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A NEW LAPLACIAN COMPARISON THEOREM AND ITS APPLICATIONS ON FINSLER MANIFOLD

MEI GUAN AND SONGTING YIN*

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Abstract. For a Finsler manifold equipped with the weighted volume form, we established a new Laplacian comparison theorem for the distance function. As applications, we obtain Bishop-Gromov type volume comparison for the geodesic balls, and Cheng type comparison theorem for the first Dirichlet eigenvalue. Moreover, we also give a simple proof for Myers theorem.

1. Introduction

Similar to Riemannian case, comparison theorems play an important role in Finsler geometry, especially in its global and analytic aspects. There are many kinds of comparison theorems obtained in the Finsler setting up to now. Shen [9] first extended comparison theorems to Finsler geometry under Ricci and S-curvature conditions. Afterwards, Wu-Xin [11], via various curvature conditions, proved Hessian comparison theorem, Laplacian comparison theorems and volume comparison theorems. The results were generalized further by Wu [10] and Zhao-Shen [15], respectively. On the other hand, by using the weighted Ricci curvature Ric_N, Ohta-Sturm [6], [7] gave another version of Laplacian and volume comparison theorems. Along this line, the second author [12] obtain Laplacian comparison theorem and volume comparison theorem on a Finsler manifold with the weighted Ricci curvature Ric_∞ bounded below. Recently, under an upper bound on Δr controlled by a function $\chi(r)$ of the distance function r, Cheng-Shen [3] obtain a relative volume comparison of Bishop-Gromov type for the geodesic balls.

Let $(M,F,d\mu)$ be a Finsler n-manifold with an arbitrary volume form $d\mu = \sigma(x)dx$. Let p be a fixed point and $r(x) = d(p,x)$ the distance function from p. Assume that $d\tilde{\mu} = \tilde{\sigma}(x)dx$, where $\tilde{\sigma}(x) = \sigma(x)e^{\tau(x,\nabla r(x))}$, and τ is the distortion with respect to $d\mu$. Then $\tilde{\sigma}(x)$ is bounded around the point p, and $d\tilde{\mu}$ gives a weighted volume form on $M\backslash\{p\}$. Further more, for any integrable function f, we have

$$\int_{M\backslash\{p\}} f d\tilde{\mu} = \int_M f d\tilde{\mu}.$$

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* Corresponding author.

So we might as well regard $d\tilde{\mu}$ as the volume form defined on M. The Finsler manifold (M,F) equipped with the weighted volume form $d\tilde{\mu}$ has some interesting properties. In particular, the radial S-curvature vanishes along the geodesic from p. See Lemma 3.1 in Section 3 below.

In this paper, our main aim is to establish a new Laplacian comparison theorem on Finsler manifolds with weighted volume form $d\tilde{\mu}$. Precisely, we have

THEOREM 1.1. *Let $(M,F,d\mu)$ be a Finsler n-manifold with an arbitrary volume form $d\mu = \sigma(x)dx$. Let p be a fixed point and $r(x) = d(p,x)$ the distance function from p. Assume that the volume form $d\tilde{\mu} = \tilde{\sigma}(x)dx$, where $\tilde{\sigma}(x) = \sigma(x)e^{\tau(x,\nabla r(x))}$, and τ is the distortion with respect to $d\mu$. If the Ricci curvature* $\mathrm{Ric} \geqslant (n-1)k$, *then the Laplacian of $r(x)$ with respect to $d\tilde{\mu}$ can be estimated as follows:*

$$\tilde{\Delta} r \leqslant (n-1)\mathrm{ct}_k(r)$$

pointwise on $M \backslash (\{p\} \cup Cut(p))$ and in the sense of distributions on $M \backslash \{p\}$. Here

$$\mathrm{ct}_k(r) = \begin{cases} \sqrt{k} \cdot \cot(\sqrt{k}r), & k > 0, \\ \dfrac{1}{r}, & k = 0, \\ \sqrt{-k} \cdot \coth(\sqrt{-k}r), & k < 0. \end{cases}$$

The equality holds if and only if the radial flag curvature $K(x, \nabla r(x)) = k$.

