Journal of Mathematical Inequalities
Volume 18, Number 1 (2024), 103–126 doi:10.7153/jmi-2024-18-07

GENERALIZED WEIGHTED HARDY'S INEQUALITIES WITH COMPACT PERTURBATIONS

HIROSHI ANDO AND TOSHIO HORIUCHI

(*Communicated by T. Burić*)

Abstract. Let Ω be a bounded domain of $\mathbf{R}^N$ $(N \geqslant 1)$ with boundary of class C^2. In the present paper we shall study a variational problem relating the weighted Hardy inequalities with sharp missing terms established in [8]. As weights we treat non-doubling functions of the distance $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ to the boundary $\partial\Omega$.

1. Introduction

Let $W(\mathbf{R}_+)$ be a class of functions

$$\{w(t) \in C^1(\mathbf{R}_+) : w(t) > 0, \lim_{t \to +0} w(t) = a \text{ for some } a \in [0, \infty]\}$$

with $\mathbf{R}_+ = (0, \infty)$. For $1 < p < \infty$, as weights of Hardy's inequalities we adopt functions $W_p(t) = w(t)^{p-1}$ with $w(t) \in P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$, where

$$\begin{cases} P(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \notin L^1((0, \eta)) \text{ for some } \eta > 0\}, \\ Q(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \in L^1((0, \eta)) \text{ for any } \eta > 0\}. \end{cases} \tag{1.1}$$

Clearly $W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$ and $P(\mathbf{R}_+) \cap Q(\mathbf{R}_+) = \emptyset$. (For the precise definitions see the section 2. See also [8], [9].) A positive continuous function $w(t)$ on $\mathbf{R}_+$ is said to be a doubling weight if there exists a positive number C such that we have

$$C^{-1} w(t) \leqslant w(2t) \leqslant C w(t) \qquad \text{for all} \quad t \in \mathbf{R}_+. \tag{1.2}$$

When $w(t)$ does not possess this property, $w(t)$ is said to be a non-doubling weight in the present paper. In one-dimensional case we typically treat a weight function $w(t)$ that may vanish or blow up in infinite order such as $e^{-1/t}$ or $e^{1/t}$ at $t = 0$. In such cases the limit of ratio $w(t)/w(2t)$ as $t \to +0$ may become 0 or $+\infty$, and hence they are regarded as non-doubling weights according to our notion.

Mathematics subject classification (2020): Primary 35J70; Secondary 35J60, 34L30, 26D10.
Keywords and phrases: Weighted Hardy's inequalities, nonlinear eigenvalue problem, weak Hardy property, p-Laplace operator with weights.
This research was partially supported by Grant-in-Aid for Scientific Research (No. 20K03670, No. 21K03304).

Paper JMI-18-07

In [8], we have established N-dimensional Hardy inequalities with non-doubling weights being functions of the distance $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ to the boundary $\partial\Omega$, where Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. In this paper we shall study a variational problem relating to those new inequalities.

We prepare more notations to describe our results. Let $1 < p < \infty$. For $W_p(t) = w(t)^{p-1}$ with $w(t) \in W(\mathbf{R}_+)$, we define a weight function $W_p(\delta(x))$ on Ω by

$$W_p(\delta(x)) = (W_p \circ \delta)(x).$$

By $L^p(\Omega; W_p(\delta))$ we denote the space of Lebesgue measurable functions with weight $W_p(\delta(x))$, for which

$$\|u\|_{L^p(\Omega;W_p(\delta))} = \left(\int_\Omega |u(x)|^p W_p(\delta(x))\, dx \right)^{1/p} < +\infty. \tag{1.3}$$

$W_0^{1,p}(\Omega; W_p(\delta))$ is given by the completion of $C_c^\infty(\Omega)$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}(\Omega;W_p(\delta))} = \||\nabla u|\|_{L^p(\Omega;W_p(\delta))} + \|u\|_{L^p(\Omega;W_p(\delta))}. \tag{1.4}$$

Then, $W_0^{1,p}(\Omega; W_p(\delta))$ becomes a Banach space with the norm $\|\cdot\|_{W_0^{1,p}(\Omega;W_p(\delta))}$. Under these preparation we recall the weighted Hardy inequalities in [8]. (See Theorem 2.1 and its corollary in Section 2.) In particular for $w(t) \in Q(\mathbf{R}_+)$, we have a simple inequality as Corollary 2.1, which is a generalization of classical Hardy's inequality:

$$\int_\Omega |\nabla u(x)|^p W_p(\delta(x))\, dx \geqslant \gamma \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\, dx \tag{1.5}$$

for $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$, where η_0 is a sufficiently small positive number, γ is some positive constant and $F_{\eta_0}(t)$ is a positive function defined in Definition 2.3. In particular if $w(t) = 1$, then $F_{\eta_0}(t) = t$ $(0 < t \leqslant \eta_0)$ and (1.5) becomes a well-known Hardy's inequality, which is valid for a bounded domain Ω of $\mathbf{R}^N$ with Lipschitz boundary (cf. [4], [6], [10], [11]). Further if Ω is convex, then $\gamma = \Lambda_p := (1 - 1/p)^p$ holds for arbitrary $1 < p < \infty$ (see [11]).

In the present paper we consider the following variational problem relating the general Hardy's inequalities established in [8]. For $\lambda \in \mathbf{R}$, $W_p(t) = w(t)^{p-1}$ and $w(t) \in W_A(\mathbf{R}_+) (\subset W(\mathbf{R}_+))$, the following variational problem (1.6) can be associated with (1.5):

$$J^w_{p,\lambda} = \inf_{u \in W_0^{1,p}(\Omega;W_p(\delta)) \setminus \{0\}} \chi^w_{p,\lambda}(u), \tag{1.6}$$

where

$$\chi^w_{p,\lambda}(u) = \frac{\int_\Omega |\nabla u(x)|^p W_p(\delta(x))\, dx - \lambda \int_\Omega |u(x)|^p W_p(\delta(x))\, dx}{\int_\Omega |u(x)|^p W_p(\delta(x)) / F_{\eta_0}(\delta(x))^p dx}. \tag{1.7}$$

Here $W_A(\mathbf{R}_+) = P_A(\mathbf{R}_+) \cup Q_A(\mathbf{R}_+)$ is a subclass of $W(\mathbf{R}_+)$ defined by Definition 2.6 and η_0 is a sufficiently small positive number such that the Hardy inequalities in Theorem 2.1 and Corollary 2.1 are valid. Note that $J^w_{p,0}$ gives the best constant in (1.5), the function $\lambda \mapsto J^w_{p,\lambda}$ is non-increasing on $\mathbf{R}$ and $J^w_{p,\lambda} \to -\infty$ as $\lambda \to \infty$.

When $p = 2$ and $w(t) = 1$, this variational problem (1.6) was originally studied in [4]. Then, the problem (1.6) was intensively studied in [2] in the case that $1 < p < \infty$ and $w(t) = t^{\alpha p/(p-1)} \in Q_A(\mathbf{R}_+)$ with $\alpha < 1 - 1/p$. In this paper we further investigate the variational problem (1.6) with non-doubling weight functions $w(t) \in W_A(\mathbf{R}_+)$ and we make clear the attainability of the infimum $J^w_{p,\lambda}$ as Theorem 3.1 and Theorem 3.2.

This paper is organized in the following way: In Subsection 2.1 we introduce a class of weight functions $W(\mathbf{R}_+)$ and two subclasses $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$ together with so-called Hardy functions, which are crucial in this paper. Further a notion of admissibilities for $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$ is introduced. In Subsection 2.2, we recall the weighted Hardy's inequalities in [8] which are crucial in this work. In Section 3, the main results are described. Theorem 3.1 and Theorem 3.2 are established in Section 4 and Section 5 respectively.

2. Preliminaries

2.1. Weight functions

First we introduce a class of weight functions according to [8] which is crucial in this paper.

DEFINITION 2.1. Let us set $\mathbf{R}_+ = (0,\infty)$ and

$$W(\mathbf{R}_+) = \{w(t) \in C^1(\mathbf{R}_+) : w(t) > 0, \lim_{t\to+0} w(t) = a \text{ for some } a \in [0,\infty]\}. \tag{2.1}$$

In the next we define two subclasses of $W(\mathbf{R}_+)$.

DEFINITION 2.2. Let us set

$$P(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \notin L^1((0,\eta)) \text{ for some } \eta > 0\}, \tag{2.2}$$

$$Q(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \in L^1((0,\eta)) \text{ for any } \eta > 0\}. \tag{2.3}$$

Here we give fundamental examples:

EXAMPLE 2.1.

1. $t^\alpha \in P(\mathbf{R}_+)$ if $\alpha \geqslant 1$ and $t^\alpha \in Q(\mathbf{R}_+)$ if $\alpha < 1$.
2. $e^{-1/t} \in P(\mathbf{R}_+)$ and $e^{1/t} \in Q(\mathbf{R}_+)$.
3. For $\alpha \in \mathbf{R}$, $t^\alpha e^{-1/t} \in P(\mathbf{R}_+)$ and $t^\alpha e^{1/t} \in Q(\mathbf{R}_+)$.

REMARK 2.1.

1. $W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$ and $P(\mathbf{R}_+) \cap Q(\mathbf{R}_+) = \emptyset$ hold.
2. If $w(t)^{-1} \notin L^1((0,\eta))$ for some $\eta > 0$, then $w(t)^{-1} \notin L^1((0,\eta))$ for any $\eta > 0$. Similarly if $w(t)^{-1} \in L^1((0,\eta))$ for some $\eta > 0$, then $w(t)^{-1} \in L^1((0,\eta))$ for any $\eta > 0$.

3. If $w(t) \in P(\mathbf{R}_+)$, then $\lim_{t\to+0} w(t) = 0$. Hence by setting $w(0) = 0$, $w(t)$ is uniquely extended to a continuous function on $[0,\infty)$. On the other hand if $w(t) \in Q(\mathbf{R}_+)$, then possibly $\lim_{t\to+0} w(t) = +\infty$.

In the next we define functions such as $F_\eta(t)$ and $G_\eta(t)$ in order to introduce variants of the Hardy potential like $F_{\eta_0}(\delta(x))^{-p}$ in (1.5).

