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# NON–LINEAR CASAZZA–KALTON–CHRISTENSEN–VAN EIJNDHOVEN PERTURBATION WITH APPLICATIONS

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*Abstract.* Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S:\mathscr{X}\to\mathscr{Y}$ be an invertible Lipschitz map. Let $T:\mathscr{X}\to\mathscr{Y}$ be a map and there exist $\lambda_1,\lambda_2\in[0,1)$ such that

$$\|Tx-Ty-(Sx-Sy)\|\leqslant\lambda_1\|Sx-Sy\|+\lambda_2\|Tx-Ty\|,\quad \forall x,y\in\mathscr{X}.$$

Then we prove that $T$ is an invertible Lipschitz map. This is non-linear version of 26 years old Casazza-Kalton-Christensen-van Eijndhoven perturbation. It also a non-linear version of 29 years old Soderlind-Campanato perturbation and 3 years old Barbagallo-Ernst-Thera perturbation. We give applications to the theory of metric frames. The notion of Lipschitz atomic decomposition for Banach spaces is also introduced.

## 1. Introduction

Let $\mathscr{X}$ be a Banach space and $I_{\mathscr{X}}$ be the identity operator on $\mathscr{X}$. Carl Neumann's classical result says that if $T:\mathscr{X}\to\mathscr{X}$ is a bounded linear operator such that $\|T-I_{\mathscr{X}}\|<1$, then $T$ is invertible [38]. Following two results are consequences of this result. They are known as *Paley-Wiener theorems*.

1. Sequences close to orthonormal bases in Hilbert spaces are Riesz bases [39, 51].
2. Sequences close to Schauder bases in Banach spaces are Schauder bases [4, 43].

History of Paley-Wiener theorems are nicely presented in [1, 42]. It was in the setting of Hilbert spaces, Paley-Wiener theorem was first generalized by Pollard [41], second generalized by Sz.-Nagy [20] and third generalized by Hilding [32]. Hilding proved the following theorem.

THEOREM 1.1. [32] (Hilding perturbation) *Let $\mathscr{H}$ be a Hilbert space. If a linear operator $T:\mathscr{H}\to\mathscr{H}$ is such that there exists $\lambda\in[0,1)$ with*

$$\|Th-h\|\leqslant\lambda\|Th\|+\lambda\|h\|,\quad \forall h\in\mathscr{H},$$

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*then $T$ is bounded, invertible and*

$$\frac{1-\lambda}{1+\lambda}\|h\| \leqslant \|Th\| \leqslant \frac{1+\lambda}{1-\lambda}\|h\|, \quad \forall h \in \mathscr{H};$$
$$\frac{1-\lambda}{1+\lambda}\|h\| \leqslant \|T^{-1}h\| \leqslant \frac{1+\lambda}{1-\lambda}\|h\|, \quad \forall h \in \mathscr{H}.$$

It took around 50 years to strengthen Theorem 1.1 to the most generality for Banach spaces.

THEOREM 1.2. [9, 11, 48] (Casazza-Kalton-Christensen-van Eijndhoven perturbation) *Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces and $S : \mathscr{X} \to \mathscr{Y}$ be a bounded invertible operator. If a linear operator $T : \mathscr{X} \to \mathscr{Y}$ is such that there exist $\lambda_1, \lambda_2 \in [0,1)$ with*

$$\|Tx - Sx\| \leqslant \lambda_1\|Sx\| + \lambda_2\|Tx\|, \quad \forall x \in \mathscr{X},$$

*then $T$ is bounded, invertible and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|Sx\| \leqslant \|Tx\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|Sx\|, \quad \forall x \in \mathscr{X};$$
$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|S\|}\|y\| \leqslant \|T^{-1}y\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}\|S^{-1}\|\|y\|, \quad \forall y \in \mathscr{Y}.$$

There is a generalization of Theorem 1.2 which is due to Guo with an extra assumption that $T$ is bounded.

THEOREM 1.3. [25] *Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces and $S : \mathscr{X} \to \mathscr{Y}$ be a bounded invertible operator. If a bounded linear operator $T : \mathscr{X} \to \mathscr{Y}$ is such that there exist $\lambda_1 \in [0,1)$ and $\lambda_2 \in [0,1]$ with*

$$\|Tx - Sx\| \leqslant \lambda_1\|Sx\| + \lambda_2\|Tx\|, \quad \forall x \in \mathscr{X},$$

*then $T$ is invertible. Further, for every $\varepsilon > 0$ satisfying $1 > \lambda_2 - \varepsilon > 0$ and $\lambda_1 + \varepsilon\|TS^{-1}\| < 1$, we have*

$$\frac{1-\lambda_1-\varepsilon\|TS^{-1}\|}{1+\lambda_2-\varepsilon}\|Sx\| \leqslant \|Tx\| \leqslant \frac{1+\lambda_1+\varepsilon\|TS^{-1}\|}{1-\lambda_2+\varepsilon}\|Sx\|, \quad \forall x \in \mathscr{X};$$
$$\frac{1-\lambda_2+\varepsilon}{1+\lambda_1+\varepsilon\|TS^{-1}\|}\frac{1}{\|S\|}\|y\| \leqslant \|T^{-1}y\| \leqslant \frac{1+\lambda_2-\varepsilon}{1-\lambda_1-\varepsilon\|TS^{-1}\|}\|S^{-1}\|\|y\|, \quad \forall y \in \mathscr{Y}.$$

Theorem 1.2 and its variants are useful in various studies such as stability of frames for Hilbert spaces [11], stability of frames and atomic decompositions for Banach spaces [46], stability of frames for Hilbert C*-modules [26], stability of G-frames [47], multipliers for Hilbert spaces [45], quantum detection problem [5], continuous frames [23], fusion frames [10], operator representations of frames (dynamics of frames) [15], pseudo-inverses of operators [21], outer inverses of operators [50], shift-invariant spaces [34], frame sequences [17], sampling [52] etc.

The main objective of this paper is to generalize Theorem 1.2 for Lipschitz functions between Banach spaces. We do this in Theorem 2.7. We show that our result generalizes Soderlind-Campanato Perturbation (Theorem 3.1) and Barbagallo-Ernst-Thera perturbation (Theorem 3.2). We then give an application to the theory of frames for metric spaces. Further, the notion of Lipschitz atomic decomposition for Banach spaces is introduced and a perturbation result is derived using Theorem 2.7.

## 2. Non-linear Casazza-Kalton-Christensen-van Eijndhoven perturbation

Let $\mathscr{M}$ be a metric space and $\mathscr{X}$ be a Banach space. Recall that a function $f:\mathscr{M}\to\mathscr{X}$ is said to be Lipschitz if there exists $b>0$ such that

$$\|f(x)-f(y)\|\leqslant b\,d(x,y),\quad \forall x,y\in\mathscr{M}.$$

A Lipschitz function $f:\mathscr{M}\to\mathscr{X}$ is said to be bi-Lipschitz if there exists $a>0$ such that

$$a\,d(x,y)\leqslant\|f(x)-f(y)\|,\quad \forall x,y\in\mathscr{M}.$$

DEFINITION 2.1. [49] Let $\mathscr{X}$ be a Banach space.