The second aim of this paper is to give some applications of the Laplacian comparison theorem (Theorem 1.1). In particular, we obtain a volume comparison theorem as follows.

THEOREM 1.2. *Let $(M,F,d\mu)$ be a Finsler n-manifold with an arbitrary volume form $d\mu = \sigma(x)dx$. Let p be a fixed point and $r(x) = d(p,x)$ the distance function from p. Assume that the volume form $d\tilde{\mu} = \tilde{\sigma}(x)dx$, where $\tilde{\sigma}(x) = \sigma(x)e^{\tau(x,\nabla r(x))}$, and τ is the distortion with respect to $d\mu$. If the Ricci curvature* $\mathrm{Ric} \geqslant (n-1)k$, *then for any $0 < r < R$ ($R \leqslant \frac{\pi}{\sqrt{k}}$ when $k > 0$), it holds that*

$$\frac{\mathrm{vol}_F^{d\tilde{\mu}} B_p^+(R)}{\mathrm{vol}_F^{d\tilde{\mu}} B_p^+(r)} \leqslant \frac{\int_0^R \mathrm{s}_k(t)^{n-1} dt}{\int_0^r \mathrm{s}_k(t)^{n-1} dt},$$

where $B_p^+(r)$ denotes the forward geodesic ball centered at p of radius r, and $s_k(t)$ is defined by (3.1) below. The equality holds if and only if the radial flag curvature $K(x, \nabla r(x)) = k$.

The paper is organized as follows. In Section 2, some fundamental concepts and formulas which are necessary for the present paper are given. In Section 3 we will prove Theorems 1.1 and 1.2, respectively. Finally, we will give some other applications of the Laplacian comparison theorem (Theorem 1.1) in Section 4.

2. Preliminaries

Let M be an $n-$dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x,y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i\partial/\partial x^i$. A *Finsler metric* on M is a function $F : TM \to [0,+\infty)$ satisfying the following properties:

(i) *Regularity*: $F(x,y)$ is smooth in $TM \setminus 0$;

(ii) *Positive homogeneity*: $F(x,\lambda y) = \lambda F(x,y)$ for $\lambda > 0$;

(iii) *Strong convexity*: The fundamental quadratic form

$$g := g_{ij}(x,y)dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$$

is positive definite.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field. Then the *covariant derivative* of X by $v \in T_xM$ with reference vector $w \in T_xM\setminus 0$ is defined by

$$D^w_v X(x) := \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.

Given two linearly independent vectors $V, W \in T_xM\setminus 0$, the flag curvature is defined by

$$K(V,W) := \frac{g_V(R^V(V,W)W,V)}{g_V(V,V)g_V(W,W) - g_V(V,W)^2},$$

where R^V is the *Riemannian curvature*:

$$R^V(X,Y)Z = D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]}Z.$$

Then the Ricci curvature for (M,F) is defined as

$$\mathrm{Ric}(V) = \sum_{i=1}^{n-1} K(V,e_i),$$

where $e_1,\cdots,e_{n-1},\frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V.

For a given volume form $d\mu = \sigma(x)dx$ and a vector $V \in T_xM\setminus 0$, the *distortion* of $(M,F,d\mu)$ is defined by

$$\tau(x,V) := \ln \frac{\sqrt{\det(g_{ij}(x,V))}}{\sigma(x)}.$$

When (M,F) is a Riemannian manifold, the distortion $\tau = \tau(x)$ becomes a smooth function on M. To measure the rate of changes of the distortion along geodesics, we define the *S-curvature* as

$$S(x,V) := \frac{d}{dt}[\tau(\gamma(t),\dot{\gamma}(t))]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0)=x$, $\dot\gamma(0)=V$.