DEFINITION 2.3. Let $\mu > 0$ and $\eta > 0$. For $w(t) \in W(\mathbf{R}_+)$, we define the followings:

1. When $w(t) \in P(\mathbf{R}_+)$,

$$F_\eta(t;w,\mu) = \begin{cases} w(t)\left(\mu + \int_t^\eta w(s)^{-1}\,ds\right) & \text{if } t \in (0,\eta), \\ w(\eta)\mu & \text{if } t \geqslant \eta, \end{cases} \tag{2.4}$$

$$G_\eta(t;w,\mu) = \begin{cases} \mu + \int_t^\eta F_\eta(s;w,\mu)^{-1}\,ds & \text{if } t \in (0,\eta), \\ \mu & \text{if } t \geqslant \eta. \end{cases} \tag{2.5}$$

2. When $w(t) \in Q(\mathbf{R}_+)$,

$$F_\eta(t;w) = \begin{cases} w(t)\int_0^t w(s)^{-1}\,ds & \text{if } t \in (0,\eta), \\ w(\eta)\int_0^\eta w(s)^{-1}\,ds & \text{if } t \geqslant \eta, \end{cases} \tag{2.6}$$

$$G_\eta(t;w,\mu) = \begin{cases} \mu + \int_t^\eta F_\eta(s;w)^{-1}\,ds & \text{if } t \in (0,\eta), \\ \mu & \text{if } t \geqslant \eta. \end{cases} \tag{2.7}$$

3. $F_\eta(t;w,\mu)$ and $F_\eta(t;w)$ are abbreviated as $F_\eta(t)$. $G_\eta(t;w,\mu)$ is abbreviated as $G_\eta(t)$.

4. For $w(t) \in P(\mathbf{R}_+)$ or $Q(\mathbf{R}_+)$, we define

$$W_p(t) = w(t)^{p-1}. \tag{2.8}$$

REMARK 2.2. In the definition (2.5), one can replace $G_\eta(t;w,\mu)$ with the more general $G_\eta(t;w,\mu,\mu') = \mu' + \int_t^\eta F_\eta(s;w,\mu)^{-1}ds$ if $t \in (0,\eta)$, $G_\eta(t;w,\mu,\mu') = \mu'$ if $t \geqslant \eta$ with $\mu' > 0$. However, for simplicity this paper uses (2.5).

Here we give fundamental examples:

EXAMPLE 2.2. Let $w(t) = t^\alpha$ for $\alpha \in \mathbf{R}$.

1. When $\alpha > 1$, $F_\eta(t) = t/(\alpha-1)$ and $G_\eta(t) = \mu + (\alpha-1)\log(\eta/t)$ for $t \in (0,\eta)$ provided that $\mu = \eta^{1-\alpha}/(\alpha-1)$.

2. When $\alpha = 1$, $F_\eta(t) = t(\mu + \log(\eta/t))$ and $G_\eta(t) = \mu - \log\mu + \log(\mu + \log(\eta/t))$ for $t \in (0,\eta)$.

3. When $\alpha < 1$, $F_\eta(t) = t/(1-\alpha)$ and $G_\eta(t) = \mu + (1-\alpha)\log(\eta/t)$ for $t \in (0,\eta)$.

By using integration by parts we see the followings:

EXAMPLE 2.3.

1. When either $w(t) = e^{-1/t} \in P(\mathbf{R}_+)$ or $w(t) = e^{1/t} \in Q(\mathbf{R}_+)$, we have $F_\eta(t) = O(t^2)$ as $t \to +0$.

2. Moreover, if $w(t) = \exp(\pm t^{-\alpha})$ with $\alpha > 0$, then $F_\eta(t) = O(t^{\alpha+1})$ as $t \to +0$. In fact, it holds that $\lim_{t\to+0} F_\eta(t)/t^{\alpha+1} = 1/\alpha$.

In a similar way we define the following:

DEFINITION 2.4. Let $\mu > 0$ and $\eta > 0$. For $w(t) \in W(\mathbf{R}_+)$, we define the followings:

1. When $w(t) \in P(\mathbf{R}_+)$,

$$f_\eta(t;w,\mu) = \begin{cases} \mu + \int_t^\eta w(s)^{-1}\,ds & \text{if } t \in (0,\eta), \\ \mu & \text{if } t \geqslant \eta. \end{cases} \tag{2.9}$$

2. When $w(t) \in Q(\mathbf{R}_+)$,

$$f_\eta(t;w) = \begin{cases} \int_0^t w(s)^{-1}\,ds & \text{if } t \in (0,\eta), \\ \int_0^\eta w(s)^{-1}\,ds & \text{if } t \geqslant \eta. \end{cases} \tag{2.10}$$

3. $f_\eta(t;w,\mu)$ and $f_\eta(t;w)$ are abbreviated as $f_\eta(t)$.

REMARK 2.3.

1. We note that for $t \in (0,\eta)$

$$\begin{cases} \frac{d}{dt}\log f_\eta(t) = -F_\eta(t)^{-1} & \text{if } w(t) \in P(\mathbf{R}_+), \\ \frac{d}{dt}\log f_\eta(t) = F_\eta(t)^{-1} & \text{if } w(t) \in Q(\mathbf{R}_+), \\ \frac{d}{dt}\log G_\eta(t) = -(F_\eta(t)G_\eta(t))^{-1}, & \\ \frac{d}{dt}G_\eta(t)^{-1} = (F_\eta(t)G_\eta(t)^2)^{-1} & \text{if } w(t) \in W(\mathbf{R}_+). \end{cases} \tag{2.11}$$

By Definition 2.3, Definition 2.4 and (2.11), we see that $F_\eta(t)^{-1} \notin L^1((0,\eta))$, $\lim_{t\to+0} G_\eta(t) = \infty$ and $(F_\eta(t)G_\eta(t))^{-1} \notin L^1((0,\eta))$, but $(F_\eta(t)G_\eta(t)^2)^{-1} \in L^1((0,\eta))$.

2. If $w(t) \in W(\mathbf{R}_+)$, then we have $\liminf_{t\to+0} F_\eta(t) = \liminf_{t\to+0} F_\eta(t)G_\eta(t) = 0$ from 1.

EXAMPLE 2.4. If either $w(t) = t^2 e^{-1/t} \in P(\mathbf{R}_+)$ or $w(t) = t^2 e^{1/t} \in Q(\mathbf{R}_+)$, then $F_\eta(t) = O(t^2)$ and $G_\eta(t) = O(1/t)$ as $t \to +0$.

Now we introduce two admissibilities for $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$.

DEFINITION 2.5.

1. A function $w(t) \in P(\mathbf{R}_+)$ is said to be admissible if there exist positive numbers η and K such that we have

$$\int_t^\eta w(s)^{-1}\, ds \leqslant e^{K/\sqrt{t}} \qquad \text{for} \quad t \in (0,\eta). \tag{2.12}$$

2. A function $w(t) \in Q(\mathbf{R}_+)$ is said to be admissible if there exist positive numbers η and K such that we have

$$\int_0^t w(s)^{-1}\, ds \geqslant e^{-K/\sqrt{t}} \qquad \text{for} \quad t \in (0,\eta). \tag{2.13}$$

DEFINITION 2.6. By $P_A(\mathbf{R}_+)$ and $Q_A(\mathbf{R}_+)$ we denote the set of all admissible functions in $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$ respectively. We set

$$W_A(\mathbf{R}_+) = P_A(\mathbf{R}_+) \cup Q_A(\mathbf{R}_+). \tag{2.14}$$

REMARK 2.4. If $w(t) \in W_A(\mathbf{R}_+)$, then there exist positive numbers η and K such that we have

$$\sqrt{t}\, G_\eta(t) \leqslant K \qquad \text{for} \quad t \in (0,\eta). \tag{2.15}$$

For the detail, see Proposition 2.1 in [8].

Here we give typical examples:

EXAMPLE 2.5. $e^{-1/t} \notin P_A(\mathbf{R}_+)$, $e^{1/t} \notin Q_A(\mathbf{R}_+)$, but $e^{-1/\sqrt{t}} \in P_A(\mathbf{R}_+)$, $e^{1/\sqrt{t}} \in Q_A(\mathbf{R}_+)$.

Verifications:

$e^{-1/t} \notin P_A(\mathbf{R}_+)$: For small $t > 0$, we have $\int_t^\eta e^{1/s}\, ds \geqslant \int_t^{2t} e^{1/s}\, ds \geqslant t e^{1/(2t)}$. But this contradicts to (2.12) for any $K > 0$.

$e^{-1/\sqrt{t}} \in P_A(\mathbf{R}_+)$: Since $e^{1/\sqrt{s}} \leqslant e^{1/\sqrt{t}}$ $(t < s < \eta)$, we have $\int_t^\eta e^{1/\sqrt{s}}\, ds \leqslant \eta e^{1/\sqrt{t}} \leqslant e^{K/\sqrt{t}}$ for some $K > 1$.

$e^{-1/t} \notin Q_A(\mathbf{R}_+)$: For $0 < s \leqslant t$, we have $\int_0^t e^{-1/s}\, ds \leqslant t e^{-1/t}$. But this contradicts to (2.13) for any $K > 0$.

$e^{-1/\sqrt{t}} \in Q_A(\mathbf{R}_+)$: For $t/2 < s < t$, we have $\int_0^t e^{-1/\sqrt{s}}\, ds \geqslant \int_{t/2}^t e^{-1/\sqrt{s}}\, ds \geqslant (t/2) e^{-\sqrt{2/t}} \geqslant e^{-K/\sqrt{t}}$ for some $K > \sqrt{2}$.

2.2. Weighted Hardy's inequalities

We define a switching function.

DEFINITION 2.7. (Switching function) For $w(t) \in W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$ we set

$$s(w) = \begin{cases} -1 & \text{if} \quad w(t) \in P(\mathbf{R}_+), \\ 1 & \text{if} \quad w(t) \in Q(\mathbf{R}_+). \end{cases} \tag{2.16}$$

Let Ω be a bounded domain of class C^2 in $\mathbf{R}^N$. Let $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. For each small $\eta > 0$, Ω_η and Σ_η denote a tubular neighborhood of $\partial\Omega$ and $\partial(\Omega \setminus \Omega_\eta)$ respectively, namely

$$\Omega_\eta = \{x \in \Omega : \delta(x) < \eta\} \quad \text{and} \quad \Sigma_\eta = \{x \in \Omega : \delta(x) = \eta\}. \tag{2.17}$$

In [8] we established a series of weighted Hardy's inequalities with sharp remainders. In particular, we have the following inequality from Theorem 3.3 in [8] by noting that $F_\eta(t) \leqslant F_{\eta_0}(t)$ for $\eta \in (0, \eta_0]$ and $t \in (0, \eta)$.