(i) Let $\mathscr{M}$ be a metric space. The collection $\operatorname{Lip}(\mathscr{M},\mathscr{X})$ is defined as $\operatorname{Lip}(\mathscr{M},\mathscr{X}) := \{f:\mathscr{M}\to\mathscr{X} \text{ is Lipschitz}\}$. For $f\in\operatorname{Lip}(\mathscr{M},\mathscr{X})$, the Lipschitz number is defined as

$$\operatorname{Lip}(f):=\sup_{x,y\in\mathscr{M},x\neq y}\frac{\|f(x)-f(y)\|}{d(x,y)}.$$

(ii) Let $(\mathscr{M},0)$ be a pointed metric space. The collection $\operatorname{Lip}_0(\mathscr{M},\mathscr{X})$ is defined as $\operatorname{Lip}_0(\mathscr{M},\mathscr{X}):=\{f:\mathscr{M}\to\mathscr{X} \text{ is Lipschitz and } f(0)=0\}$. For $f\in\operatorname{Lip}_0(\mathscr{M},\mathscr{X})$, the Lipschitz norm is defined as

$$\|f\|_{\operatorname{Lip}_0}:=\sup_{x,y\in\mathscr{M},x\neq y}\frac{\|f(x)-f(y)\|}{d(x,y)}.$$

THEOREM 2.2. [49] *Let $\mathscr{X}$ be a Banach space.*

(i) *If $\mathscr{M}$ is a metric space, then $\operatorname{Lip}(\mathscr{M},\mathscr{X})$ is a semi-normed vector space w.r.t. the semi-norm $\operatorname{Lip}(\cdot)$.*

(ii) *If $(\mathscr{M},0)$ is a pointed metric space, then $\operatorname{Lip}_0(\mathscr{M},\mathscr{X})$ is a Banach space w.r.t. the norm $\|\cdot\|_{\operatorname{Lip}_0}$. Further, $\operatorname{Lip}_0(\mathscr{X}):=\operatorname{Lip}_0(\mathscr{X},\mathscr{X})$ is a unital Banach algebra. In particular, if $T\in\operatorname{Lip}_0(\mathscr{X})$ satisfies $\|T-I_{\mathscr{X}}\|_{\operatorname{Lip}_0}<1$, then $T$ is invertible and $T^{-1}\in\operatorname{Lip}_0(\mathscr{X})$.*

We now develop perturbation result for Lipschitz functions using various results. Our developments are motivated from the linear version of improvement of Paley-Wiener theorem by van Eijndhoven [48]. Since $\mathrm{Lip}_0(\mathscr{X})$ is a unital Banach algebra, we can talk about the notion of spectrum and resolvent. In the remaining part of the paper, the spectrum of an element $T \in \mathrm{Lip}_0(\mathscr{X})$ is denoted by $\sigma(T)$ and the resolvent by $\rho(T)$.

THEOREM 2.3. *Let $A$ be a closed non-empty subset of $\mathbb{C}$, such that for any $r \in A$, there is a sequence $\{r_n\}_n \subseteq A$ converging to $r$ such that $|r_n| > |r|$ for any $n \in \mathbb{N}$. Let $T$ be a Lipschitz operator on $\mathscr{X}$ with $T0 = 0$. If there is $\varepsilon > 0$ such that*

$$\|(Tx - rx) - \alpha(Ty - ry)\| \geqslant \varepsilon\|x - y\|, \quad \forall r \in A, \forall x, y \in \mathscr{X}, \tag{1}$$

*then the distance between $A$ and the spectrum of $T$ is at least $\varepsilon$.*

*Proof.* The proof of this Theorem rely on the following lemma.

RESOLVENT LEMMA. *Let $T$ be an element from an unitary Banach algebra $\mathscr{A}$, and $\lambda$ be a complex number lying in the resolvent set of $T$; denote by $\mathrm{dist}(\lambda, \sigma(T))$ the euclidean distance between the number $\lambda$ and the non-empty compact spectrum $\sigma(T)$ of the operator $T$. Then*

$$\|(T - \lambda I_{\mathscr{A}})^{-1}\| \geqslant \frac{1}{\mathrm{dist}(\lambda, \sigma(T))}.$$

The proof of Theorem 2.3 is done in two steps, each one requesting the use of the Resolvent Lemma.

First, one proves that $A$ and the spectrum of $T$ are disjoints. Let us assume the contrary, and pick $r$ a member of $A \cap \sigma(T)$ of the highest modulus. Then the numbers $r_n$ are in the resolvent set of $T$, so $(T - r_n I_{\mathscr{X}})^{-1}$ is a Lipschitz operator, and relation (1) implies that

$$\|(T - r_n I_{\mathscr{X}})^{-1}\| = \mathrm{Lip}_0((T - r_n I_{\mathscr{X}})^{-1}) \leqslant \frac{1}{\varepsilon}.$$

But $r_n$ is converging to $r$, an element of the spectrum of $T$, so by the Resolvent lemma $\|(T - r_n I_{\mathscr{X}})^{-1}\|$ goes to the infinity, a contradiction.

The next step is to prove that the distance between the (now we know being disjoint) sets $A$ and $\sigma(T)$ is indeed superior to $\varepsilon$. To this end, we pick $r$ from $A$; relation (1) implies that

$$\|(T - r I_{\mathscr{X}})^{-1}\| = \mathrm{Lip}_0((T - r I_{\mathscr{X}})^{-1}) \leqslant \frac{1}{\varepsilon}.$$

The Resolvent Lemma reads that

$$\|(T - \lambda I_{\mathscr{X}})^{-1}\| \geqslant \frac{1}{\mathrm{dist}(\lambda, \sigma(T))};$$

by combining the two previous inequalities, one proves the conclusion of the Theorem 2.3. $\square$

THEOREM 2.4. *Let $\mathscr{X}$ be a Banach space, $T : \mathscr{X} \to \mathscr{X}$ be a map, $T0 = 0$ and there exist $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx - Ty - (x-y)\| \leqslant \lambda_1 \|x-y\| + \lambda_2 \|Tx - Ty\|, \quad \forall x, y \in \mathscr{X}. \tag{2}$$

*Then*

(i) *$T$ is Lipschitz and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|x-y\| \leqslant \|Tx - Ty\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|x-y\|, \quad \forall x, y \in \mathscr{X}. \tag{3}$$

(ii) *We have*

$$\left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right) \subseteq \rho(T).$$

(iii) *$T$ is invertible and*

$$\frac{1-\lambda_2}{1+\lambda_1}\|x-y\| \leqslant \|T^{-1}x - T^{-1}y\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}\|x-y\|, \quad \forall x, y \in \mathscr{X}.$$

(iv) *We have*

$$\frac{1-\lambda_1}{1+\lambda_2} \leqslant \|T\|_{Lip_0} \leqslant \frac{1+\lambda_1}{1-\lambda_2} \quad \textit{and} \quad \frac{1-\lambda_2}{1+\lambda_1} \leqslant \|T^{-1}\|_{Lip_0} \leqslant \frac{1+\lambda_2}{1-\lambda_1}.$$

*Proof.* Let $x, y \in \mathscr{X}$. Then using Inequality (2)

$$\begin{aligned}\|Tx - Ty\| &\leqslant \|Tx - Ty - (x-y)\| + \|x-y\| \leqslant \lambda_1\|x-y\| + \lambda_2\|Tx-Ty\| + \|x-y\| \\ &= (1+\lambda_1)\|x-y\| + \lambda_2\|Tx-Ty\| \\ &\implies \|Tx-Ty\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|x-y\|\end{aligned}$$

and

$$\begin{aligned}\|x-y\| &\leqslant \|Tx - Ty - (x-y)\| + \|Tx-Ty\| \leqslant \lambda_1\|x-y\| + \lambda_2\|Tx-Ty\| + \|Tx-Ty\| \\ &= \lambda_1\|x-y\| + (1+\lambda_2)\|Tx-Ty\| \\ &\implies \frac{1-\lambda_1}{1+\lambda_2}\|x-y\| \leqslant \|Tx-Ty\|.\end{aligned}$$