For a smooth function $u: M\to\mathbb{R}$, the *gradient vector* of u at x is defined as

$$\nabla u(x) := \begin{cases} g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^j}\frac{\partial}{\partial x^i}, & du(x)\neq 0,\\ 0, & du(x)=0.\end{cases}$$

Set $M_V := \{x\in M|V(x)\neq 0\}$ for a vector field V on M, and $M_u := M_{\nabla u}$. For a smooth vector field V on M and $x\in M_V$, we define $\nabla V(x)\in T_x^*M\otimes T_xM$ by using the covariant derivative as

$$\nabla V(v) := D_v^V V(x)\in T_xM, \qquad v\in T_xM.$$

For a smooth function $u: M\to R$ and $x\in M_u$, Set $\nabla^2 u(x) := \nabla(\nabla u)(x)$. Define the Hessian of u by [11]

$$H(u)(X,Y) = XYu - \nabla_X Yu, \quad X,Y\in T_xM.$$

Then we have

$$H(u)(X,Y) = g_{\nabla u}(D_X^{\nabla u}\nabla u, Y) = g_{\nabla u}(D_Y^{\nabla u}\nabla u, X) = H(u)(Y,X).$$

Let $V = V^i\frac{\partial}{\partial x^i}$ be a C^∞ vector field on M. The *divergence* of V with respect to an arbitrary volume form $d\mu$ is defined by

$$\mathrm{div}V := \sum_{i=1}^n\left(\frac{\partial V^i}{\partial x^i} + V^i\frac{\partial\Phi}{\partial x^i}\right),$$

where $d\mu = e^\Phi dx$. Then the *Finsler-Laplacian* of u can be defined by

$$\Delta u := \mathrm{div}(\nabla u).$$

Let $(M,F,d\mu)$ be a Finsler n-manifold. Define the canonical energy functional as ([8], p. 210)

$$\mathscr{E}(u) := \frac{\int_M F^*(du)^2 d\mu}{\int_M u^2 d\mu},$$

where F^* is the dual Finsler metric with respect to F. It follows that a function $u\in W_0^{1,2}(M)$ satisfies $d_u\mathscr{E}=0$ with $\lambda=\mathscr{E}(u)$ if and only if

$$\Delta u = -\lambda u.$$

In this case, λ and u called an eigenvalue and an eigenfunction of $(M,F,d\mu)$, respectively. It is shown in [4] that $u\in C^{1,\alpha}(M_u)\cap C^\infty(M)$.

3. The proofs of Theorem 1.1 and 1.2

LEMMA 3.1. *Let $(M,F,d\mu)$ be a Finsler n-manifold with an arbitrary volume form $d\mu=\sigma(x)dx$. Let p be a fixed point and $r(x)=d(p,x)$ the distance function from p. Assume that the volume form $d\tilde\mu=\tilde\sigma(x)dx$, where $\tilde\sigma(x)=\sigma(x)e^{\tau(x,\nabla r(x))}$, and τ is the distortion with respect to $d\mu$. Then the S-curvature vanishes along the geodesics from p.*

Proof. Computing the distortion under the volume form $d\tilde{\mu}$, we have

$$\begin{aligned}\tilde{\tau}(x,y) &= \log\frac{\sqrt{\det(g_{ij}(x,y))}}{\tilde{\sigma}(x)} = \log\frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)e^{\tau(x,\nabla r(x))}}\\ &= \log\frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)} - \tau(x,\nabla r(x))\\ &= \tau(x,y) - \tau(x,\nabla r(x)).\end{aligned}$$