THEOREM 2.1. *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$ and $w(t) \in W_A(\mathbf{R}_+)$. Assume that $\mu > 0$ and η_0 is a sufficiently small positive number. Then, for $\eta \in (0, \eta_0]$ there exist positive numbers $C = C(w, p, \eta, \mu)$ and $L' = L'(w, p, \eta, \mu)$ such that for $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$ we have*

$$\int_{\Omega_\eta} \left(|\nabla u(x)|^p - \Lambda_p \frac{|u(x)|^p}{F_{\eta_0}(\delta(x))^p} \right) W_p(\delta(x))\, dx$$
$$\geqslant C \int_{\Omega_\eta} \frac{|u(x)|^p W_p(\delta(x))}{F_\eta(\delta(x))^p G_\eta(\delta(x))^2}\, dx + s(w) L' \int_{\Sigma_\eta} |u(x)|^p W_p(\eta)\, d\sigma_\eta, \tag{2.18}$$

where $d\sigma_\eta$ denotes surface elements on Σ_η.

Similarly we have the following inequality from Corollary 3.3 in [8].

COROLLARY 2.1. *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$ and $w(t) \in W_A(\mathbf{R}_+)$. Assume that $\mu > 0$ and η_0 is a sufficiently small positive number. Then, for $\eta \in (0, \eta_0]$ there exist positive numbers $\gamma = \gamma(w, p, \eta, \mu)$ and $L' = L'(w, p, \eta, \mu)$ such that for $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$ we have*

$$\int_\Omega \left(|\nabla u(x)|^p - \gamma \frac{|u(x)|^p}{F_\eta(\delta(x))^p} \right) W_p(\delta(x))\, dx \geqslant s(w) L' \int_{\Sigma_\eta} |u(x)|^p W_p(\eta)\, d\sigma_\eta, \tag{2.19}$$

where $d\sigma_\eta$ denotes surface elements on Σ_η.

REMARK 2.5. In Theorem 3.3 and Corollary 3.3 in [8], it was assumed that $u(x) \in W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$. However, since we have the inequalities (2.18) and (2.19) for $u(x) \in C_c^\infty(\Omega)$, by Lemma 4.5 and Remark 4.1 as stated later, we see that the inequalities (2.18) and (2.19) hold for $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$. Therefore we have Theorem 2.1 and Corollary 2.1.

REMARK 2.6. These inequalities are closely related to the weighted Hardy-Sobolev inequalities with sharp remainder terms (cf. [1], [3], [4], [5], [7], [9], [12]).

3. Main results

Let η_0 be a sufficiently small positive number such that the Hardy's inequalities in Theorem 2.1 and Corollary 2.1 are valid. Let $w(t) \in W(\mathbf{R}_+)$ and $W_p(t) = w(t)^{p-1}$ with $1 < p < \infty$. Moreover, we assume that

$$w'(t) \geqslant 0 \quad \text{for all } t \in (0,\eta_0) \quad \text{or} \quad w'(t) \leqslant 0 \quad \text{for all } t \in (0,\eta_0). \tag{3.1}$$

Then we have the following.

LEMMA 3.1. *Assume that $w(t) \in W(\mathbf{R}_+)$ satisfies (3.1). Then it holds that*

$$\lim_{t \to +0} F_{\eta_0}(t) = 0. \tag{3.2}$$

In particular, $F_{\eta_0}(t)$ is bounded in $\mathbf{R}_+$.

The proof of Lemma 3.1 is stated at the end of this section.

For $\lambda \in \mathbf{R}$, let us recall the variational problem associated with (1.5):

$$J^w_{p,\lambda} = \inf_{u \in W_0^{1,p}(\Omega; W_p(\delta)) \setminus \{0\}} \chi^w_{p,\lambda}(u), \tag{3.3}$$

where

$$\chi^w_{p,\lambda}(u) = \frac{\int_\Omega |\nabla u(x)|^p W_p(\delta(x))\,dx - \lambda \int_\Omega |u(x)|^p W_p(\delta(x))\,dx}{\int_\Omega |u(x)|^p W_p(\delta(x))/F_{\eta_0}(\delta(x))^p\,dx}.$$

Our main result is the following:

THEOREM 3.1. *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$ and $w(t) \in W_A(\mathbf{R}_+)$ satisfies (3.1). Then, there exists a constant $\lambda^* \in \mathbf{R}$ such that:*

1. *If $\lambda \leqslant \lambda^*$, then $J^w_{p,\lambda} = \Lambda_p$. If $\lambda > \lambda^*$, then $J^w_{p,\lambda} < \Lambda_p$.*

Here

$$\Lambda_p = \left(1 - \frac{1}{p}\right)^p. \tag{3.4}$$

Moreover, it holds that:

2. *If $\lambda < \lambda^*$, then the infimum $J^w_{p,\lambda}$ in (3.3) is not attained.*
3. *If $\lambda > \lambda^*$, then the infimum $J^w_{p,\lambda}$ in (3.3) is attained.*

In particular we have the following inequality:

COROLLARY 3.1. *Under the same assumptions as in Theorem 3.1, there exists a constant* $\lambda \in \mathbf{R}$ *such that for* $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$

$$\int_\Omega |\nabla u(x)|^p W_p(\delta(x))\,dx \geqslant \Lambda_p \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx + \lambda \int_\Omega |u(x)|^p W_p(\delta(x))\,dx. \tag{3.5}$$

REMARK 3.1.

1. For the case of $w(t) = 1$ and $\lambda = 0$, the value of the infimum $J^1_{p,0}$ in (3.3) and its attainability are studied in [10].
2. For the case of $w(t) = 1$ and $p = 2$, it is shown that the infimum $J^1_{2,\lambda}$ in (3.3) is attained if and only if $\lambda > \lambda^*$. See [4]. If $p \neq 2$ and $\lambda = \lambda^*$, then it is an open problem whether the infimum $J^w_{p,\lambda}$ in (3.3) is achieved.
3. For the case of $w(t) = t^{\alpha p/(p-1)} \in Q_A(\mathbf{R}_+)$ with $\alpha < 1 - 1/p$, Theorem 3.1 is shown in [2].
4. In the assertion 3 of Theorem 3.1, the minimizer $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$ for the variational problem (3.3) is a non-trivial weak solution of the following Euler-Lagrange equation:

$$-\mathrm{div}\left(W_p(\delta)|\nabla u|^{p-2}\nabla u\right) - \lambda W_p(\delta)|u|^{p-2}u = J^w_{p,\lambda}\frac{W_p(\delta)}{F_{\eta_0}(\delta)^p}|u|^{p-2}u \quad \text{in } \mathscr{D}'(\Omega).$$

When $p = 2$ and $\lambda = \lambda^*$ hold, we have the following that is rather precise.

THEOREM 3.2. *In addition to the assumption of Theorem 3.1, we assume that* $p = 2$ *and* $\lambda = \lambda^*$. *Let* $\eta_0 > 0$ *be a sufficiently small number as in Theorem 2.1. Moreover we assume that*

$$\lim_{t \to +0} F_{\eta_0}(t) G_{\eta_0}(t)^2 = 0. \tag{3.6}$$

Then, J^w_{2,λ^*} *is not achieved.*

REMARK 3.2. By Theorem 3.1, $J^w_{2,\lambda^*} = 1/4$ holds.

EXAMPLE 3.1. Let $w(t) = t^{\alpha p/(p-1)}$ for $\alpha \in \mathbf{R}$. Then $W_p(t) = t^{\alpha p}$. If $\alpha \geqslant 1 - 1/p$, then $w(t) \in P_A(\mathbf{R}_+)$, if $\alpha < 1 - 1/p$, then $w(t) \in Q_A(\mathbf{R}_+)$. Clearly (3.1) is valid. We have that as $t \to +0$

$$F_{\eta_0}(t) = \begin{cases} O(t) & \text{for } \alpha \neq 1 - 1/p, \\ O\big(t\log(1/t)\big) & \text{for } \alpha = 1 - 1/p, \end{cases}$$

$$G_{\eta_0}(t) = \begin{cases} O\big(\log(1/t)\big) & \text{for } \alpha \neq 1 - 1/p, \\ O\big(\log\log(1/t)\big) & \text{for } \alpha = 1 - 1/p. \end{cases}$$

Therefore (3.6) holds.

EXAMPLE 3.2. Let either $w(t) = e^{-1/\sqrt{t}} \in P_A(\mathbf{R}_+)$ or $w(t) = e^{1/\sqrt{t}} \in Q_A(\mathbf{R}_+)$. Then (3.1) and (3.6) hold. In fact, we have that as $t \to +0$

$$F_{\eta_0}(t) = O(t^{3/2}), \quad G_{\eta_0}(t) = O(t^{-1/2}), \quad F_{\eta_0}(t)G_{\eta_0}(t)^2 = O(t^{1/2}).$$

Here we give the proof of Lemma 3.1.

Proof of Lemma 3.1. First we assume that $w(t) \in P(\mathbf{R}_+)$. Let ε be any number satisfying $0 < \varepsilon < 2\eta_0$. For $0 < t < \varepsilon/2$ we have that

$$F_{\eta_0}(t) = w(t)\left(\mu + \int_{\varepsilon/2}^{\eta_0} w(s)^{-1} ds\right) + w(t)\int_t^{\varepsilon/2} w(s)^{-1} ds. \tag{3.7}$$

Since $w(t)^{-1} \notin L^1((0,\eta_0))$, it follows that $\lim_{t\to+0} w(t) = 0$ from the Definition 2.1, and hence $w(t)$ is non-decreasing in $(0,\eta_0]$ by (3.1). Then we have

$$w(t)\int_t^{\varepsilon/2} w(s)^{-1} ds \leqslant w(t)\int_t^{\varepsilon/2} w(t)^{-1} ds = \frac{\varepsilon}{2} - t < \frac{\varepsilon}{2}. \tag{3.8}$$

By $\lim_{t\to+0} w(t) = 0$, there exists a $\delta > 0$ such that for $0 < t < \delta$

$$w(t) < \frac{\varepsilon}{2\left(\mu + \int_{\varepsilon/2}^{\eta_0} w(s)^{-1} ds\right)}. \tag{3.9}$$

From (3.7), (3.8) and (3.9) it follows that for $0 < t < \min\{\varepsilon/2, \delta\}$

$$F_{\eta_0}(t) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows (3.2). Secondly we assume that $w(t) \in Q(\mathbf{R}_+)$. If $w'(t) \geqslant 0$ for $t \in (0,\eta_0)$, then $\lim_{t\to+0} w(t) = a < \infty$, and so

$$F_{\eta_0}(t) = w(t)\int_0^t w(s)^{-1} ds \to 0 \quad \text{as } t \to +0$$

by $w(t) \in L^1((0,\eta_0))$. If $w'(t) \leqslant 0$ for $t \in (0,\eta_0)$, then we see that for $t \in (0,\eta_0]$

$$F_{\eta_0}(t) = w(t)\int_0^t w(s)^{-1} ds \leqslant w(t)\int_0^t w(t)^{-1} ds = t,$$

which implies (3.2). It concludes the proof. $\square$

4. Proof of Theorem 3.1

In this section, we give the proof of Theorem 3.1.