Hence $T$ is Lipschitz and (i) holds. Let $\alpha \leqslant 0$. Then

$$\begin{aligned}\|Tx - Ty - \alpha(x-y)\| &= \|(1-\alpha)(x-y) - (x-y-(Tx-Ty))\| \\ &\geqslant (1-\alpha)\|x-y\| - \|Tx-Ty-(x-y)\| \\ &\geqslant (1-\alpha)\|x-y\| - \lambda_1\|x-y\| - \lambda_2\|Tx-Ty\| \\ &= (1-\alpha-\lambda_1)\|x-y\| - \lambda_2\|Tx-Ty\| \\ &\geqslant (1-\alpha-\lambda_1)\|x-y\| - \lambda_2\|Tx-Ty-\alpha(x-y)\| + \lambda_2\alpha\|x-y\| \\ &= (1-\alpha-\lambda_1+\lambda_2\alpha)\|x-y\| - \lambda_2\|Tx-Ty-\alpha(x-y)\|\end{aligned}$$

which implies

$$\|Tx-Ty-\alpha(x-y)\| \geqslant \frac{1-\alpha-\lambda_1+\lambda_2\alpha}{1+\lambda_2}\|x-y\|$$
$$= \frac{1-\lambda_1-(1-\lambda_2)\alpha}{1+\lambda_2}\|x-y\| \geqslant \frac{1-\lambda_1}{1+\lambda_2}\|x-y\|.$$

By applying Theorem 2.3 for $\alpha_1 = 0$ and $\beta_1 = \frac{1-\lambda_1}{1+\lambda_2}$ we get $(-\infty, \frac{1-\lambda_1}{1+\lambda_2}) \subseteq \rho(T)$. Since $0 \in (-\infty, \frac{1-\lambda_1}{1+\lambda_2})$, $T$ is invertible. Using Inequality (3) we then get

$$\frac{1-\lambda_1}{1+\lambda_2}\|T^{-1}u-T^{-1}v\| \leqslant \|u-v\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|T^{-1}u-T^{-1}v\|, \quad \forall u,v \in \mathscr{X}$$

which gives (iii). Finally (iv) follows from (i) and (iii). $\square$

THEOREM 2.5. *Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S \in Lip_0(\mathscr{X},\mathscr{Y})$ be invertible. Let $T : \mathscr{X} \to \mathscr{Y}$ be a map, $T0 = 0$ and there exist $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx-Ty-(Sx-Sy)\| \leqslant \lambda_1\|Sx-Sy\| + \lambda_2\|Tx-Ty\|, \quad \forall x,y \in \mathscr{X}. \tag{4}$$

*Then*

(i) *$T$ is Lipschitz and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|Sx-Sy\| \leqslant \|Tx-Ty\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|Sx-Sy\|, \quad \forall x,y \in \mathscr{X}.$$

(ii) *$\alpha S - T$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$.*

(iii) *$T$ is invertible and*

$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|S\|_{Lip_0}}\|u-v\| \leqslant \|T^{-1}u-T^{-1}v\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}\|S^{-1}\|_{Lip_0}\|u-v\|, \quad \forall u,v \in \mathscr{Y}.$$

(iv) *We have*

$$\frac{1-\lambda_1}{1+\lambda_2}\|S\|_{Lip_0} \leqslant \|T\|_{Lip_0} \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|S\|_{Lip_0} \quad \textit{and}$$
$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|S\|_{Lip_0}} \leqslant \|T^{-1}\|_{Lip_0} \leqslant \frac{1+\lambda_2}{1-\lambda_1}\|S^{-1}\|_{Lip_0}.$$

*Proof.* Define $R := TS^{-1}$. Then Inequality (4) gives

$$\|TS^{-1}u-TS^{-1}v-(SS^{-1}u-SS^{-1}v)\| \leqslant \lambda_1\|SS^{-1}u-SS^{-1}v\| + \lambda_2\|TS^{-1}u-TS^{-1}v\|, \quad \forall u,v \in \mathscr{Y},$$

i.e.,

$$\|Ru-Rv-(u-v)\| \leqslant \lambda_1\|u-v\| + \lambda_2\|Ru-Rv\|, \quad \forall u,v \in \mathscr{Y}.$$

By applying Theorem 2.4 to $R$ we get the following.

(i) $R$ is Lipschitz hence $T$ is Lipschitz. Further,

$$\frac{1-\lambda_1}{1+\lambda_2}\|Sx-Sy\| \leqslant \|R(Sx)-R(Sy)\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|Sx-Sy\|, \quad \forall x,y \in \mathscr{X}.$$

But $\|R(Sx)-R(Sy)\| = \|Tx-Ty\|$, $\forall x,y \in \mathscr{X}$.

(ii) $\alpha I_{\mathscr{X}} - R$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$. Since $S$ is invertible we then have $\alpha S - T$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$.

(iii) $R$ is invertible hence $T$ is invertible. Further,

$$\begin{aligned}\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|S\|_{\mathrm{Lip}_0}}\|u-v\| \leqslant \frac{1}{\|S\|_{\mathrm{Lip}_0}}\|R^{-1}u-R^{-1}v\| &\leqslant \|S^{-1}(R^{-1}u)-S^{-1}(R^{-1}v)\| \\ &= \|T^{-1}u-T^{-1}v\| \leqslant \|S^{-1}\|_{\mathrm{Lip}_0}\|R^{-1}u-R^{-1}v\| \\ &\leqslant \frac{1+\lambda_2}{1-\lambda_1}\|S^{-1}\|_{\mathrm{Lip}_0}\|u-v\|, \quad \forall u,v \in \mathscr{Y}.\end{aligned}$$

(iv) This follows easily from (i) and (iii). $\square$

Our next task is to derive the results by removing the condition $T0 = 0$.

THEOREM 2.6. *Let $\mathscr{X}$ be a Banach space, $T : \mathscr{X} \to \mathscr{X}$ be a map and there exist $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx-Ty-(x-y)\| \leqslant \lambda_1\|x-y\| + \lambda_2\|Tx-Ty\|, \quad \forall x,y \in \mathscr{X}.$$

*Then*

(i) *$T$ is Lipschitz and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|x-y\| \leqslant \|Tx-Ty\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|x-y\|, \quad \forall x,y \in \mathscr{X}.$$

(ii) *$\alpha I_{\mathscr{X}} - T$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$.*

(iii) *$T$ is invertible and*

$$\frac{1-\lambda_2}{1+\lambda_1}\|x-y\| \leqslant \|T^{-1}x-T^{-1}y\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}\|x-y\|, \quad \forall x,y \in \mathscr{X}.$$

(iv) *We have*

$$\begin{aligned}&\frac{1-\lambda_1}{1+\lambda_2} \leqslant Lip(T) \leqslant \frac{1+\lambda_1}{1-\lambda_2} \quad \textit{and} \\ &\frac{1-\lambda_2}{1+\lambda_1} \leqslant Lip(T^{-1}) \leqslant \frac{1+\lambda_2}{1-\lambda_1}.\end{aligned}$$

*Proof.* Define

$$\tilde{T}x := Tx - T0, \quad \forall x \in \mathscr{X}.$$

Then $\tilde{T}0 = 0$ and

$$\begin{aligned}\|\tilde{T}x - \tilde{T}y - (x-y)\| = \|Tx - Ty - (x-y)\| &\leqslant \lambda_1\|x-y\| + \lambda_2\|Tx - Ty\| \\ &= \lambda_1\|x-y\| + \lambda_2\|\tilde{T}x - \tilde{T}y\|, \quad \forall x,y \in \mathscr{X}.\end{aligned}$$

Applying Theorem 2.4 and using the fact that 'a map is bijective if and only if its translate is bijective', proof is complete. □

THEOREM 2.7. *Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S \in Lip(\mathscr{X}, \mathscr{Y})$ be invertible. Let $T : \mathscr{X} \to \mathscr{Y}$ be a map and there exist $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx - Ty - (Sx - Sy)\| \leqslant \lambda_1\|Sx - Sy\| + \lambda_2\|Tx - Ty\|, \quad \forall x,y \in \mathscr{X}.$$