Let γ be a geodesic such that $\gamma(0)=p$. Then, along the geodesic γ, we have

$$\tilde{\tau}(\gamma,\dot{\gamma}) = \tilde{\tau}(x,\nabla r(x)) = 0,$$

and thus

$$\tilde{S}(\gamma,\dot{\gamma}) = \tilde{S}(x,\nabla r(x)) = 0. \quad \square$$

Proof of Theorem 1.1. We follow the standard arguments. See [11] for reference. Suppose that r is smooth at $q \in M$. Let $\gamma : [0, r(q)] \to M$ be the normal geodesic from p to q. Choose the local $g_{\nabla r}$-orthonormal basis $\{e_1,\cdots,e_{n-1},e_n=\dot{\gamma}\}$ at T_qM. we get local vector fields $\{e_1(s),\cdots,e_{n-1}(s),e_n(s)=\dot{\gamma}\}$ by parallel transport along geodesic rays. For any $1 \leqslant i \leqslant n-1$, there is a unique Jacobi vector field J_i such that $J_i(0)=0, J_i(r(q))=e_i$. Set $W_i(s) = \frac{\mathrm{s}_k(s)}{\mathrm{s}_k(r(q))}e_i(s)$, where

$$\mathrm{s}_k(r) := \begin{cases} \dfrac{\sin(\sqrt{k}r)}{\sqrt{k}}, & k>0;\\ r, & k=0;\\ \dfrac{\sinh(\sqrt{-k}r)}{\sqrt{-k}}, & k<0.\end{cases} \tag{3.1}$$

Obviously, $W_i(0)=0$, $W_i(r(q)) = J_i(r(q))$. Using the basic index lemma, we have

$$\begin{aligned}\mathrm{tr}_{\nabla r}H(r)\big|_q &= \sum_{i=1}^{n-1} I_\gamma(J_i,J_i) \leqslant \sum_{i=1}^{n-1} I_\gamma(W_i,W_i)\\ &= \frac{1}{\mathrm{s}_k(r(q))^2}\int_0^{r(q)}\left[(n-1)\mathrm{s}_k'(s)^2 - \mathrm{Ric}(\dot{\gamma})\mathrm{s}_k(s)^2\right]ds\\ &= \frac{1}{\mathrm{s}_k(r(q))^2}\int_0^{r(q)}\left[(n-1)\mathrm{s}_k'(s)^2 - (n-1)k\mathrm{s}_k(s)^2\right]ds\\ &= (n-1)\mathrm{ct}_k(r(q)).\end{aligned} \tag{3.2}$$

Now from Lemma 3.1, we have

$$\tilde{\tau}(\gamma,\dot{\gamma}) = 0, \quad \tilde{S}(\gamma,\dot{\gamma}) = 0$$

along the geodesic γ. It follows that

$$\tilde{\Delta} r = \mathrm{tr}_{\nabla r}H(r) - \tilde{S}(\nabla r) = \mathrm{tr}_{\nabla r}H(r). \tag{3.3}$$

Then from (3.2) and (3.3), we have

$$\tilde{\Delta} r \leqslant (n-1)\mathrm{ct}_k(r).$$

By the well known Calabi skill, we can get the inequality in distributional sense on $M\backslash\{p\}$. This finishes the proof of the first part.

In what follows, we consider the second part. If the equality holds,

$$\tilde{\Delta} r = (n-1)\mathrm{ct}_k(r),$$

then from (3.3) we have

$$\mathrm{tr}_{\nabla r} H(r) = (n-1)\mathrm{ct}_k(r). \tag{3.4}$$

Direct differentiating on both sides with respect to r in (3.4), we obtain

$$\frac{\partial}{\partial r}\mathrm{tr}_{\nabla r} H(r) + \frac{(\mathrm{tr}_{\nabla r} H(r))^2}{n-1} = -(n-1)k. \tag{3.5}$$

Let $S_p(r(x))$ be the forward geodesic sphere of radius $r(x)$ centered at p. Choosing the local $g_{\nabla r}$-orthonormal frame $E_1,\cdots,E_{n-1}$ of $S_p(r(x))$ near x, we get local vector fields $E_1,\cdots,E_{n-1},E_n = \nabla r$ by parallel transport along geodesic rays. Thus, it follows from [11] that

$$\frac{\partial}{\partial r}\mathrm{tr}_{\nabla r} H(r) = -\mathrm{Ric}(\nabla r) - \sum_{i,j=1}^{n-1}[H(r)(E_i,E_j)]^2, \tag{3.6}$$