4.1. Upper bound of $J^w_{p,\lambda}$

First, we prove the assertion 1 of Theorem 3.1. As test functions we adopt for $\varepsilon > 0$ and $0 < \eta \leqslant \eta_0/2$

$$u_\varepsilon(t) = \begin{cases} f_{\eta_0}(t)^{1+s(w)\varepsilon-1/p} & (0 < t \leqslant \eta), \\ f_{\eta_0}(\eta)^{1+s(w)\varepsilon-1/p}(2\eta - t)/\eta & (\eta < t \leqslant 2\eta), \\ 0 & (2\eta < t \leqslant \eta_0). \end{cases} \tag{4.1}$$

We note that

$$u'_\varepsilon(t) = \begin{cases} (1+s(w)\varepsilon-1/p)\, f_\eta(t)^{s(w)\varepsilon-1/p} s(w)/w(t) & (0 < t < \eta), \\ -f_{\eta_0}(\eta)^{1+s(w)\varepsilon-1/p}/\eta & (\eta < t < 2\eta), \\ 0 & (2\eta < t \leqslant \eta_0). \end{cases} \tag{4.2}$$

We have

$$\begin{aligned} \int_0^\eta |u'_\varepsilon(t)|^p W_p(t)\, dt &= \left(1 - \frac{1}{p} + s(w)\varepsilon\right)^p \int_0^\eta f_{\eta_0}(t)^{s(w)\varepsilon p - 1} \frac{1}{w(t)}\, dt \\ &= \left(1 - \frac{1}{p} + s(w)\varepsilon\right)^p \frac{f_{\eta_0}(\eta)^{s(w)\varepsilon p}}{p\varepsilon}. \end{aligned} \tag{4.3}$$

In a similar way

$$\int_0^\eta \frac{|u_\varepsilon(t)|^p W_p(t)}{F_{\eta_0}(t)^p}\, dt = \int_0^\eta f_{\eta_0}(t)^{s(w)\varepsilon p-1} \frac{1}{w(t)}\, dt = \frac{f_{\eta_0}(\eta)^{s(w)\varepsilon p}}{p\varepsilon}. \tag{4.4}$$

Noting that $f_{\eta_0}(t)^{s(w)\varepsilon p}$ is bounded by the definitions of $s(w)$ and $f_{\eta_0}(t)$, it follows from Lemma 3.1 that

$$\begin{aligned} \int_0^\eta |u_\varepsilon(t)|^p W_p(t)\, dt &= \int_0^\eta f_{\eta_0}(t)^{p-1+s(w)\varepsilon p} w(t)^{p-1}\, dt \\ &= \int_0^\eta F_{\eta_0}(t)^{p-1} f_{\eta_0}(t)^{s(w)\varepsilon p}\, dt < +\infty. \end{aligned} \tag{4.5}$$

Hence we have

$$\begin{aligned} \int_0^{2\eta} |u'_\varepsilon(t)|^p W_p(t)\, dt &= \left(1 - \frac{1}{p} + s(w)\varepsilon\right)^p \frac{f_{\eta_0}(\eta)^{s(w)\varepsilon p}}{p\varepsilon} + C(\varepsilon,\eta), \\ \int_0^{2\eta} \frac{|u_\varepsilon(t)|^p W_p(t)}{F_{\eta_0}(t)^p}\, dt &= \frac{f_{\eta_0}(\eta)^{s(w)\varepsilon p}}{p\varepsilon} + D(\varepsilon,\eta), \\ \int_0^{2\eta} |u_\varepsilon(t)|^p W_p(t)\, dt &= \int_0^\eta F_{\eta_0}(t)^{p-1} f_{\eta_0}(t)^{s(w)\varepsilon p}\, dt + E(\varepsilon,\eta), \end{aligned}$$

where $C(\varepsilon,\eta)$, $D(\varepsilon,\eta)$ and $E(\varepsilon,\eta)$ are given by

$$C(\varepsilon,\eta) = f_{\eta_0}(\eta)^{p+s(w)\varepsilon p-1}\eta^{-p}\int_\eta^{2\eta} W_p(t)\,dt,$$
$$D(\varepsilon,\eta) = f_{\eta_0}(\eta)^{p+s(w)\varepsilon p-1}\int_\eta^{2\eta} \frac{(2\eta-t)^p W_p(t)}{F_{\eta_0}(t)^p\eta^p}\,dt,$$
$$E(\varepsilon,\eta) = f_{\eta_0}(\eta)^{p+s(w)\varepsilon p-1}\int_\eta^{2\eta} \frac{(2\eta-t)^p W_p(t)}{\eta^p}\,dt,$$

and they remain bounded as $\varepsilon \to +0$. Therefore we see that

$$\frac{\int_0^{2\eta} |u'_\varepsilon(t)|^p W_p(t)\,dt}{\int_0^{2\eta} |u_\varepsilon(t)|^p W_p(t)/F_{\eta_0}(t)^p dt} \to \Lambda_p \quad \text{as } \varepsilon \to +0, \tag{4.6}$$

and we also have

$$\frac{\int_0^{2\eta} |u_\varepsilon(t)|^p W_p(t)\,dt}{\int_0^{2\eta} |u_\varepsilon(t)|^p W_p(t)/F_{\eta_0}(t)^p dt} \to 0 \quad \text{as } \varepsilon \to +0. \tag{4.7}$$

As a result we have the following lemma.

LEMMA 4.1. *Let $1 < p < \infty$, $0 < \eta \leqslant \eta_0/2$ and $w(t) \in W(\mathbf{R}_+)$. For any $\kappa > 0$, there exists a function $h(t) \in W_0^{1,p}((0,2\eta);W_p)$ such that*

$$\frac{\int_0^{2\eta} |h'(t)|^p W_p(t)\,dt}{\int_0^{2\eta} |h(t)|^p W_p(t)/F_{\eta_0}(t)^p dt} \leqslant \Lambda_p + \kappa. \tag{4.8}$$

Proof. By $L^p((0,\eta);W_p)$ we denote the space of Lebesgue measurable functions with weight $W_p(t)$, for which

$$\|u\|_{L^p((0,\eta);W_p)} = \left(\int_0^\eta |u(t)|^p W_p(t)\,dt\right)^{1/p} < +\infty.$$

$W_0^{1,p}((0,\eta);W_p)$ is given by the completion of $C_c^\infty((0,\eta))$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}((0,\eta);W_p)} = \|u'\|_{L^p((0,\eta);W_p)} + \|u\|_{L^p((0,\eta);W_p)}.$$

Then $W_0^{1,p}((0,\eta);W_p)$ becomes a Banach space with the norm $\|\cdot\|_{W_0^{1,p}((0,\eta);W_p)}$.

Let us set $h(t) = u_\varepsilon(t)$ for a sufficiently small $\varepsilon > 0$. Then $h(t)$ satisfies the estimate (4.8). It suffices to check that $h(t) \in W_0^{1,p}((0,2\eta);W_p)$. If $w(t) \in Q(\mathbf{R}_+)$, then $\lim_{t\to+0} f_{\eta_0}(t) = 0$ and $\lim_{t\to+0} u_\varepsilon(t) = \lim_{t\to+0} f_{\eta_0}(t)^{1+s(w)\varepsilon-1/p} = 0$. Therefore $h(t)$ is clearly approximated by test functions in $C_c^\infty((0,2\eta))$.

If $w(t) \in P(\mathbf{R}_+)$, then we employ the following lemma:

LEMMA 4.2. *Assume that* $1 < p < \infty$ *and* $w(t) \in P(\mathbf{R}_+)$. *For* $\varepsilon > 0$, $\eta > 0$ *and* $\eta_0 > 0$ *satisfying* $0 < \eta \leqslant \eta_0/2$, *let us set*

$$\varphi_\varepsilon(t) = \ 0\ (0 \leqslant t \leqslant \varepsilon); \quad \frac{f_{\eta_0}(\varepsilon) - f_{\eta_0}(t)}{f_{\eta_0}(\varepsilon) - f_{\eta_0}(\eta)}\ (\varepsilon \leqslant t \leqslant \eta); \quad 1\ (\eta \leqslant t \leqslant 2\eta). \tag{4.9}$$

Then, as $\varepsilon \to +0$, $\varphi_\varepsilon \to 1$ *in* $L^p((0,2\eta);W_p)$ *and* $\varphi_\varepsilon' \to 0$ *in* $L^p((0,2\eta);W_p)$.