*Then*

(i) *$T$ is Lipschitz and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|Sx - Sy\| \leqslant \|Tx - Ty\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|Sx - Sy\|, \quad \forall x,y \in \mathscr{X}.$$

(ii) *$\alpha S - T$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$.*

(iii) *$T$ is invertible and*

$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{Lip(S)}\|u - v\| \leqslant \|T^{-1}u - T^{-1}v\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}Lip(S^{-1})\|u - v\|, \quad \forall u,v \in \mathscr{Y}.$$

(iv) *We have*

$$\begin{aligned}\frac{1-\lambda_1}{1+\lambda_2}Lip(S) \leqslant Lip(T) \leqslant \frac{1+\lambda_1}{1-\lambda_2}Lip(S) \quad &\textit{and} \\ \frac{1-\lambda_2}{1+\lambda_1}\frac{1}{Lip(S)} \leqslant Lip(T^{-1}) \leqslant \frac{1+\lambda_2}{1-\lambda_1}Lip(S^{-1}).&\end{aligned}$$

*Proof.* Define $R := TS^{-1}$ and the proof is similar to proof of Theorem 2.5. □

Following two corollaries are motivated from [32].

COROLLARY 2.8. *Let $p \geqslant 1$. Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S \in Lip(\mathscr{X}, \mathscr{Y})$ be invertible. Let $T : \mathscr{X} \to \mathscr{Y}$ be a map and there exist $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx - Ty - (Sx - Sy)\| \leqslant ((\lambda_1\|Sx - Sy\|)^p + (\lambda_2\|Tx - Ty\|)^p)^{\frac{1}{p}}, \quad \forall x,y \in \mathscr{X}. \tag{5}$$

*Then*

(i) *$T$ is Lipschitz and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|Sx-Sy\| \leqslant \|Tx-Ty\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|Sx-Sy\|, \quad \forall x,y \in \mathscr{X}.$$

(ii) *$\alpha S - T$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$.*

(iii) *$T$ is invertible and*

$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{Lip(S)}\|u-v\| \leqslant \|T^{-1}u-T^{-1}v\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}Lip(S^{-1})\|u-v\|, \quad \forall u,v \in \mathscr{Y}.$$

(iv) *We have*

$$\frac{1-\lambda_1}{1+\lambda_2}Lip(S) \leqslant Lip(T) \leqslant \frac{1+\lambda_1}{1-\lambda_2}Lip(S) \quad \text{and}$$
$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{Lip(S)} \leqslant Lip(T^{-1}) \leqslant \frac{1+\lambda_2}{1-\lambda_1}Lip(S^{-1}).$$

*Proof.* Note that if $r,s \geqslant 0$, then

$$(r^p+s^p)^{\frac{1}{p}} \leqslant r+s \quad \text{if} \quad p \geqslant 1.$$

Hence Inequality (5) gives

$$\begin{aligned}\|Tx-Ty-(Sx-Sy)\| &\leqslant ((\lambda_1\|Sx-Sy\|)^p+(\lambda_2\|Tx-Ty\|)^p)^{\frac{1}{p}}\\ &\leqslant \lambda_1\|Sx-Sy\|+\lambda_2\|Tx-Ty\|, \quad \forall x,y \in \mathscr{X}.\end{aligned}$$

Result follows by applying Theorem 2.7. □

COROLLARY 2.9. *Let $0 < p < 1$. Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S \in Lip(\mathscr{X},\mathscr{Y})$ be invertible. Let $T: \mathscr{X} \to \mathscr{Y}$ be a map and there exist $\lambda_1,\lambda_2 \in \left[0, 2^{1-\frac{1}{p}}\right)$ such that*

$$\|Tx-Ty-(Sx-Sy)\| \leqslant ((\lambda_1\|Sx-Sy\|)^p+(\lambda_2\|Tx-Ty\|)^p)^{\frac{1}{p}}, \quad \forall x,y \in \mathscr{X}.$$

*Then*

(i) *$T$ is Lipschitz and*

$$\frac{1-2^{\frac{1}{p}-1}\lambda_1}{1+2^{\frac{1}{p}-1}\lambda_2}\|Sx-Sy\| \leqslant \|Tx-Ty\| \leqslant \frac{1+2^{\frac{1}{p}-1}\lambda_1}{1-2^{\frac{1}{p}-1}\lambda_2}\|Sx-Sy\|, \quad \forall x,y \in \mathscr{X}.$$

(ii) *$\alpha S - T$ is invertible for all* $\alpha \in \left(-\infty, \frac{1-2^{\frac{1}{p}-1}\lambda_1}{1+2^{\frac{1}{p}-1}\lambda_2}\right)$.

(iii) *$T$ is invertible and*

$$\frac{1-2^{\frac{1}{p}-1}\lambda_2}{1+2^{\frac{1}{p}-1}\lambda_1}\frac{1}{Lip(S)}\|u-v\| \leqslant \|T^{-1}u-T^{-1}v\| \leqslant \frac{1+2^{\frac{1}{p}-1}\lambda_2}{1-2^{\frac{1}{p}-1}\lambda_1}Lip(S^{-1})\|u-v\|,$$
$$\forall u,v \in \mathscr{Y}.$$

(iv) *We have*

$$\frac{1-2^{\frac{1}{p}-1}\lambda_1}{1+2^{\frac{1}{p}-1}\lambda_2}Lip(S) \leqslant Lip(T) \leqslant \frac{1+2^{\frac{1}{p}-1}\lambda_1}{1-2^{\frac{1}{p}-1}\lambda_2}Lip(S) \quad \textit{and}$$
$$\frac{1-2^{\frac{1}{p}-1}\lambda_2}{1+2^{\frac{1}{p}-1}\lambda_1}\frac{1}{Lip(S)} \leqslant Lip(T^{-1}) \leqslant \frac{1+2^{\frac{1}{p}-1}\lambda_2}{1-2^{\frac{1}{p}-1}\lambda_1}Lip(S^{-1}).$$

*Proof.* Note that if $r,s \geqslant 0$, then

$$(r^p+s^p)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p}-1}(r+s) \quad \text{if} \quad p<1.$$

Hence Inequality (5) gives

$$\begin{aligned}\|Tx-Ty-(Sx-Sy)\| &\leqslant ((\lambda_1\|Sx-Sy\|)^p+(\lambda_2\|Tx-Ty\|)^p)^{\frac{1}{p}}\\ &\leqslant 2^{\frac{1}{p}-1}\lambda_1\|Sx-Sy\|+2^{\frac{1}{p}-1}\lambda_2\|Tx-Ty\|, \quad \forall x,y \in \mathscr{X}.\end{aligned}$$

Result follows by applying Theorem 2.7. □

Next we generalize Corollary 1 in [11].