Therefore, from (3.5) and (3.6), we derive

$$\begin{aligned} -(n-1)k &= \frac{\partial}{\partial r}\mathrm{tr}_{\nabla r} H(r) + \frac{1}{n-1}(\mathrm{tr}_{\nabla r} H(r))^2 \\ &\leqslant \frac{\partial}{\partial r}\mathrm{tr}_{\nabla r} H(r) + \sum_{i,j=1}^{n-1}[H(r)(E_i,E_j)]^2 \\ &= -\mathrm{Ric}(\nabla r) \leqslant -(n-1)k. \end{aligned}$$

Thus, we obtain

$$\sum_{i,j=1}^{n-1}[H(r)(E_i,E_j)]^2 = \frac{1}{n-1}(\mathrm{tr}_{\nabla r} H(r))^2. \tag{3.7}$$

By Schwarz inequality, it follows from (3.4) and (3.7) that, for $1 \leqslant i,j \leqslant n-1$,

$$\Big(H(r)(E_i,E_j)\Big) = \begin{pmatrix} \mathrm{ct}_k(r) & 0 & \cdots & 0 \\ 0 & \mathrm{ct}_k(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{ct}_k(r) \end{pmatrix} \tag{3.8}$$

Now we calculate the flag curvature of (M,F). By (3.8) we observe that $\{E_i\}_{i=1}^{n-1}$ are $(n-1)$ eigenvectors of $\nabla^2 r$. That is,

$$D_{E_i}^{\nabla r}\nabla r = \mathrm{ct}_k(r)E_i, \qquad i=1,\cdots,n-1.$$

Since ∇r is a geodesic field on (M,F), the flag curvature $K(\nabla r;\cdot)$ is equal to the sectional curvature of the weighted Riemannian manifold $(M,g_{\nabla r})$. Note that $\{E_i\}_{i=1}^{n-1}$ are $(n-1)$ eigenvectors of $\nabla^2 r$ and parallel along the geodesic ray. By a straightforward computation, we get, for $1\leqslant i\leqslant n-1$,

$$\begin{aligned}
K(\nabla r;E_i) &= R^{\nabla r}(E_i,\nabla r,E_i,\nabla r) = g_{\nabla r}(R^{\nabla r}(E_i,\nabla r)\nabla r,E_i)\\
&= g_{\nabla r}(D_{E_i}^{\nabla r}D_{\nabla r}^{\nabla r}\nabla r - D_{\nabla r}^{\nabla r}D_{E_i}^{\nabla r}\nabla r - D_{[E_i,\nabla r]}^{\nabla r}\nabla r,E_i)\\
&= -g_{\nabla r}(D_{\nabla r}^{\nabla r}(\mathrm{ct}_k(r))E_i + D^{\nabla r}_{D_{E_i}^{\nabla r}\nabla r - D_{\nabla r}^{\nabla r}E_i}\nabla r,E_i)\\
&= -g_{\nabla r}(\mathrm{ct}_k'(r)E_i + D^{\nabla r}_{\mathrm{ct}_k(r)E_i}\nabla r,E_i)\\
&= -\mathrm{ct}_k'(r) - \mathrm{ct}_k(r)g_{\nabla r}(D_{E_i}^{\nabla r}\nabla r,E_i)\\
&= -\mathrm{ct}_k'(r) - \mathrm{ct}_k(r)^2\\
&= k.
\end{aligned}$$

Conversely, if the radial flag curvature $K(x,\nabla r(x)) = k$, then by Hessian comparison theorem [11], we have

$$\begin{cases} H(r)(E_i,E_j) = \mathrm{ct}_k(r)\delta_{ij}, & i,j\leqslant n-1,\\ H(r)(\nabla r,\nabla r) = 0, \end{cases}$$

which means that

$$\tilde{\Delta} r = \mathrm{tr}_{\nabla r}H(r) = (n-1)\mathrm{ct}_k(r). \quad \square$$

