} \tag{3.47}$$

Where

$$\hat{w}_1 = \sum_{j=1}^{n}\frac{r_j^2}{w_j} - \sum_{j=1}^{n}\frac{u_j^2}{w_j} \quad and \quad \hat{w}_2 = \sum_{j=1}^{n}\frac{r_j^3}{w_j^2} - \sum_{j=1}^{n}\frac{u_j^3}{w_j^2}.$$

Proof. Using (2.35) for $H = g$, $x_j = \frac{r_j}{w_j}$ and $y_j = \frac{u_j}{w_j}$, we get (3.47). □

DEFINITION 3.4. ([4]) (Kullback-Leibler divergence) Let $\mathbf{u} = (u_1,u_2,\ldots,u_n)$ and $\mathbf{w} = (w_1,w_2,\ldots,w_n)$ be two positive probability distributions, then the Kullback-Leibler divergence is defined as

$$D_{kl}(\mathbf{u},\mathbf{w}) = \sum_{j=1}^{n} u_j \log\frac{u_j}{w_j}.$$

COROLLARY 3.5. *Let* $[d_1,d_2] \subseteq \mathbb{R}^+$ *and* $\mathbf{r} = (r_1,r_2,\ldots,r_n)$, $\mathbf{u} = (u_1,u_2,\ldots,u_n)$, *and* $\mathbf{w} = (w_1,w_2,\ldots,w_n)$ *be positive probability distributions such that* $\frac{r_j}{w_j}, \frac{u_j}{w_j} \in [d_1,d_2]$ *for* $j = 1,2,\ldots,n$. *Also, let* G_i $(i = 1,2,3,4)$ *be Green functions as defined in* (1.6)–(1.9). *If* (3.45) *holds, then*

$$\begin{aligned}&D_{kl}(\mathbf{r},\mathbf{w}) - D_{kl}(\mathbf{u},\mathbf{w})\\ &\leqslant \left(\log d_k + 1 + \frac{d_k^2 - 2d_k(d_1+d_2)}{2d_1d_2}\right)\sum_{j=1}^{n}(r_j-u_j)\\ &\quad + \frac{d_1+d_2}{2d_1d_2}\left(\sum_{j=1}^{n}\left(\frac{r_j^2}{w_j} - \frac{u_j^2}{w_j}\right)\right) - \frac{1}{6d_1d_2}\left(\sum_{j=1}^{n}\left(\frac{r_j^3}{w_j^2} - \frac{u_j^3}{w_j^2}\right)\right), \quad for \quad k = 1,2.\end{aligned} \tag{3.48}$$

Proof. Let $H(y) = y\log y$, $\forall y \in [d_1,d_2]$. Then H is a 4-convex because $H''''(y) = \frac{2}{y^3} > 0$, therefore using (2.16) for $H(y) = y\log y$ and $x_j = \frac{r_j}{w_j}, y_j = \frac{u_j}{w_j}$, we get (3.48). □

COROLLARY 3.6. *Let* $[d_1,d_2] \subseteq \mathbb{R}^+$ *and* $\mathbf{r} = (r_1,r_2,\ldots,r_n)$, $\mathbf{u} = (u_1,u_2,\ldots,u_n)$, *and* $\mathbf{w} = (w_1,w_2,\ldots,w_n)$ *be positive probability distributions such that* $\frac{r_j}{w_j}, \frac{u_j}{w_j} \in [d_1,d_2]$ *for* $j = 1,2,\ldots,n$. *If* (3.45) *holds, then*

$$\begin{aligned}
&D_{kl}(\mathbf{r},\mathbf{w}) - D_{kl}(\mathbf{u},\mathbf{w}) \\
&\geqslant (1+\log d_k)\sum_{j=1}^{n}\left(r_j-u_j\right) \\
&\quad + \left(\frac{1}{2}\check{w}_1 - d_k\sum_{j=1}^{n}(r_j-u_j)\right) g''\left(\frac{\frac{1}{6}\check{w}_2 - \frac{d_k^2}{2}\sum_{j=1}^{n}(r_j-u_j)}{\frac{1}{2}\check{w}_1 - d_k\sum_{j=1}^{n}(r_j-u_j)}\right), \quad for \quad k = 1,2.
\end{aligned} \tag{3.49}$$