COROLLARY 2.10. *Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S \in Lip(\mathscr{X},\mathscr{Y})$ be invertible. Let $T : \mathscr{X} \to \mathscr{Y}$ be a Lipschitz map and there exists $\lambda \in [0,1)$ such that*

$$\|Tx-Ty-(Sx-Sy)\| \leqslant \lambda\|Sx-Sy\|+\|Tx-Ty\|, \quad \forall x,y \in \mathscr{X}.$$

*Then $T$ is invertible and*

$$Lip(T^{-1}) \leqslant \frac{2}{1-\lambda}Lip(S^{-1}).$$

*Proof.* Define $R := TS^{-1}$. Then

$$\|Ru-Rv-(u-v)\| \leqslant \lambda\|u-v\|+\|Ru-Rv\|, \quad \forall u,v \in \mathscr{Y}.$$

Note that $\mathrm{Lip}(R) \neq 0$. Let

$$0 < \varepsilon < \min\left\{1, \frac{1-\lambda}{\mathrm{Lip}(R)}\right\}.$$

Define $\lambda_1 := \lambda - \varepsilon \mathrm{Lip}(R)$ and $\lambda_2 := 1-\varepsilon$. Then $\lambda_1, \lambda_2 \in [0,1)$ and

$$\begin{aligned}\|Ru - Rv - (u-v)\| &\leqslant \lambda \|u-v\| + \|Ru - Rv\| \\ &\leqslant \lambda \|u-v\| + \|Ru-Rv\| + \varepsilon(\mathrm{Lip}(R)\|u-v\| - \|Ru-Rv\|) \\ &= (\lambda + \varepsilon \mathrm{Lip}(R))\|u-v\| + (1-\varepsilon)\|Ru-Rv\|, \quad \forall u,v \in \mathscr{Y}.\end{aligned}$$

By applying Theorem 2.6 we get that $R$ is Lipschitz, invertible and

$$\mathrm{Lip}(R^{-1}) \leqslant \frac{2-\varepsilon}{1-(\lambda+\varepsilon\mathrm{Lip}(R))}.$$

Since $\varepsilon$ can be made arbitrarily small, we must have

$$\mathrm{Lip}(R^{-1}) \leqslant \frac{2}{1-\lambda}.$$

Substituting the expression of $R$ gives

$$\frac{1}{\mathrm{Lip}(S^{-1})}\mathrm{Lip}(T^{-1}) \leqslant \mathrm{Lip}(ST^{-1}) = \mathrm{Lip}(R^{-1}) \leqslant \frac{2}{1-\lambda}. \quad \square$$

We finally derive the following non-linear version of Theorem 1.3.

THEOREM 2.11. *Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S \in Lip(\mathscr{X},\mathscr{Y})$ be invertible. Let $T : \mathscr{X} \to \mathscr{Y}$ be a Lipschitz map and there exist $\lambda_1 \in [0,1)$ and $\lambda_2 \in [0,1]$ such that*

$$\|Tx - Ty - (Sx - Sy)\| \leqslant \lambda_1\|Sx-Sy\| + \lambda_2\|Tx-Ty\|, \quad \forall x,y \in \mathscr{X}.$$

*Then $T$ is Lipschitz invertible. Further, for every $\varepsilon > 0$ satisfying $1 > \lambda_2 - \varepsilon > 0$ and $\lambda_1 + \varepsilon Lip(TS^{-1}) < 1$, we have*

(i)

$$\frac{1-\lambda_1-\varepsilon Lip(TS^{-1})}{1+\lambda_2-\varepsilon}\|Sx-Sy\| \leqslant \|Tx-Ty\| \leqslant \frac{1+\lambda_1+\varepsilon Lip(TS^{-1})}{1-\lambda_2+\varepsilon}\|Sx-Sy\|, \quad \forall x,y \in \mathscr{X}.$$

(ii) *$\alpha S - T$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1-\varepsilon Lip(TS^{-1})}{1+\lambda_2-\varepsilon}\right)$.*

(iii) *$T$ is invertible and*

$$\begin{aligned}\frac{1-\lambda_2+\varepsilon}{1+\lambda_1+\varepsilon Lip(TS^{-1})}\frac{1}{Lip(S)}\|u-v\| &\leqslant \|T^{-1}u - T^{-1}v\| \\ &\leqslant \frac{1+\lambda_2-\varepsilon}{1-\lambda_1-\varepsilon Lip(TS^{-1})}Lip(S^{-1})\|u-v\|, \quad \forall u,v \in \mathscr{Y}.\end{aligned}$$

(iv) *We have*

$$\frac{1-\lambda_1-\varepsilon Lip(TS^{-1})}{1+\lambda_2-\varepsilon}Lip(S) \leqslant Lip(T) \leqslant \frac{1+\lambda_1+\varepsilon Lip(TS^{-1})}{1-\lambda_2+\varepsilon}Lip(S) \quad \textit{and}$$
$$\frac{1-\lambda_2+\varepsilon}{1+\lambda_1+\varepsilon Lip(TS^{-1})}\frac{1}{Lip(S)} \leqslant Lip(T^{-1}) \leqslant \frac{1+\lambda_2-\varepsilon}{1-\lambda_1-\varepsilon Lip(TS^{-1})}Lip(S^{-1}).$$

*Proof.* Define $R := TS^{-1}$. Then for every $\varepsilon > 0$ satisfying $1 > \lambda_2 - \varepsilon > 0$ and $\lambda_1 + \varepsilon \mathrm{Lip}(TS^{-1}) < 1$,

$$\begin{aligned}\|Ru-Rv-(u-v)\| &\leqslant \lambda_1\|u-v\| + \lambda_2\|Ru-Rv\| \\ &= \lambda_1\|u-v\| + (\lambda_2-\varepsilon)\|Ru-Rv\| + \varepsilon\|Ru-Rv\| \\ &\leqslant \lambda_1\|u-v\| + (\lambda_2-\varepsilon)\|Ru-Rv\| + \varepsilon \mathrm{Lip}(R)\|u-v\| \\ &= (\lambda_1+\varepsilon \mathrm{Lip}(R))\|u-v\| + (\lambda_2-\varepsilon)\|Ru-Rv\|, \quad \forall u,v \in \mathscr{Y}.\end{aligned}$$

Remaining parts of the proof is similar to the proof of Theorem 2.5. □

It is an easy observation that the constant $\lambda_1$ in Theorem 2.11 can not be strengthened. We are therefore left with the following open problem.

QUESTION 2.12. Can the constant $\lambda_2$ be strengthened further in Theorem 2.11?

## 3. Applications

Our first two applications of Theorem 2.7 are easy proofs of Soderlind-Campanato perturbation and Barbagallo-Ernst-Thera perturbation.

THEOREM 3.1. [6, 44] (Soderlind-Campanato perturbation) *Let $\mathscr{X}$ be a real Banach space, $A : \mathscr{X} \to \mathscr{X}$ be a map and there exist $\alpha > 0, 0 \leqslant \beta < 1$ such that*

$$\|\alpha Ax - \alpha Ay - (x-y)\| \leqslant \beta\|x-y\|, \quad \forall x,y \in \mathscr{X}.$$

*Then $A$ is Lipschitz, invertible and $Lip(A^{-1}) \leqslant \frac{\alpha}{1-\beta}$.*

*Proof.* Set $T = \alpha A$ and $\lambda_1 = \beta$ in Theorem 2.7. Then $\frac{1}{\alpha}\mathrm{Lip}(A^{-1}) = \mathrm{Lip}(T^{-1}) \leqslant \frac{1}{1-\lambda_1} = \frac{1}{1-\beta}$. □

THEOREM 3.2. [3] (Barbagallo-Ernst-Thera perturbation) *Let $\mathscr{X}$ be a real Banach space, $A : \mathscr{X} \to \mathscr{X}$ be a map and there exist $\alpha > 0, 0 \leqslant \beta < 1$ such that*

$$\|Ax - Ay - (\alpha x - \alpha y)\| \leqslant \beta\|Ax - Ay\|, \quad \forall x,y \in \mathscr{X}. \tag{6}$$

*Then*

(i) *If $\beta < 1/2$, then $A$ is Lipschitz, invertible and $Lip(A^{-1}) \leqslant \frac{1-\beta}{\alpha(1-2\beta)}$.*

(ii) *If* $\mathscr{X}$ *is a Hilbert space, then* $A$ *is Lipschitz, invertible and* $Lip(A^{-1}) \leqslant \frac{1+\beta}{\alpha}$.