REMARK. Notice that under the weighted volume form the S-curvature only vanishes along the geodesic from p. Therefore, for an arbitrary smooth function $f(x)$, it does not hold that

$$\tilde{\Delta} f = \mathrm{tr}_{\nabla r}H(f).$$

Let $(M,F,d\mu)$ be a Finsler n-manifold. For a fixed point $p\in M$, define

$$\begin{aligned}
I_p &:= \{v\in T_pM \mid F(v)=1\}, & c(v) &:= \sup\{t>0 \mid d_F(p,\exp(tv)) = t\}\\
\mathbf{C}(p) &:= \{c(v)v \mid c(v)<\infty, v\in I_p\}, & C(p) &:= \exp\mathbf{C}(p), \quad i_p := \inf\{c(v)\mid v\in I_p\},\\
\mathbf{D}(p) &:= \{tv \mid 0\leqslant t<c(v), v\in I_p\}, & D(p) &:= \exp\mathbf{D}(p).
\end{aligned}$$

Then $D(p) = M\backslash C(p)$. Let $\{\theta^\alpha \mid \alpha = 1,\cdots n-1\}$ be the local coordinates that are intrinsic to I_p. For any $q\in D(p)$, the polar coordinates of q are defined by $(\rho,\theta) = (\rho(q),\theta^1(q),\cdots,\theta^{n-1}(q))$, where $\rho(q) = F(v)$, $\theta^\alpha(q) = \theta^\alpha(\frac{v}{F(v)})$ and $v = \exp_p^{-1}(q)$.

Let $\mathbf{D}_p(r) := \{v\in I_p \mid rv\in \mathbf{D}_p\}$. It is easy to see that $\mathbf{D}_p(r) = I_p$ for $r<i_p$. Set

$$\sigma_p(r) := \int_{\mathbf{D}_p(r)} \sigma(r,\theta)d\theta.$$

Denote by $B_p^+(r)$ the forward geodesic ball centered at p of radius r. Then for $r < i_p$,

$$\mathrm{vol}_F^{d\mu}(B_p^+(r)) = \int_0^r \sigma_p(r)dr.$$

Proof of Theorem 1.2. Since $\mathrm{Ric} \geqslant (n-1)k$, by Laplacian comparison theorem (Theorem 1.1), we have

$$\tilde{\Delta} r \leqslant (n-1)\mathrm{ct}_k(r) = (n-1)\frac{\mathbf{s}_k'(r)}{\mathbf{s}_k(r)}.$$

Let (r,θ) be the polar coordinate around p and write the volume form by $d\tilde{\mu} = \tilde{\sigma}(r,\theta)drd\theta$. Then

$$\tilde{\Delta} r = \frac{\partial}{\partial r}\log\tilde{\sigma}(r,\theta).$$

which yields

$$\frac{\partial}{\partial r}\log\tilde{\sigma} \leqslant \frac{\partial}{\partial r}\log\mathbf{s}_k(r)^{n-1}.$$

Set $\hat{\sigma} := \mathbf{s}_k(r)^{n-1}$ and define $f(r) = \frac{\tilde{\sigma}}{\hat{\sigma}}$. Then

$$f'(r) = \frac{\tilde{\sigma}'\hat{\sigma} - \tilde{\sigma}\hat{\sigma}'}{\hat{\sigma}^2} = \frac{\tilde{\sigma}}{\hat{\sigma}}\frac{\partial}{\partial r}(\log\tilde{\sigma} - \log\hat{\sigma}) \leqslant 0.$$

Hence, $f(r)$ is non-increasing monotonically on r. It follows that

$$\frac{\tilde{\sigma}(R,\theta)}{\hat{\sigma}(R)} \leqslant \frac{\tilde{\sigma}(r,\theta)}{\hat{\sigma}(r)}, \quad r \leqslant R.$$