Proof. Since $\lim_{t\to+0} f_{\eta_0}(t) = \infty$, clearly $\varphi_\varepsilon(t) \to 1$ in $L^p((0,2\eta);W_p)$ as $\varepsilon \to +0$, and $\int_0^{2\eta} |\varphi_\varepsilon'(t)|^p W_p(t)\,dt = (f_{\eta_0}(\varepsilon) - f_{\eta_0}(\eta))^{1-p} \to 0$ as $\varepsilon \to +0$. Then we see the assertion. □

End of the proof of Lemma 4.1. For $0 < \overline{\varepsilon} < \eta$, we set $h_{\overline{\varepsilon}}(t) = \varphi_{\overline{\varepsilon}}(t)h(t)$, where $\varphi_{\overline{\varepsilon}}(t)$ is defined by (4.9) with $\varepsilon = \overline{\varepsilon}$. Then $\operatorname{supp} h_{\overline{\varepsilon}}(t) \subset [\overline{\varepsilon}, 2\eta]$. By virtue of Lemma 4.2, we also see that $h_{\overline{\varepsilon}}(t) \to h(t)$ in $W^{1,p}((0,2\eta);W_p)$ as $\overline{\varepsilon} \to +0$. In fact, noting that $h_{\overline{\varepsilon}}'(t) = \varphi_{\overline{\varepsilon}}'(t)h(t) + \varphi_{\overline{\varepsilon}}(t)h'(t)$, we have

$$\begin{aligned}\int_0^{2\eta} &|h_{\overline{\varepsilon}}'(t) - h'(t)|^p W_p(t)\,dt \\ &\leqslant C_p \left(\int_0^{2\eta} (1 - \varphi_{\overline{\varepsilon}}(t))^p |h'(t)|^p W_p(t)\,dt + \int_0^{2\eta} |\varphi_{\overline{\varepsilon}}'(t)|^p |h(t)|^p W_p(t)\,dt \right)\end{aligned}$$

with some constant $C_p > 0$ depending only on p. The first term obviously goes to 0 as $\overline{\varepsilon} \to +0$. As for the second, noting that $s(w) = -1$ and $0 < \varepsilon < 1$, we have

$$\begin{aligned}\int_0^{2\eta} |\varphi_{\overline{\varepsilon}}'(t)|^p |h(t)|^p W_p(t)\,dt &= \int_{\overline{\varepsilon}}^{\eta} |\varphi_{\overline{\varepsilon}}'(t)|^p |h(t)|^p W_p(t)\,dt \\ &= \frac{1}{(f_{\eta_0}(\overline{\varepsilon}) - f_{\eta_0}(\eta))^p} \int_{\overline{\varepsilon}}^{\eta} \frac{f_{\eta_0}(t)^{p-1+ps(w)\varepsilon}}{w(t)}\,dt \\ &= \frac{1}{p(1-\varepsilon)} \frac{f_{\eta_0}(\overline{\varepsilon})^{p(1-\varepsilon)} - f_{\eta_0}(\eta)^{p(1-\varepsilon)}}{(f_{\eta_0}(\overline{\varepsilon}) - f_{\eta_0}(\eta))^p}.\end{aligned}$$

Since $\lim_{t\to+0} f_{\eta_0}(t) = \infty$, we see that $\int_0^{2\eta} |\varphi_{\overline{\varepsilon}}'(t)|^p |h(t)|^p W_p(t)\,dt \to 0$ as $\overline{\varepsilon} \to +0$. Since $h_{\overline{\varepsilon}}(t)$ is clearly approximated by test functions in $C_c^\infty((0,2\eta))$, the assertion $h(t) \in W_0^{1,p}((0,2\eta);W_p)$ follows. □

LEMMA 4.3. *Let* Ω *be a bounded domain of class* C^2 *in* $\mathbf{R}^N$. *Let* $1 < p < \infty$ *and* $w(t) \in W(\mathbf{R}_+)$. *Then it holds that*

$$J^w_{p,\lambda} \leqslant \Lambda_p \tag{4.10}$$

for all $\lambda \in \mathbf{R}$.

Proof. For each small $\eta > 0$, by Ω_η we denote a tubular neighborhood of $\partial\Omega$;

$$\Omega_\eta = \{x \in \Omega : \delta(x) = \operatorname{dist}(x, \partial\Omega) < \eta\}. \tag{4.11}$$

Since the boundary $\partial\Omega$ is of class C^2, there exists an $\eta_0 > 0$ such that for any $\eta \in (0,\eta_0)$ and every $x \in \Omega_\eta$ we have a unique point $\sigma(x) \in \partial\Omega$ satisfying $\delta(x) = |x - \sigma(x)|$. The mapping

$$\Omega_\eta \ni x \mapsto (\delta(x),\sigma(x)) = (t,\sigma) \in (0,\eta)\times\partial\Omega$$

is a C^2 diffeomorphism, and its inverse is given by

$$(0,\eta)\times\partial\Omega \ni (t,\sigma) \mapsto x(t,\sigma) = \sigma + t\cdot n(\sigma) \in \Omega_\eta,$$

where $n(\sigma)$ is the inward unit normal to $\partial\Omega$ at $\sigma \in \partial\Omega$. For each $t \in (0,\eta)$, the mapping

$$\partial\Omega \ni \sigma \mapsto \sigma_t(\sigma) = x(t,\sigma) \in \Sigma_t = \{x \in \Omega : \delta(x) = t\}$$

is also a C^2 diffeomorphism of $\partial\Omega$ onto Σ_t, and its Jacobian satisfies

$$|\mathrm{Jac}\,\sigma_t(\sigma) - 1| \leqslant ct \qquad \text{for any } \sigma \in \partial\Omega, \tag{4.12}$$

where c is a positive constant depending only on η_0, $\partial\Omega$ and the choice of local coordinates. Since $n(\sigma)$ is orthogonal to Σ_t at $\sigma_t(\sigma) = \sigma + t\cdot n(\sigma) \in \Sigma_t$, it follows that for every integrable function $v(x)$ in Ω_η

$$\begin{aligned}\int_{\Omega_\eta} v(x)dx &= \int_0^\eta dt \int_{\Sigma_t} v(\sigma_t)d\sigma_t \\ &= \int_0^\eta dt \int_{\partial\Omega} v(x(t,\sigma))|\mathrm{Jac}\,\sigma_t(\sigma)|d\sigma, \end{aligned}\tag{4.13}$$

where $d\sigma$ and $d\sigma_t$ denote surface elements on $\partial\Omega$ and Σ_t, respectively. Hence (4.13) together with (4.12) implies that for every integrable function $v(x)$ in Ω_η

$$\int_0^\eta (1-ct)dt \int_{\partial\Omega} |v(x(t,\sigma))|d\sigma \leqslant \int_{\Omega_\eta} |v(x)|dx \tag{4.14}$$

$$\leqslant \int_0^\eta (1+ct)dt \int_{\partial\Omega} |v(x(t,\sigma))|d\sigma. \tag{4.15}$$

Let $\kappa > 0$, and let $\eta \in (0,\eta_0)$. Take $h(t) \in W_0^{1,p}((0,\eta);W_p)$ be a function satisfying (4.8) with replacing 2η by η for simplicity. Define

$$u(x) = \begin{cases} h(\delta(x)) & \text{if} \quad x \in \Omega_\eta, \\ 0 & \text{if} \quad x \in \Omega\setminus\Omega_\eta. \end{cases} \tag{4.16}$$

Then we have $\mathrm{supp}\, u \subset \Omega_\eta$. Since $|\nabla u(x)| = |h'(\delta(x))|$ for $x \in \Omega_\eta$ by $|\nabla\delta(x)| = 1$, it follows from (4.15) that

$$\int_{\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\,dx \leqslant (1+c\eta)|\partial\Omega| \int_0^\eta |h'(t)|^p W_p(t)\,dt, \tag{4.17}$$

which implies $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$ by Lemma 4.1. On the other hand, by (4.14) and (4.16) we have that

$$\int_{\Omega_\eta} |u(x)|^p \frac{W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx \geqslant (1-c\eta)|\partial\Omega| \int_0^\eta |h(t)|^p \frac{W_p(t)}{F_{\eta_0}(t)^p} dt. \tag{4.18}$$

By combining (4.17), (4.18) and trivial estimate

$$\int_{\Omega_\eta} |u(x)|^p W_p(\delta(x))\, dx \leqslant \Big(\sup_{0<t<\eta} F_{\eta_0}(t)\Big)^p \int_{\Omega_\eta} |u(x)|^p \frac{W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\, dx, \tag{4.19}$$

we obtain that

$$\chi^w_{p,\lambda}(u) \leqslant \frac{1+c\eta}{1-c\eta} \frac{\int_0^\eta |h'(t)|^p W_p(t)\, dt}{\int_0^\eta |h(t)|^p W_p(t)/F_{\eta_0}(t)^p dt} + |\lambda| \Big(\sup_{0<t<\eta} F_{\eta_0}(t)\Big)^p.$$

This together with Lemma 4.1 implies that

$$J^w_{p,\lambda} \leqslant \frac{1+c\eta}{1-c\eta}(\Lambda_p + \kappa) + |\lambda| \Big(\sup_{0<t<\eta} F_{\eta_0}(t)\Big)^p. \tag{4.20}$$

Letting $\eta \to +0$ and $\kappa \to +0$ in (4.20), then (4.10) follows from Lemma 3.1. Therefore it concludes the proof. □

LEMMA 4.4. *Let Ω be a bounded domain of class C^2 in $\mathbf{R}^N$. Let $1 < p < \infty$ and $w(t) \in W_A(\mathbf{R}_+)$. Then there exists a $\lambda \in \mathbf{R}$ such that $J^w_{p,\lambda} = \Lambda_p$.*

Proof. Let $\eta_0 > 0$ be a sufficiently small number as in Theorem 2.1. Take and fix any $u(x) \in W_0^{1,p}(\Omega; W_p(\delta)) \setminus \{0\}$. Then, for $\eta \in (0, \eta_0]$

$$\begin{aligned} &\int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx \\ &\quad = \int_{\Omega_\eta} \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx + \int_{\Omega\setminus\Omega_\eta} \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx. \end{aligned} \tag{4.21}$$

Since there exists a positive number C_η independent of $u(x)$ such that

$$\int_{\Omega\setminus\Omega_\eta} |u(x)|^p \frac{W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx \leqslant C_\eta \int_{\Omega\setminus\Omega_\eta} |u(x)|^p W_p(\delta(x))\, dx, \tag{4.22}$$

by using Hardy's inequality (2.18) we have

$$\begin{aligned} \Lambda_p \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx \leqslant &\int_{\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\, dx - s(w)L' \int_{\Sigma_\eta} |u(\sigma_\eta)|^p W_p(\eta)\, d\sigma_\eta \\ &+ \Lambda_p C_\eta \int_{\Omega\setminus\Omega_\eta} |u(x)|^p W_p(\delta(x))\, dx. \end{aligned} \tag{4.23}$$

In order to control the integrand on the surface Σ_η we prepare the following:

LEMMA 4.5. *Let Ω be a bounded domain of class C^2 in $\mathbf{R}^N$. Let $1 < p < \infty$ and $w(t) \in W(\mathbf{R}_+)$. Assume that η_0 is a sufficiently small positive number and $\eta \in (0, \eta_0/3)$. Then, for any $\varepsilon > 0$ there exists a positive number $C_{\varepsilon,\eta}$ such that we have for any $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$*

$$\|u\|^p_{L^p(\Sigma_\eta; W_p(\delta))} \leqslant \varepsilon \||\nabla u|\|^p_{L^p(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))} + C_{\varepsilon,\eta} \|u\|^p_{L^p(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))}. \tag{4.24}$$

Here we denote by $\overline{\Omega_\eta}$ the closure of Ω_η.