*Proof.* Set $T = \frac{1}{\alpha}A$ and $\lambda_2 = \beta$ in Theorem 2.7. Then

(i) $\alpha\mathrm{Lip}(A^{-1}) = 1+\lambda_2 = 1+\beta \leqslant \frac{1-\beta}{1-2\beta}$.

(ii) $\alpha\mathrm{Lip}(A^{-1}) = 1+\lambda_2 = 1+\beta$. □

We now give applications to the theory of frames. Paley-Wiener theorem for orthonormal basis in Hilbert spaces inspired the study of perturbation of frames for Hilbert spaces. This was first derived by Christensen in his two papers [12, 13]. This motivated the perturbation of frames and atomic decompositions for Banach spaces [16]. Crucial result used in all these perturbation results is the Neumann series. Later, using Theorem 1.2, Casazza and Christensen [11] improved the results obtained in paper [13]. Using Theorem 1.2 Stoeva made a systematic study of perturbations of frames for Banach spaces [46]. For the sake of completeness, we note that Theorem 1.2 was used in the study of perturbations of frames for Hilbert C*-modules [26].

Large body of work on frames for Hilbert spaces (see [14, 22, 27, 28]) lead to the well developed theory of frames (known as Banach frames and $\mathscr{X}_d$-frames) for Banach spaces (see [7, 8, 18, 24]) lead to the beginning of frames for metric spaces (known as metric frames) [35].

For stating these definitions we need the definition of BK-space (Banach scalar valued sequence space or Banach co-ordinate space).

DEFINITION 3.3. [2] A sequence space $\mathscr{M}_d$ is said to be a BK-space if all the coordinate functionals are continuous, i.e., whenever $\{x_n\}_n$ is a sequence in $\mathscr{M}_d$ converging to $x \in \mathscr{M}_d$, then each coordinate of $x_n$ converges to each coordinate of $x$.

DEFINITION 3.4. [35] Let $\mathscr{M}$ be a metric space and $\mathscr{M}_d$ be an associated BK-space. Let $\{f_n\}_n$ be a collection in $\mathrm{Lip}(\mathscr{M},\mathbb{K})$ (where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) and $S:\mathscr{M}_d \to \mathscr{M}$. If:

(i) $\{f_n(x)\}_n \in \mathscr{M}_d$, for each $x \in \mathscr{M}$,

(ii) There exist positive $a,b$ such that $a\,d(x,y) \leqslant \|\{f_n(x)-f_n(y)\}_n\|_{\mathscr{M}_d} \leqslant b\,d(x,y)$, $\forall x,y \in \mathscr{M}$,

(iii) $S$ is Lipschitz and $S(\{f_n(x)\}_n) = x$, for each $x \in \mathscr{M}$,

then we say that $(\{f_n\}_n, S)$ is a *metric frame* for $\mathscr{M}$ with respect to $\mathscr{M}_d$. We say constant $a$ as lower frame bound and constant $b$ as upper frame bound.

As noted in [35], we observe that if $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$ w.r.t. $\mathscr{M}_d$, then Definition 3.4 tells that the analysis map

$$\theta_f : \mathscr{M} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \mathscr{M}_d$$

is well-defined and bi-Lipschitz. Further, $S\theta_f = I_{\mathscr{M}}$. Now we improve Theorem 4 in [11] to metric frames for Banach spaces.

THEOREM 3.5. *Let $(\{f_n\}_n, S)$ be a metric frame with lower frame bound $a$ and upper frame bound $b$ for a Banach space $\mathscr{X}$ w.r.t. $\mathscr{M}_d$. Let $T : \mathscr{M}_d \to \mathscr{X}$ be a Lipschitz map and suppose that there exist $\lambda_1, \lambda_2, \mu \geqslant 0$ such that $\max\{\lambda_2, \lambda_1 + \mu b\} < 1$ and*

$$\begin{aligned}
&\|S\{c_n\}_n - S\{d_n\}_n - (T\{c_n\}_n - T\{d_n\}_n)\| \\
&\leqslant \lambda_1 \|S\{c_n\}_n - S\{d_n\}_n\| + \lambda_2 \|T\{c_n\}_n - T\{d_n\}_n\| \\
&\quad + \mu \|\{c_n - d_n\}_n\|, \quad \forall \{c_n\}_n, \{d_n\}_n \in \mathscr{M}_d.
\end{aligned} \tag{7}$$

*Then there exists a collection $\{g_n\}_n$ in $\mathrm{Lip}(\mathscr{X}, \mathbb{K})$ such that $(\{g_n\}_n, T)$ is a metric frame for $\mathscr{X}$ with lower and upper frame bounds*

$$\frac{a(1-\lambda_2)}{1+\lambda_1+\mu b}, \quad \frac{b(1+\lambda_2)}{1-(\lambda_1+\mu b)}$$

*respectively.*

*Proof.* Given $x, y \in \mathscr{X}$, by taking $\{c_n\}_n$ as $\theta_f x$ and $\{d_n\}_n$ as $\theta_f y$ in Inequality (7) we get

$$\begin{aligned}
&\|S\theta_f x - S\theta_f y - (T\theta_f x - T\theta_f y) \\
&\| \leqslant \lambda_1 \|S\theta_f x - S\theta_f y\| + \lambda_2 \|T\theta_f x - T\theta_f y\| + \mu \|\theta_f x - \theta_f y\|, \quad \forall x, y \in \mathscr{X}.
\end{aligned}$$

But $S\theta_f x = x, \forall x \in \mathscr{X}$ and hence

$$\begin{aligned}
\|x - y - (T\theta_f x - T\theta_f y)\| &\leqslant \lambda_1 \|x-y\| + \lambda_2 \|T\theta_f x - T\theta_f y\| + \mu \|\theta_f x - \theta_f y\| \\
&\leqslant (\lambda_1 + \mu\, \mathrm{Lip}(\theta_f)) \|x-y\| + \lambda_2 \|T\theta_f x - T\theta_f y\| \\
&\leqslant (\lambda_1 + \mu b) \|x-y\| + \lambda_2 \|T\theta_f x - T\theta_f y\|, \quad \forall x, y \in \mathscr{X}.
\end{aligned}$$

Theorem 2.7 now says that the map $T\theta_f$ is Lipschitz invertible and

$$\frac{1-\lambda_2}{1+\lambda_1+\mu b} \leqslant \mathrm{Lip}(T\theta_f)^{-1} \leqslant \frac{1+\lambda_2}{1-(\lambda_1+\mu b)}.$$

Define $g_n := f_n (T\theta_f)^{-1}$ for all $n \in \mathbb{N}$. Then $g_n$ is Lipschitz for all $n$, $\{g_n(x)\}_n \in \mathscr{M}_d$, for each $x \in \mathscr{M}$ and

$$\begin{aligned}
\|\{g_n(x) - g_n(y)\}_n\| &= \|\{f_n((T\theta_f)^{-1}(x)) - f_n((T\theta_f)^{-1}(y))\}_n\| \\
&\leqslant b \|(T\theta_f)^{-1}(x) - (T\theta_f)^{-1}(y)\| \\
&\leqslant b \frac{1+\lambda_2}{1-(\lambda_1+\mu b)} \|x-y\|, \quad \forall x, y \in \mathscr{X},
\end{aligned}$$

$$\begin{aligned}
a \frac{1-\lambda_2}{1+\lambda_1+\mu b} \|x-y\| &\leqslant a \|(T\theta_f)^{-1}(x)) - (T\theta_f)^{-1}(y)\| \\
&\leqslant \|\{f_n((T\theta_f)^{-1}(x)) - f_n((T\theta_f)^{-1}(y))\}_n\| \\
&= \|\{g_n(x) - g_n(y)\}_n\|, \quad \forall x, y \in \mathscr{X}
\end{aligned}$$

which establish lower and upper frame bounds. Further,

$$T\{g_n(x)\}_n = T\{f_n(T\theta_f)^{-1}(x)\}_n = T\theta_f((T\theta_f)^{-1}(x)) = x, \quad \forall x \in \mathscr{X}.$$

Therefore $(\{g_n\}_n, T)$ is a metric frame for $\mathscr{X}$. $\square$

We now give another application of Theorem 2.7. For this purpose, we introduce the notion of non-linear atomic decompositions.