Where

$$\check{w}_1 = \sum_{j=1}^{n}\frac{r_j^2}{w_j} - \sum_{j=1}^{n}\frac{u_j^2}{w_j} \quad and \quad \check{w}_2 = \sum_{j=1}^{n}\frac{r_j^3}{w_j^2} - \sum_{j=1}^{n}\frac{u_j^3}{w_j^2}.$$

Proof. Using (2.35) for $H(y) = y\log y, \forall y \in [d_1,d_2]$, $x_j = \frac{r_j}{w_j}$ and $y_j = \frac{u_j}{w_j}$, we get (3.49). □

DEFINITION 3.7. ([4,12]) (Shannon-entropy) Let $\mathbf{u} = (u_1,u_2,\ldots,u_n)$ be a positive probability distribution. Then the Shannon-entropy is defined by

$$E_s(\mathbf{u}) = -\sum_{j=1}^{n} u_j \log u_j.$$

COROLLARY 3.8. *Let* $[d_1,d_2] \subseteq \mathbb{R}^+$ *and* $\mathbf{r}=(r_1,r_2,\ldots,r_n)$ *and* $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ *be positive probability distributions such that* $u_j,r_j \in [d_1,d_2]$ *for* $j = 1,2,\ldots,n$. *Also, let* G_i $(i = 1,2,3,4)$ *be Green functions as defined in* (1.6)–(1.9).

If

$$\sum_{j=1}^{n} G_i(r_j,s) - \sum_{j=1}^{n} G_i(u_j,s) \geqslant 0 \quad for \quad i \in \{1,2,3,4\}, \tag{3.50}$$

then the following inequality holds:

$$E_s(\mathbf{r}) - E_s(\mathbf{u}) \leqslant \frac{d_1+d_2}{2d_1d_2}\sum_{j=1}^{n}\left(u_j^2 - r_j^2\right) - \frac{1}{6d_1d_2}\sum_{j=1}^{n}(u_j^3 - r_j^3). \tag{3.51}$$

Proof. Let $H(u) = u\log u$, $u \in [d_1,d_2]$. Then $H''''(u) = \frac{2}{u^3} > 0$, which shows that H is a 4-convex. Since $\mathbf{u}$ and $\mathbf{r}$ are positive probability distributions, therefore the equality

$$\sum_{j=1}^{n}(u_j - r_j) = 0 \tag{3.52}$$

holds. So using (2.16) for $H(u) = u\log u$ and $w_j = 1$ for $j = 1,2,3,\ldots,n$, we get (3.51). $\square$

COROLLARY 3.9. *Let $[d_1,d_2] \subseteq \mathbb{R}^+$ and $\mathbf{r}=(r_1,r_2,\ldots,r_n)$ and $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ be positive probability distributions such that $u_j, r_j \in [d_1,d_2]$ for $j = 1,2,\ldots,n$ and (3.50) holds. Then the following inequality holds:*

$$E_s(\mathbf{r}) - E_s(\mathbf{u}) \geqslant \left(\frac{1}{2}\sum_{j=1}^{n} u_j^2 - \sum_{j=1}^{n} r_j^2\right) H''\left(\frac{\sum_{j=1}^{n} u_j^3 - 6\sum_{j=1}^{n} r_j^3}{3\left(\sum_{j=1}^{n} u_j^2 - 2\sum_{j=1}^{n} r_j^2\right)}\right). \tag{3.53}$$

Proof. Since $\mathbf{u}$ and $\mathbf{r}$ are positive probability distributions, therefore the equality (3.52) holds. So using (2.35) for $H(u) = u\log u$ and $w_j = 1$ for $j = 1,2,3,\ldots,n$, we get (3.53). $\square$

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