REMARK 4.1. By Rellich's theorem and Hardy type inequality, we see that the imbedding $W_0^{1,p}(\Omega; W_p(\delta)) \hookrightarrow L^p(\Omega; W_p(\delta))$ is compact. Therefore, by this lemma we see that a trace operator $W_0^{1,p}(\Omega; W_p(\delta)) \to L^p(\Sigma_\eta; W_p(\delta))$ is also compact.

Proof. For $\eta \in (0, \max_{x \in \Omega} \delta(x)/3)$, let $W^{1,p}(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))$ be given by the completion of $C^\infty(\Omega_{3\eta} \setminus \overline{\Omega_\eta})$ with respect to the norm defined by

$$\|u\|_{W^{1,p}(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))} = \||\nabla u|\|_{L^p(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))} + \|u\|_{L^p(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))}.$$

Since $W_p(\delta(x)) > 0$ in $\overline{\Omega_{3\eta} \setminus \overline{\Omega_\eta}}$, $W^{1,p}(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))$ is well-defined and becomes a Banach space with the norm $\| \cdot \|_{W^{1,p}(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta))}$.

Hence the inequality (4.24) follows from the standard theory for a trace operator

$$W^{1,p}(\Omega_{3\eta} \setminus \overline{\Omega_\eta}; W_p(\delta)) \to L^p(\Sigma_\eta; W_p(\delta)).$$

Here we give a simple proof of it. We use the following cut-off function $\psi(x) \in C^\infty(\Omega)$ such that $\psi(x) \geqslant 0$ and

$$\psi(x) = \begin{cases} 1 & (x \in \Omega_{2\eta}), \\ 0 & (x \in \Omega \setminus \Omega_{3\eta}). \end{cases} \tag{4.25}$$

We retain the notations in the proof of Lemma 4.3. Take and fix a $u(x) \in W_0^{1,p}(\Omega; W_p(\delta))$ and assume $u(x) \geqslant 0$. Then,

$$\begin{aligned}
\int_{\Sigma_\eta} u(\sigma_\eta)^p W_p(\eta)\, d\sigma_\eta &= \int_{\partial\Omega} u(x(\eta,\sigma))^p W_p(\eta) |\mathrm{Jac}\, \sigma_\eta(\sigma)|\, d\sigma \\
&= -\int_{\partial\Omega} d\sigma \int_\eta^{3\eta} \frac{\partial}{\partial t} \left(u(x(t,\sigma))^p \psi(x(t,\sigma)) W_p(t) |\mathrm{Jac}\, \sigma_t(\sigma)| \right) dt \\
&= -\int_{\partial\Omega} d\sigma \int_\eta^{3\eta} \frac{\partial}{\partial t} \left(u(x(t,\sigma))^p \right) \cdot \psi(x(t,\sigma)) W_p(t) |\mathrm{Jac}\, \sigma_t(\sigma)|\, dt \\
&\quad - \int_{\partial\Omega} d\sigma \int_\eta^{3\eta} u(x(t,\sigma))^p \cdot \frac{\partial}{\partial t} \left(\psi(x(t,\sigma)) W_p(t) |\mathrm{Jac}\, \sigma_t(\sigma)| \right) dt \\
&= I_1 + I_2.
\end{aligned}$$

Note that $x(t,\sigma), W_p(t), \mathrm{Jac}\, \sigma_t(\sigma) \in C^1$ in $t \in (\eta, 3\eta)$ and

$$\int_{\Omega_{3\eta} \setminus \Omega_\eta} u(x)^p\, dx = \int_{\partial\Omega} d\sigma \int_\eta^{3\eta} u(x(t,\sigma))^p |\mathrm{Jac}\, \sigma_t(\sigma)|\, dt.$$

Then, we have for some $C_\eta > 0$ independent of $u(x)$

$$|I_2| \leqslant C_\eta \int_{\Omega_{3\eta}\setminus\Omega_\eta} u(x)^p W_p(\delta(x))\,dx.$$

As for I_1, for any $\varepsilon > 0$ there is a positive number C_ε independent of $u(x)$ and η such that we have

$$|I_1| \leqslant \varepsilon \int_{\Omega_{3\eta}\setminus\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\,dx + C_\varepsilon \int_{\Omega_{3\eta}\setminus\Omega_\eta} u(x)^p W_p(\delta(x))\,dx.$$

Therefore we obtain (4.24). It concludes the proof of Lemma 4.5. $\square$

End of the proof of Lemma 4.4. From (4.23) and Lemma 4.5, it follows that

$$\begin{aligned}
&\int_{\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\,dx \\
&\geqslant \Lambda_p \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta)^p}\,dx - \Lambda_p C_\eta \int_{\Omega\setminus\Omega_\eta} |u(x)|^p W_p(\delta(x))\,dx \\
&\quad - L'\left(\varepsilon \int_{\Omega_{3\eta}\setminus\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\,dx + C_{\varepsilon,\eta} \int_{\Omega_{3\eta}\setminus\Omega_\eta} |u(x)|^p W_p(\delta(x))\,dx\right),
\end{aligned}$$

and so

$$\begin{aligned}
&\int_{\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\,dx + L'\varepsilon \int_{\Omega_{3\eta}\setminus\Omega_\eta} |\nabla u(x)|^p W_p(\delta(x))\,dx \\
&\quad \geqslant \Lambda_p \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx - \left(L'C_{\varepsilon,\eta} + \Lambda_p C_\eta\right) \int_{\Omega\setminus\Omega_\eta} |u(x)|^p W_p(\delta(x))\,dx.
\end{aligned}$$

Now we set $L'\varepsilon = 1$ and $C' = -\left(L'C_{\varepsilon,\eta} + \Lambda_p C_\eta\right) < 0$, and we have the desired estimate:

$$\int_\Omega |\nabla u(x)|^p W_p(\delta(x))\,dx \geqslant \Lambda_p \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx + C' \int_\Omega |u(x)|^p W_p(\delta(x))\,dx,$$

which implies that

$$\chi^w_{p,\lambda}(u) \geqslant \Lambda_p$$

for $\lambda \leqslant C'$. Consequently, it holds that $J^w_{p,\lambda} \geqslant \Lambda_p$ for $\lambda \leqslant C'$. This together with (4.10) implies the desired conclusion. It completes the proof of Lemma 4.4. $\square$

Proof of the assertion 1 *of Theorem* 3.1. By Lemma 4.4 and $\lim_{\lambda\to\infty} J^w_{p,\lambda} = -\infty$, the set $\{\lambda \in \mathbf{R} : J^w_{p,\lambda} = \Lambda_p\}$ is non-empty and upper bounded. Hence the $\sup\{\lambda \in \mathbf{R} : J^w_{p,\lambda} = \Lambda_p\}$ exists finitely. Put

$$\lambda^* = \sup\{\lambda \in \mathbf{R} : J^w_{p,\lambda} = \Lambda_p\}. \tag{4.26}$$

Since the function $\lambda \mapsto J^w_{p,\lambda}$ is non-increasing on $\mathbf{R}$, it follows from Lemma 4.3 and Lemma 4.4 that $J^w_{p,\lambda} = \Lambda_p$ for $\lambda < \lambda^*$ and $J^w_{p,\lambda} < \Lambda_p$ for $\lambda > \lambda^*$. Since $J^w_{p,\lambda}$ is clearly Lipschitz continuous on $\mathbf{R}$ with respect to λ, we have the equality $J^w_{p,\lambda^*} = \Lambda_p$. Therefore the assertion 1 of Theorem 3.1 is valid. $\square$

4.2. $J^w_{p,\lambda}$ is not attained when $\lambda < \lambda^*$

Next, we prove the assertion 2 of Theorem 3.1.

Proof of the assertion 2 *of Theorem* 3.1. Suppose that for some $\lambda < \lambda^*$ the infimum $J^w_{p,\lambda}$ in (3.3) is attained at an element $u \in W_0^{1,p}(\Omega; W_p(\delta)) \setminus \{0\}$. Then, by the assertion 1 of Theorem 3.1, we have that

$$\chi^w_{p,\lambda}(u) = J^w_{p,\lambda} = \Lambda_p \tag{4.27}$$

and for $\lambda < \bar{\lambda} < \lambda^*$

$$\chi^w_{p,\bar{\lambda}}(u) \geqslant J^w_{p,\bar{\lambda}} = \Lambda_p. \tag{4.28}$$

From (4.27) and (4.28) it follows that

$$(\bar{\lambda} - \lambda) \int_\Omega |u(x)|^p W_p(\delta(x))\, dx \leqslant 0.$$

Since $\bar{\lambda} - \lambda > 0$, we conclude that

$$\int_\Omega |u(x)|^p W_p(\delta(x))\, dx = 0,$$

which contradicts $u \neq 0$ in $W_0^{1,p}(\Omega; W_p(\delta))$. Therefore it completes the proof. □

4.3. Attainability of $J^w_{p,\lambda}$ when $\lambda > \lambda^*$

At last, we prove the assertion 3 of Theorem 3.1. Let η_0 be sufficiently small as in Theorem 2.1 and let $\eta \in (0, \eta_0]$. Let $\{u_k\}$ be a minimizing sequence for the variational problem (3.3) normalized so that

$$\int_\Omega \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\, dx = 1 \quad \text{for all } k. \tag{4.29}$$

Since $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega; W_p(\delta))$, by taking a suitable subsequence, we may assume that there exists a $u \in W_0^{1,p}(\Omega; W_p(\delta))$ such that

$$\nabla u_k \xrightarrow{weak} \nabla u \quad \text{in } (L^p(\Omega; W_p(\delta)))^N, \tag{4.30}$$

$$u_k \xrightarrow{weak} u \quad \text{in } L^p(\Omega; W_p(\delta)/F_{\eta_0}(\delta)^p), \tag{4.31}$$

$$u_k \longrightarrow u \quad \text{in } L^p(\Omega; W_p(\delta)) \tag{4.32}$$

and

$$u_k \longrightarrow u \quad \text{in } L^p(\Sigma_\eta; W_p(\delta)) \tag{4.33}$$

by Remark 4.1. Under these preparation we establish the properties of concentration and compactness for the minimizing sequence, respectively.