Integrating it over $\mathbf{D}_p(r)$ with respect to $d\theta$, we get

$$\frac{\tilde{\sigma}_p(R)}{\hat{\sigma}(R)} \leqslant \frac{\tilde{\sigma}_p(r)}{\hat{\sigma}(r)}, \quad r \leqslant R.$$

By a standard argument, we then have

$$\frac{\int_0^R \tilde{\sigma}_p(t)dt}{\int_0^R \hat{\sigma}(t)dt} \leqslant \frac{\int_0^r \tilde{\sigma}_p(t)dt}{\int_0^r \hat{\sigma}(t)dt}, \quad r \leqslant R.$$

Namely,

$$\frac{\mathrm{vol}_F^{d\tilde{\mu}} B_p^+(R)}{\mathrm{vol}_F^{d\tilde{\mu}} B_p^+(r)} \leqslant \frac{\int_0^R \mathrm{s}_k(t)^{n-1}dt}{\int_0^r \mathrm{s}_k(t)^{n-1}dt}.$$

If the equality holds, then all inequalities become equalities. In particular, we have

$$\tilde{\Delta} r = (n-1)\mathbf{ct}_k(r).$$

Therefore, by Theorem 1.1, we conclude that the equality holds if and only if the radial flag curvature $K(x,\nabla r(x)) = k$. $\square$

4. Some other applications

Using a similar argument as the Riemannian case, the celebrated Myers theorem has been generalized to Finsler geometry (see [1], p. 194). By using the Laplacian comparison theorem (Theorem 1.1), we can give a simple proof of Myers theorem.

THEOREM 4.1. [1], [5] *Let (M,F) be a Finsler n-manifold. If the Ricci curvature* $\mathrm{Ric} \geqslant (n-1)k > 0$, *the M is compact and*

$$Diam(M) \leqslant \frac{\pi}{\sqrt{k}}.$$

Proof. Let $d\mu = \sigma(x)dx$ be an arbitrary volume form, and τ be the distortion with respect to $d\mu$. Define

$$d\tilde{\mu} = \tilde{\sigma}(x)dx, \quad \tilde{\sigma}(x) = \sigma(x)e^{\tau(x,\nabla r(x))}.$$

Let $r(x) = d(p,x)$ be the distance function from p. Then, by Theorem 1.1, the Laplacian of $r(x)$ with respect to $d\tilde{\mu}$ satisfies

$$\tilde{\Delta} r \leqslant (n-1)\sqrt{k}\cot(\sqrt{k}r). \tag{4.1}$$

Notice that if $x \in M \backslash \mathrm{Cut}(p) \cup \{p\}$, then $r(x)$ is smooth and $\tilde{\Delta} r$ is bounded. If $Diam(M) > \frac{\pi}{\sqrt{k}}$, then there exists some point x_0 such that r is smooth at x_0 and $r(x_0) = \frac{\pi}{\sqrt{k}}$. However, we find that the left side in (4.1) is bounded at x_0 while the right side turns to $-\infty$. It is a contradiction. This ends the proof. □

Recall that Cheng [2] obtained some eigenvalue comparison theorems in Riemannian geometry. The results are partially generalized to Finsler geometry[13], [14]. Next we will give an eigenvalue comparison theorem by using the Laplacian comparison (Theorem 1.1).

THEOREM 4.2. *Let $(M,F,d\mu)$ be a complete Finsler n-manifold with an arbitrary volume form $d\mu = \sigma(x)dx$. Let p be a fixed point and $r(x) = d(p,x)$ the distance function from p. Assume that the volume form $d\tilde{\mu} = \tilde{\sigma}(x)dx$, where $\tilde{\sigma}(x) = \sigma(x)e^{\tau(x,\nabla r(x))}$, and τ is the distortion with respect to $d\mu$. If the Ricci curvature satisfies* $\mathrm{Ric} \geqslant (n-1)k$, *then the first Dirichlet eigenvalue of Finsler Laplacian with respect to $(M,F,d\tilde{\mu})$ satisfies*

$$\tilde{\lambda}_1(B_p^+(r)) \leqslant \lambda_1(V_n(k,r)),$$

where $V_n(k,r)$ denotes a (forward or backward) geodesic ball with radius r in the n-dimensional simply connected Finsler spaces with flag curvature k and vanishing S curvature. The equality holds if and only if the radial flag curvature $K(x,\nabla r(x)) = k$.