DEFINITION 3.6. Let $\mathscr{X}$ be a Banach space $\mathscr{X}$ and $\mathscr{M}_d$ be a BK-space. Let $\{f_n\}_n$ be a sequence in $\operatorname{Lip}(\mathscr{X}, \mathbb{K})$ and $\{\tau_n\}_n$ to be a sequence in $\mathscr{X}$ If:

(i) $\{f_n(x)\}_n \in \mathscr{M}_d$, for each $x \in \mathscr{X}$,

(ii) There exist positive $a, b$ such that

$$a\|x-y\| \leqslant \|\{f_n(x) - f_n(y)\}_n\|_{\mathscr{M}_d} \leqslant b\|x-y\|, \quad \forall x, y \in \mathscr{X},$$

(iii) $x = \sum_{n=1}^{\infty} f_n(x)\tau_n$, for each $x \in \mathscr{X}$,

then we say that $(\{f_n\}_n, \{\tau_n\}_n)$ is a *Lipschitz atomic decomposition* for $\mathscr{X}$ with respect to $\mathscr{M}_d$. We say constant $a$ as lower Lipschitz atomic bound and constant $b$ as upper Lipschitz atomic bound.

In [8] it is proved that not every Banach space admits an atomic decomposition. Motivated from this, we ask the following open problem.

QUESTION 3.7. Classify Banach spaces which admit Lipschitz atomic decompositions.

Following proposition gives various examples of Lipschitz atomic decompositions.

PROPOSITION 3.8. *Let $(\{g_n\}_n, \{\omega_n\}_n)$ be an atomic decomposition for a Banach space $\mathscr{Y}$ w.r.t. BK-space $\mathscr{M}_d$. Let $\mathscr{X}$ be a Banach space and let $A : \mathscr{X} \to \mathscr{Y}$ be a bi-Lipschitz map such that there exists a linear map $A : \mathscr{Y} \to \mathscr{X}$ satisfying $BA = I_{\mathscr{X}}$. Then $(\{f_n := g_nA\}_n, \{\tau_n := B\omega_n\}_n)$ is a Lipschitz atomic decomposition $\mathscr{X}$ w.r.t. $\mathscr{M}_d$. In particular, if a Banach space admits a Schauder basis, then it admits a Lipschitz atomic decomposition.*

Particular case of Proposition 3.8 gives the following example.

EXAMPLE 3.9. Let $(\{g_n\}_n, \{\omega_n\}_n)$ be an atomic decomposition for a Banach space $\mathscr{X}$ w.r.t. a BK-space $\mathscr{M}_d$. Let $T : \mathscr{X} \to \mathscr{X}$ be any bi-Lipschitz map. Define $A : \mathscr{X} \ni x \mapsto (x, Tx) \in \mathscr{X} \oplus \mathscr{X}$ and $B : \mathscr{X} \oplus \mathscr{X} \ni (x, y) \mapsto x \in \mathscr{X}$. Then $A$ is bi-Lipschitz, $B$ is linear and satisfies $BA = I_{\mathscr{X}}$. Hence $(\{f_n := g_nA\}_n, \{\tau_n := B\omega_n\}_n)$ is a Lipschitz atomic decomposition for $\mathscr{X}$ w.r.t. $\mathscr{M}_d$.

At this point it seems that it is best to develop some theory of Lipschitz atomic decompositions before giving an application of Theorem 2.7.

Proposition 2.3 in [8] shows that under certain conditions, there is a close relationship between Banach frames and atomic decompositions. Following is the non-linear version of that result.

PROPOSITION 3.10. *Let $\mathscr{X}$ be a Banach space and $\mathscr{M}_d$ be a BK-space. Let $\{f_n\}_n$ be a sequence in* $\operatorname{Lip}(\mathscr{X},\mathbb{K})$ *and $S:\mathscr{M}_d\to\mathscr{X}$ be a bounded linear operator. If the standard unit vectors $\{e_n\}_n$ form a Schauder basis for $\mathscr{M}_d$, then the following are equivalent.*

(i) $(\{f_n\}_n, S)$ *is a metric frame for $\mathscr{X}$.*

(ii) $(\{f_n\}_n, \{Se_n\}_n)$ *is a Lipschitz atomic decomposition for $\mathscr{X}$ w.r.t. $\mathscr{M}_d$.*

*Proof.* We set $\tau_n = Se_n, \forall n\in\mathbb{N}$ and see that

$$\sum_{n=1}^{\infty} f_n(x)\tau_n = \sum_{n=1}^{\infty} f_n(x)Se_n = S\left(\sum_{n=1}^{\infty} f_n(x)e_n\right) = S(\{f_n(x)\}_n), \quad \forall x\in\mathscr{X}. \quad \square$$

Well established dilation theory of frames for Hilbert spaces says that frames are images of Riesz bases under projections [19, 28, 33]. This result has been extended to frames and atomic decompositions for Banach spaces [8, 29–31, 36]. In the next theorem we derive a dilation result for Lipschitz atomic decompositions. We need a proposition to use in the theorem.

PROPOSITION 3.11. [37] *A sequence $\{\tau_n\}_n$ in a Banach space $\mathscr{X}$ is a Schauder basis for $\mathscr{X}$ if and only if the following three conditions hold.*

(i) $\tau_n\neq 0$ *for all $n$.*

(ii) *There exists $b>0$ such that for every sequence $\{a_k\}_k$ of scalars and every pair of natural numbers $n<m$, we have*

$$\left\|\sum_{k=1}^{n} a_k\tau_k\right\| \leqslant b\left\|\sum_{k=1}^{m} a_k\tau_k\right\|.$$

(iii) $\overline{\operatorname{span}}\{\tau_n\}_n = \mathscr{X}$.

THEOREM 3.12. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz atomic decomposition for a Banach space $\mathscr{X}$ w.r.t. $\mathscr{M}_d$. Then there is a Banach space $\mathscr{Z}$ with a Schauder basis $\{\omega_n\}_n$, an injective map $\theta:\mathscr{X}\to\mathscr{Z}$ and a map $P:\mathscr{Z}\to\mathscr{Z}$ satisfying $P(\mathscr{Z})=\mathscr{X}$, $P^2=P$ and $P\omega_n=\theta\tau_n, \forall n\in\mathbb{N}$.*

*Proof.* We generalize the idea of proof of Theorem 2.6 in [8] (which is motivated from the arguments in [40]) to non-linear setting. Let $c_{00}$ be the vector space of scalar sequences with only finitely many non-zero terms. Let $\{e_n\}_n$ be the standard unit vectors in $c_{00}$.