PROPOSITION 4.1. *Let Ω be a bounded domain of class C^2 in $\mathbf{R}^N$. Let $1 < p < \infty$ and $w(t) \in W_A(\mathbf{R}_+)$. Let $\lambda \in \mathbf{R}$. Let $\{u_k\}$ be a minimizing sequence for (3.3) satisfying (4.29) and (4.30) $\sim$ (4.33) with $u = 0$. Then it holds that*

$$\nabla u_k \longrightarrow 0 \quad in \ \ (L^p_{\mathrm{loc}}(\Omega; W_p(\delta)))^N \tag{4.34}$$

and

$$J^w_{p,\lambda} = \Lambda_p. \tag{4.35}$$

Proof. Let $\eta_0 > 0$ be a sufficiently small number as in Theorem 2.1 and let $\eta \in (0, \eta_0]$. By Hardy's inequality (2.18) and (4.29) we have that

$$\begin{aligned}
&\int_{\Omega_\eta} |\nabla u_k(x)|^p W_p(\delta(x))\, dx \\
&\quad \geqslant \Lambda_p \int_{\Omega_\eta} \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\, dx + s(w)L' \int_{\Sigma_\eta} |u_k(\sigma_\eta)|^p W_p(\eta)\, d\sigma_\eta \\
&\quad = \Lambda_p \left(1 - \int_{\Omega \setminus \Omega_\eta} \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx\right) + s(w)L' \int_{\Sigma_\eta} |u_k(\sigma_\eta)|^p W_p(\eta)\, d\sigma_\eta,
\end{aligned}$$

and so

$$\begin{aligned}
\chi^w_{p,\lambda}(u_k) \geqslant \Lambda_p &\left(1 - \int_{\Omega \setminus \Omega_\eta} \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\, dx\right) + s(w)L' \int_{\Sigma_\eta} |u_k(\sigma_\eta)|^p W_p(\eta)\, d\sigma_\eta \\
&+ \int_{\Omega \setminus \Omega_\eta} |\nabla u_k(x)|^p W_p(\delta(x))\, dx - \lambda \int_\Omega |u_k(x)|^p W_p(\delta(x))\, dx.
\end{aligned} \tag{4.36}$$

Since there exists a positive number C_η independent of u_k such that

$$\int_{\Omega \setminus \Omega_\eta} \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p} dx \leqslant C_\eta \int_\Omega |u_k(x)|^p W_p(\delta(x))\, dx,$$

it follows from (4.32) with $u = 0$ that

$$\lim_{k \to \infty} \int_{\Omega \setminus \Omega_\eta} \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\, dx = 0. \tag{4.37}$$

Hence, letting $k \to \infty$ in (4.36), by (4.37), (4.32) and (4.33) with $u = 0$, we obtain that

$$0 \leqslant \limsup_{k \to \infty} \int_{\Omega \setminus \Omega_\eta} |\nabla u_k(x)|^p W_p(\delta(x))\, dx \leqslant J^w_{p,\lambda} - \Lambda_p.$$

Since $J^w_{p,\lambda} - \Lambda_p \leqslant 0$ by Lemma 4.3, we conclude that $J^w_{p,\lambda} - \Lambda_p = 0$ and

$$\lim_{k \to \infty} \int_{\Omega \setminus \Omega_\eta} |\nabla u_k(x)|^p W_p(\delta(x))\, dx = 0. \tag{4.38}$$

These show (4.34) and (4.35). Consequently it completes the proof. $\square$

PROPOSITION 4.2. *Let Ω be a bounded domain of class C^2 in $\mathbf{R}^N$. Let $1<p<\infty$, $w(t)\in W_A(\mathbf{R}_+)$ and $\lambda\in\mathbf{R}$. Let $\{u_k\}$ be a minimizing sequence for (3.3) satisfying (4.29) and (4.30) $\sim$ (4.33) with $u\neq 0$. Then it holds that*

$$J^w_{p,\lambda}=\min\big(\Lambda_p,\chi^w_{p,\lambda}(u)\big). \tag{4.39}$$

In addition, if $J^w_{p,\lambda}<\Lambda_p$, then it holds that

$$J^w_{p,\lambda}=\chi^w_{p,\lambda}(u), \tag{4.40}$$

namely u is a minimizer for (3.3), and

$$u_k\longrightarrow u \quad \text{in } W^{1,p}_0(\Omega;W_p(\delta)). \tag{4.41}$$

Proof. Let $\eta_0>0$ be a sufficiently small number as in Theorem 2.1 and let $\eta\in(0,\eta_0]$. Then we have (4.36) by the same arguments as in the proof of Proposition 4.1. Since there exists a positive number C_η independent of u_k such that

$$\int_{\Omega\setminus\Omega_\eta}\frac{|u_k(x)-u(x)|^pW_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx\leqslant C_\eta\int_\Omega|u_k(x)-u(x)|^pW_p(\delta(x))\,dx,$$

(4.32) implies that

$$\lim_{k\to\infty}\int_{\Omega\setminus\Omega_\eta}\frac{|u_k(x)|^pW_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx=\int_{\Omega\setminus\Omega_\eta}\frac{|u(x)|^pW_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx. \tag{4.42}$$

Since it follows from (4.30) that $\nabla u_k\longrightarrow\nabla u$ weakly in $(L^p(\Omega\setminus\overline{\Omega_\eta};W_p(\delta)))^N$, by weakly lower semi-continuity of the L^p-norm, we see that

$$\begin{aligned}\liminf_{k\to\infty}\int_{\Omega\setminus\Omega_\eta}|\nabla u_k(x)|^pW_p(\delta(x))\,dx&\geqslant\Big(\liminf_{k\to\infty}\||\nabla u_k|\|_{L^p(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))}\Big)^p\\&\geqslant\||\nabla u|\|^p_{L^p(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))}\\&=\int_{\Omega\setminus\Omega_\eta}|\nabla u(x)|^pW_p(\delta(x))\,dx.\end{aligned} \tag{4.43}$$

Hence, by letting $k\to\infty$ in (4.36), from (4.32), (4.33), (4.42) and (4.43) it follows that

$$\begin{aligned}J^w_{p,\lambda}\geqslant{}&\Lambda_p\Big(1-\int_{\Omega\setminus\Omega_\eta}\frac{|u(x)|^pW_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx\Big)+s(w)L'\int_{\Sigma_\eta}|u(\sigma_\eta)|^pW_p(\eta)\,d\sigma_\eta\\&+\int_{\Omega\setminus\Omega_\eta}|\nabla u(x)|^pW_p(\delta(x))\,dx-\lambda\int_\Omega|u(x)|^pW_p(\delta(x))\,dx.\end{aligned} \tag{4.44}$$

If $w(t)\in Q(\mathbf{R}_+)$, then $s(w)=1$, hence we can omit the integrand on the surface Σ_η. On the other hand if $w(t)\in P(\mathbf{R}_+)$, then $\lim_{t\to+0}W_p(t)=\lim_{t\to+0}w(t)^{p-1}=0$. Thus, letting $\eta\to+0$ in (4.44), we obtain that

$$\begin{aligned}J^w_{p,\lambda}\geqslant{}&\Lambda_p\Big(1-\int_\Omega\frac{|u(x)|^pW_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx\Big)\\&+\int_\Omega|\nabla u(x)|^pW_p(\delta(x))\,dx-\lambda\int_\Omega|u(x)|^pW_p(\delta(x))\,dx.\end{aligned} \tag{4.45}$$

Since it holds that

$$0 < \int_\Omega |u(x)|^p \frac{W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx \leqslant \liminf_{k\to\infty} \int_\Omega \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx = 1 \tag{4.46}$$

by $u \neq 0$, (4.29), (4.31) and weakly lower semi-continuity of the L^p-norm, we have from (4.45) and (4.46) that

$$\begin{aligned} J^w_{p,\lambda} &\geqslant \Lambda_p\left(1 - \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx\right) + \chi^w_{p,\lambda}(u) \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx \\ &\geqslant \min\big(\Lambda_p, \chi^w_{p,\lambda}(u)\big). \end{aligned} \tag{4.47}$$

This together with Lemma 4.3 implies (4.39). Moreover, by (4.39) and (4.47), we conclude that

$$J^w_{p,\lambda} = \Lambda_p\left(1 - \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx\right) + \chi^w_{p,\lambda}(u) \int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx. \tag{4.48}$$

In addition, if $J^w_{p,\lambda} < \Lambda_p$, then $J^w_{p,\lambda} = \chi^w_{p,\lambda}(u)$ by (4.39), and so, it follows from (4.48) and (4.29) that

$$\int_\Omega \frac{|u(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx = 1 = \lim_{k\to\infty} \int_\Omega \frac{|u_k(x)|^p W_p(\delta(x))}{F_{\eta_0}(\delta(x))^p}\,dx. \tag{4.49}$$

(4.31) and (4.49) imply that

$$u_k \longrightarrow u \quad \text{in } L^p(\Omega, W_p(\delta)/F_{\eta_0}(\delta)^p). \tag{4.50}$$

Further, by (4.29), (4.32), (4.40) and (4.49), we obtain that

$$\begin{aligned} \int_\Omega |\nabla u_k(x)|^p W_p(\delta(x))\,dx &= \chi^w_{p,\lambda}(u_k) + \lambda \int_\Omega |u_k(x)|^p W_p(\delta(x))\,dx \\ &\longrightarrow \chi^w_{p,\lambda}(u) + \lambda \int_\Omega |u(x)|^p W_p(\delta(x))\,dx = \int_\Omega |\nabla u(x)|^p W_p(\delta(x))\,dx. \end{aligned}$$

This together with (4.30) implies that

$$\nabla u_k \longrightarrow \nabla u \quad \text{in } (L^p(\Omega; W_p(\delta)))^N. \tag{4.51}$$

(4.51) and (4.32) show (4.41). Consequently it completes the proof. □

Proof of the assertion 3 *of Theorem* 3.1. Let $\lambda > \lambda^*$. Then $J^w_{p,\lambda} < \Lambda_p$ by the assertion 1 of Theorem 3.1. Let $\{u_k\}$ be a minimizing sequence for (3.3) satisfying (4.29) $\sim$ (4.33). Then we see that $u \neq 0$ by Proposition 4.1. Therefore, by applying Proposition 4.2, we conclude that $\chi^w_{p,\lambda}(u) = J^w_{p,\lambda}$, namely u is a minimizer for (3.3). It finishes the proof. □

5. Proof of Theorem 3.2

For $M > 0$ and $w(t) \in W(\mathbf{R}_+)$, we define the following operator:

$$L_M^w(u(x)) = -\operatorname{div}(w(\delta(x))\nabla u(x)) - J_{2,\lambda^*}^w \frac{w(\delta(x))u(x)}{F_{\eta_0}(\delta(x))^2} + Mw(\delta(x))u(x). \tag{5.1}$$

Our proof of Theorem 3.2 is relied on the maximum principle and the following non-existence result on the operator L_M^w:

LEMMA 5.1. *Let Ω be a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $w(t) \in W_A(\mathbf{R}_+)$ and $w(t)$ satisfies the condition (3.6). If $u(x)$ is a non-negative function in $W_0^{1,2}(\Omega; w(\delta)) \cap C(\overline{\Omega})$ and satisfies the inequality*

$$L_M^w(u(x)) \geqslant 0 \qquad in \;\; \Omega \tag{5.2}$$

in the sense of distributions for some positive number M, then $u(x) \equiv 0$.