Proof. If $V_n(k,r)$ is a geodesic ball in a Riemannian space form, we let φ be the negative first eigenfunction in $\overline{V_n(k,r)}$. Since all simply connected spaces forms are two-point homogenous, φ is a radial function. Namely, $\varphi(\bar{x}) = \varphi(\bar{d}(\bar{p},\bar{x}))$, where $\bar{p}$

is the center of $\overline{V_n(k,r)}$, and $\bar{d}(\bar{p},\bar{x})$ is the distance function from $\bar{p}$ to $\bar{x}$. Moreover, we have [2]

$$\begin{cases} \varphi''(t)+(n-1)\mathrm{ct}_k(t)\varphi'(t)+\lambda_1(V_n(k,r))\varphi(t)=0, \\ \varphi(r)=0, \qquad \varphi'(t)>0, \quad t\in(0,r). \end{cases} \tag{4.2}$$

We remark that, for both forward and backward geodesic balls with constant flag curvature and vanishing S curvature, the first Dirichlet eigenvalue are the same (see [13] for details).

Let $\rho(x)=d(p,x)$ be the forward distance function of (M,F), $u(x)=\varphi(\rho(x))$. Since $du=\varphi' d\rho$ and $\varphi'>0$, we find $\nabla u=\varphi'\nabla\rho$. Using Theorem 1.1 and noting that $F(\nabla\rho)=1$, it follows from (4.2) that

$$\begin{aligned} \tilde{\Delta}u &= \mathrm{div}(\nabla u) \\ &= \mathrm{div}(\varphi'\nabla\rho) \\ &= \varphi''(\rho)+\varphi'(\rho)\tilde{\Delta}\rho \\ &\leqslant \varphi''(\rho)+(n-1)\mathrm{ct}_k(\rho)\varphi'(\rho) \\ &= -\lambda_1(V_n(k,r))u. \end{aligned}$$

Note that $u|_{B_p^+(r)}<0$ and $u|_{\partial B_p^+(r)}=0$. It follows that

$$\begin{aligned} \int_{B_p^+(r)}(F^*(du))^2 d\tilde{\mu} &= \int_{B_p^+(r)} du(\nabla u)d\tilde{\mu} \\ &= -\int_{B_p^+(r)} u\tilde{\Delta}u d\tilde{\mu} \\ &\leqslant \lambda_1(V_n(k,r))\int_{B_p^+(r)} u^2 d\tilde{\mu}. \end{aligned} \tag{4.3}$$

Thus, from (4.3) we have

$$\begin{aligned} \tilde{\lambda}_1(B_p^+(r)) &= \inf_{f\in C_0^\infty(B_p^+(r))\setminus\{0\}} \frac{\int_{B_p^+(r)} F^*(df)^2 d\tilde{\mu}}{\int_{B_p^+(r)} f^2 d\tilde{\mu}} \\ &\leqslant \frac{\int_{B_p^+(r)} F^*(du)^2 d\tilde{\mu}}{\int_{B_p^+(r)} u^2 d\tilde{\mu}} \\ &\leqslant \lambda_1(V_n(k,r)). \end{aligned}$$

If the equality holds, then we have

$$\tilde{\Delta}\rho=(n-1)\mathbf{ct}_k(\rho).$$

From Theorem 1.1, this means that the radial flag curvature $K(x,\nabla r(x))=k$. □

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Mei Guan
School of Artificial Intelligence and Big Data
Hefei University
Hefei 230601, Anhui, China
e-mail: gm29@hfuu.edu.can

Songting Yin
Department of Mathematics and Computer Science
Tongling University
Tongling 244000, Anhui, China
e-mail: yst419@163.com