*Case* $(i)$: $\tau_n \neq 0$, for all $n$. We define a norm on $c_{00}$ as follows. Let $\{a_n\}_n \in c_{00}$. Define

$$\left\|\sum_{n=1}^{\infty} a_n e_n\right\| := \max_n \left\|\sum_{k=1}^{n} a_k \tau_k\right\|. \tag{8}$$

Proposition 3.11 then tells that $\{e_n\}_n$ is a Schauder basis for the completion of $c_{00}$, call as $\mathscr{Z}$ w.r.t. just defined norm. Define

$$\theta : \mathscr{X} \ni x \mapsto \theta x := \sum_{n=1}^{\infty} f_n(x) e_n \in \mathscr{Z}.$$

From the first condition of the definition of Lipschitz atomic decomposition, from Definition 8 and from the construction of $\mathscr{Z}$, it follows that $\theta$ is well-defined. From the third condition of definition of atomic decomposition, $\theta$ is injective. We next define

$$\Gamma : \mathscr{Z} \ni \sum_{n=1}^{\infty} a_n e_n \mapsto \Gamma\left(\sum_{n=1}^{\infty} a_n e_n\right) := \sum_{n=1}^{\infty} a_n \tau_n \in \mathscr{X}.$$

By verifying $\Gamma$ is bounded linear on dense subset $c_{00}$ of $\mathscr{Z}$, we see that $\Gamma$ is bounded on $\mathscr{Z}$. Then

$$\Gamma\theta x = \Gamma\left(\sum_{n=1}^{\infty} f_n(x) e_n\right) = \sum_{n=1}^{\infty} f_n(x)\tau_n = x, \quad \forall x \in \mathscr{X}. \tag{9}$$

So if we define $P := \theta\Gamma$, then $P^2 = \theta\Gamma\theta\Gamma = \theta\Gamma = P$. Equation (9) tells that $P(\mathscr{Z}) = \mathscr{X}$. We next see that $Pe_n = \theta\Gamma e_n = \theta\tau_n$, $\forall n$. Thus we can take $\omega_n = e_n$, for all $n$ to get the result.

*Case* $(ii)$: $\tau_n = 0$, for some $n$. Let $J = \{n : \tau_n \neq 0\}$. We now apply case (i) to the collection $\{a_n\}_{n\in J}$. Let $\theta$, $\mathscr{Z}$, $\Gamma$ and $P$ be as in the case (i). Without affecting the definition of atomic decomposition, we can take $f_n = 0$ for all $n \in J^c$. Now consider the space $\mathscr{Z} \oplus \ell^2(J^c)$ and let $\{\rho_n\}_{n\in J^c}$ be an orthonormal basis for $\ell^2(J^c)$. Define $Q : \mathscr{Z} \oplus \ell^2(J^c) \ni z \oplus y \mapsto Q(z\oplus y) := Pz \oplus 0 \in \mathscr{Z} \oplus \ell^2(J^c)$. Now the space $\mathscr{Z} \oplus \ell^2(J^c)$ has Schauder basis $\{\tau_n \oplus 0, 0 \oplus \rho_m\}_{n\in J, m\in J^c}$ and $Q$ satisfy the conclusions. Thus we can take $\omega_n = e_n$, for all $n \in J$ and $\omega_n = \rho_n$, for all $n \in J^c$ to get the result. $\square$

We leave the further study of Lipschitz atomic decomposition to future work and end the paper with an application of Theorem 2.7.

THEOREM 3.13. *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz atomic decomposition with lower Lipschitz atomic bound* $a$ *and upper Lipschitz atomic bound* $b$ *for* $\mathscr{X}$ *w.r.t.* $\mathscr{M}_d$. *Let* $\{\omega_n\}_n$ *be a collection in* $\mathscr{X}$ *and suppose that there exist* $\lambda_1, \lambda_2, \mu \geqslant 0$ *such that the following conditions hold.*

(i) *For each* $x \in \mathscr{X}$, *the series* $\sum_{n=1}^{\infty} f_n(x)\omega_n$ *converges in* $\mathscr{X}$.

(ii) $\max\{\lambda_2, \lambda_1 + \mu b\} < 1$ *and*

(iii)

$$\left\|\sum_{n=1}^{\infty}(c_n-d_n)(\tau_n-\omega_n)\right\| \leqslant \lambda_1\left\|\sum_{n=1}^{\infty}(c_n-d_n)\tau_n\right\| + \lambda_2\left\|\sum_{n=1}^{\infty}(c_n-d_n)\omega_n\right\| + \mu\|\{c_n-d_n\}_n\|, \quad \forall \{c_n\}_n, \{d_n\}_n \in \mathscr{M}_d. \tag{10}$$

*Then there exists a collection* $\{g_n\}_n$ *in* $\mathrm{Lip}(\mathscr{X}, \mathbb{K})$ *such that* $(\{g_n\}_n, \{\omega_n\}_n)$ *is a Lipschitz atomic decomposition for* $\mathscr{X}$ *with lower and upper Lipschitz atomic bounds*

$$\frac{a(1-\lambda_2)}{1+\lambda_1+\mu b}, \quad \frac{b(1+\lambda_2)}{1-(\lambda_1+\mu b)}$$

*respectively.*

*Proof.* From the first condition we get that the map $T : \mathscr{X} \ni x \mapsto \sum_{n=1}^{\infty} f_n(x)\omega_n \in \mathscr{X}$ is well-defined. Now using Inequality (10),

$$\begin{aligned}
\|x-y-(Tx-Ty)\| &= \left\|\sum_{n=1}^{\infty}(f_n(x)-f_n(y))\tau_n - \sum_{n=1}^{\infty}(f_n(x)-f_n(y))\omega_n\right\| \\
&= \left\|\sum_{n=1}^{\infty}(f_n(x)-f_n(y))(\tau_n-\omega_n)\right\| \\
&\leqslant \lambda_1\left\|\sum_{n=1}^{\infty}(f_n(x)-f_n(y))\tau_n\right\| + \lambda_2\left\|\sum_{n=1}^{\infty}(f_n(x)-f_n(y))\omega_n\right\| \\
&\quad + \mu\|\{f_n(x)-f_n(y)\}_n\| \\
&= \lambda_1\|x-y\| + \lambda_2\|Tx-Ty\| + \mu\|\{f_n(x)-f_n(y)\}_n\| \\
&\leqslant (\lambda_1+\mu b)\|x-y\| + \lambda_2\|Tx-Ty\|, \quad \forall x, y \in \mathscr{X}.
\end{aligned}$$

Theorem 2.7 then says that $T$ is Lipschitz, invertible and

$$\frac{1-\lambda_2}{1+\lambda_1+\mu b} \leqslant \mathrm{Lip}(T)^{-1} \leqslant \frac{1+\lambda_2}{1-(\lambda_1+\mu b)}.$$

Define $g_n := f_n T^{-1}$, $\forall n \in \mathbb{N}$. Then $\{g_n(x)\}_n \in \mathscr{M}_d$, for each $x \in \mathscr{X}$ and

$$\begin{aligned}
\|\{g_n(x)-g_n(y)\}_n\| &= \|\{f_n(T^{-1}x)-f_n(T^{-1}y)\}_n\|_n \leqslant b\|T^{-1}x-T^{-1}y\| \\
&\leqslant b\frac{1+\lambda_2}{1-(\lambda_1+\mu b)}\|x-y\|, \quad \forall x, y \in \mathscr{X},
\end{aligned}$$

$$a\frac{1-\lambda_2}{1+\lambda_1+\mu b}\|x-y\| \leqslant a\|T^{-1}x-T^{-1}y\| \leqslant \|\{f_n(T^{-1}x)-f_n(T^{-1}y)\}_n\|$$
$$= \|\{g_n(x)-g_n(y)\}_n\|, \quad \forall x,y \in \mathscr{X}.$$

Finally

$$\sum_{n=1}^{\infty} g_n(x)\omega_n = \sum_{n=1}^{\infty} f_n(T^{-1}x)\omega_n = T(T^{-1}x) = x, \quad \forall x \in \mathscr{X}. \quad \square$$

We conclude the paper with the following remarks.

REMARK 3.14. So far in the literature, there are three ways to prove Theorem 1.2 one given in [9], another in [48] and yet another in [11]. As we mentioned earlier, we have done the non-linear version of arguments used in [48]. We hope that arguments used in [9] and [11] can be generalized to give different proofs of Theorem 2.7.

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