Admitting this lemma for the moment, we prove Theorem 3.2.

Proof of Theorem 3.2. If the infimum J_{2,λ^*}^w in (3.3) is achieved by a function $u(x)$ then it is also achieved by $|u(x)|$. Therefore there exists $u(x) \in W_0^{1,2}(\Omega; w(\delta))$, $u(x) \geqslant 0$ such that

$$-\operatorname{div}(w(\delta(x))\nabla u(x)) - J_{2,\lambda^*}^w \frac{w(\delta(x))u(x)}{F_{\eta_0}(\delta(x))^2} - \lambda^* w(\delta(x))u(x) = 0.$$

By the standard regularity theory of the elliptic type, we see that $u(x) \in C(\overline{\Omega})$, and by the maximum principle, $u(x) > 0$ in Ω. Then $u(x)$ clearly satisfies the inequality (5.2) for some $M > 0$, and hence the assertion of Theorem 3.2 is a consequence of Lemma 5.1. □

Proof of Lemma 5.1. Assume by contradiction that there exists a non-negative function $u(x)$ as in Lemma 5.1. By the maximum principle, we see $u(x) > 0$ in Ω. Let us set

$$v_s(t) = f_{\eta_0}(t)^{1/2} G_{\eta_0}(t)^{-s} \qquad \text{for} \quad s > 1/2.$$

Then we have $v_s(t) \in W_0^{1,2}((0,\eta_0); w)$ and $v_s(\delta(x)) \in W_0^{1,2}(\Omega_{\eta_0}; w(\delta))$. We assume that η_0 is sufficiently small so that $\delta(x) \in C^2(\Omega_{\eta_0})$, and Theorem 2.1 holds in Ω_{η_0}. Since $|\nabla \delta(x)| = 1$, we have for $\delta = \delta(x)$

$$\operatorname{div}(w(\delta)\nabla(v_s(\delta))) = w(\delta)v_s'(\delta)\Delta\delta + w'(\delta)v_s'(\delta) + w(\delta)v_s''(\delta).$$

With somewhat more calculations we have

$$\begin{aligned}\operatorname{div}(w(\delta)\nabla(v_s(\delta))) &= f_{\eta_0}(\delta)^{-1/2} G_{\eta_0}(\delta)^{-s} \left(s(w)/2 + sG_{\eta_0}(\delta)^{-1}\right)\Delta\delta \\ &\quad + w(\delta)^{-1} f_{\eta_0}(\delta)^{-3/2} G_{\eta_0}(\delta)^{-s} \left(-1/4 + s(s+1)G_{\eta_0}(\delta)^{-2}\right).\end{aligned}$$

Since $J^w_{2,\lambda^*} = 1/4$ by Remark 3.2, we have

$$L^w_M(v_s(\delta)) = -w(\delta)^{-1} f_{\eta_0}(\delta)^{-3/2} G_{\eta_0}(\delta)^{-s-2} \times \left\{ s(s+1) + F_{\eta_0}(\delta)\left(s(w)G_{\eta_0}(\delta)^2/2 + sG_{\eta_0}(\delta)\right)\Delta\delta - MF_{\eta_0}(\delta)^2 G_{\eta_0}(\delta)^2 \right\}.$$

From Lemma 3.1, Remark 2.3, 1 and (3.6) it follows that

$$F_{\eta_0}(t),\ G_{\eta_0}(t)^{-1},\ F_{\eta_0}(t)G_{\eta_0}(t),\ F_{\eta_0}(t)G_{\eta_0}(t)^2 \longrightarrow 0 \quad \text{as } t \to +0.$$

Therefore we have

$$L^w_M(v_s(\delta(x))) \leqslant 0 \qquad \text{in } \Omega_{\eta_0}.$$

Now we choose a small $\varepsilon > 0$ so that $\varepsilon v_s(\delta(x)) \leqslant u(x)$ on Σ_{η_0}, and set $w_s(\delta(x)) = \varepsilon v_s(\delta(x)) - u(x)$. Then $w_s^+(\delta(x)) = \max\left(w_s(\delta(x)), 0\right) \in W_0^{1,2}(\Omega_{\eta_0}; w(\delta))$, and we see that

$$L^w_M(w_s(\delta(x))) \leqslant 0 \qquad \text{in } \Omega_{\eta_0}.$$

Hence we have for $\delta = \delta(x)$

$$\int_{\Omega_{\eta_0}} \left(|\nabla w_s^+(\delta)|^2 w(\delta) - \frac{w(\delta) w_s^+(\delta)^2}{4F_{\eta_0}(\delta)^2} + Mw(\delta)w_s^+(\delta)^2 \right) dx \leqslant 0.$$

But, by Theorem 2.1, we have

$$\int_{\Omega_{\eta_0}} \left(|\nabla w_s^+(\delta(x))|^2 w(\delta(x)) - \frac{w(\delta(x)) w_s^+(\delta(x))^2}{4F_{\eta_0}(\delta(x))^2} \right) dx \geqslant 0.$$

Therefore we have $w_s^+(\delta(x)) = 0$ in Ω_{η_0}, and so $\varepsilon v_s(\delta(x)) \leqslant u(x)$ in Ω_{η_0} for any $s > 1/2$. By letting $s \to 1/2$, $\varepsilon f_{\eta_0}(\delta(x))^{1/2} G_{\eta_0}(\delta(x))^{-1/2} \leqslant u(x)$ holds in Ω_{η_0}. Namely

$$\frac{u(x)^2 w(\delta(x))}{F_{\eta_0}(\delta(x))^2} \geqslant \varepsilon^2 \frac{1}{F_{\eta_0}(\delta(x)) G_{\eta_0}(\delta(x))} \quad \text{in } \Omega_{\eta_0}.$$

Since it holds that $\left(F_{\eta_0}(\delta(x))G_{\eta_0}(\delta(x))\right)^{-1} \notin L^1(\Omega_{\eta_0})$ by Remark 2.3, 1, we have that $u(x) \notin L^2(\Omega_{\eta_0}; w(\delta)/F_{\eta_0}(\delta)^2)$. This together with Hardy's inequality (2.18) contradicts to that $u(x) \in W_0^{1,2}(\Omega; w(\delta))$. $\square$

REFERENCES

[1] H. ANDO, T. HORIUCHI, *Missing terms in the weighted Hardy-Sobolev inequalities and its application*, Kyoto J. Math., vol. **52**, no. 4 (2012), 759–796.

[2] H. ANDO, T. HORIUCHI, *Weighted Hardy's inequalities and the variational problem with compact perturbations*, Mathematical Journal of Ibaraki University, vol. **52** (2020), 15–26.

[3] H. ANDO, T. HORIUCHI, E. NAKAI, *Weighted Hardy inequalities with infinitely many sharp missing terms*, Mathematical Journal of Ibaraki University, vol. **46** (2014), 9–30.

[4] H. BREZIS, M. MARCUS, *Hardy's inequalities revisited*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome **25**, no. 1–2 (1997), 217–237.

[5] Z. CHEN, Y. SHEN, *Sharp Hardy-Sobolev inequalities with general weights and remainder terms*, Journal of inequalities and applications, (2009), Article ID 419845, 24 pages, doi:10.1155/2009/419845.

[6] E. B. DAVIS, *The Hardy constant*, Quart. J. Math. Oxford, (2) vol. **46** (1995), 417–431.

[7] A. DETALLA, T. HORIUCHI, H. ANDO, *Missing terms in Hardy-Sobolev inequalities and its application*, Far East Journal of Mathematical Sciences, vol. **14**, no. 3 (2004), 333–359.

[8] T. HORIUCHI, *Hardy's inequalities with non-doubling weights and sharp remainders*, Scientiae Mathematicae Japonicae, **85**, no. 2 (2023), 125–147; Scientiae Mathematicae Japonicae, in Edition Electronica, e-2022-2.

[9] X. LIU, T. HORIUCHI, H. ANDO, *One dimensional weighted Hardy's inequalities and application*, Journal of Mathematical Inequalities, vol. **14** (2020), no. 4, 1203–1222.

[10] M. MARCUS, V. J. MIZEL, Y. PINCHOVER, *On the best constant for Hardy's inequality in* $\mathbb{R}^n$, Trans. Amer. Math. Soc., vol. **350**, no. 8 (1998), 3237–3255.

[11] T. MATSKEWICH, P. E. SOBOLEVSKII, *The best possible constant in a generalized Hardy's inequality for convex domains in* $\mathbb{R}^n$, Nonlinear Analysis TMA, vol. **28** (1997), 1601–1610.

[12] V. G. MAZ'JA, *Soboloev spaces* (2nd edition), Springer, (2011).

(Received May 29, 2022)

Hiroshi Ando
Department of Mathematics
Faculty of Science, Ibaraki University
Mito, Ibaraki, 310-8512, Japan
e-mail: hiroshi.ando.math@vc.ibaraki.ac.jp

Toshio Horiuchi
Department of Mathematics
Faculty of Science, Ibaraki University
Mito, Ibaraki, 310-8512, Japan
e-mail: toshio.horiuchi.math@vc.ibaraki.ac.jp