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ON A MORE ACCURATE REVERSE HARDY–HILBERT'S INEQUALITY WITH TWO PARTIAL SUMS

AIZHEN WANG AND BICHENG YANG*

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Abstract. By means of the weight coefficients, Hermite-Hadamard's inequality, Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate reverse Hardy-Hilbert's inequality with two partial sums is given. The equivalent statements of the best possible constant factor related to several parameters are provided, and some particular inequalities are deduced.

1. Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geqslant 0$, $0<\sum_{m=1}^{\infty}a_m^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$, we have the following well known Hardy-Hilbert$'$s inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [4], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

The more accurate extension of (1) was provided as follows (cf. [4], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

In 2006, by introducing parameters $\lambda_i\in(0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda\in(0,4]$, an extension of (1) was provided by Krnic et al. [16] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{3}$$

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* Corresponding author.

where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B(u,v) = \int_0^\infty \frac{t^{v-1}}{(1+t)^{u+v}}dt \quad (u,v>0)$$

is the beta function. For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (3) reduces to (1); for $p=q=2$, $\lambda_1=\lambda_2=\frac{1}{2}$, (3) reduces to Yang's inequality in [30].

Recently, applying inequality (3), Adiyasuren et. al. [1] gave a Hardy-Hilbert's inequality involving two partial sums as follows: For $\lambda_i \in (0,1]\cap(0,\lambda)$ $(\lambda \in (0.2];$ $i=1,2)$, $\lambda_1+\lambda_2=\lambda$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q\right]^{\frac{1}{q}}, \tag{4}$$

where, the constant factor $\lambda_1\lambda_2 B(\lambda_1,\ \lambda_2)$ is the best possible, and the partial sums $A_m = \sum_{i=1}^m a_i$ and $B_n = \sum_{k=1}^n b_k$, satisfying

$$0 < \sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q < \infty.$$

Inequality (1) with the integral analogues played an important role in analysis and its applications (cf. [2,3,5,12,17,19,26,27,29,31,37]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [4], Theorem 351): If $K(t)$ $(t>0)$ is a decreasing function, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s)=\int_0^\infty K(t)t^{s-1}dt<\infty$, then

$$\int_0^\infty x^{s-1}\left(\sum_{n=1}^{\infty}K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty}a_n^p. \tag{5}$$

Some new extensions of (5) were provided by [20–22, 32, 33].

In 2016, by means of the techniques of real analysis, Hong et al. [11] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. The further results were provided by [7–10, 13–15, 23–25, 28, 34–36]. In 2021, He et. al [6] gave a mare accurate Hardy-Hilbert inequality involving two partial sums.

In this paper, following the way of [6, 11], by means of the weight coefficients and the idea of introduced parameters, applying Hermite-Hadamard's inequality, Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate reverse Hardy-Hilbert's inequality with two partial sums is given. The equivalent conditions of the best possible constant factor related to several parameters are provided, and some particular inequalities are deduced.

2. Some lemmas

In what follows, we suppose that $0<p<1$ $(q<0)$, $\frac{1}{p}+\frac{1}{q}=1$, $\eta_i \in [0,\frac{1}{4}]$ $(i=1,2)$, $\eta_1+\eta_2=\eta \in [0,\frac{1}{2}]$, $\lambda \in (0,3]$, $\lambda_i \in (0,\frac{3}{2}]\cap(0,\lambda)$ $(i=1,2)$, $\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. We also assume that $a_m, b_n \geqslant 0$ $(m,n \in \mathbf{N}=\{1,2,\cdots\})$, $A_m = \sum_{i=1}^m a_i$, $B_n=\sum_{k=1}^n b_k$, satisfying $A_m = o(e^{t(m-\eta_1)})$, $B_n = o(e^{t(n-\eta_2)})$ $(t>0;\, m,n\to\infty)$ and

$$0<\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\widehat{\lambda}_1)-1}a_m^p<\infty, \quad 0<\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\widehat{\lambda}_2)-1}b_n^q<\infty.$$

LEMMA 1. (cf. [31], (2.2.3)) *(i) If* $(-1)^i \frac{d^i}{dt^i}g(t)>0$, $t\in[m,\infty)$ $(m\in\mathbf{N})$ *with* $g^{(i)}(\infty)=0$ $(i=0,1,2,3)$, P_i, B_i *are the Bernoulli functions and the Bernoulli numbers of i-order, then*

$$\int_m^{\infty} P_{2q-1}(t)g(t)dt = -\varepsilon_q \frac{B_{2q}}{2q}g(m) \quad (\varepsilon_q\in(0,1),\ q=1,2,\cdots) \tag{6}$$

In particular, for $q=1$, *in view of* $B_2=\frac{1}{6}$, *we have*

$$-\frac{1}{12}g(m)<\int_m^{\infty}P_1(t)g(t)dt<0; \tag{7}$$

for $q=2$, *in view of* $B_4=-\frac{1}{30}$, *we have*

$$0<\int_m^{\infty}P_1(t)g(t)dt<\frac{1}{120}g(m). \tag{8}$$

(ii) (cf. [31], (2.3.2)) *If* $f(t)(>0)\in C[m,\infty)$, $f^{(i)}(\infty)=0$ $(i=0,1,2,3)$, *then we have the following Euler-Maclaurin summation formula:*

$$\sum_{k=m}^{\infty}f(k) = \int_m^{\infty}f(t)dt+\frac{1}{2}f(m)+\int_m^{\infty}P_1(t)f'(t)dt, \tag{9}$$

$$\int_m^{\infty}P_1(t)f'(t)dt = -\frac{1}{12}f'(m)+\frac{1}{6}\int_m^{\infty}P_1(t)f''(t)dt. \tag{10}$$

LEMMA 2. *For* $s\in(0,3]$, $s_2\in(0,\frac{3}{2}]\cap(0,s)$, $k_s(s_i)=B(s_i,s-s_i)$ $(i=1,2)$, *define the following weight coefficient:*

$$\varpi_s(s_2,m) := (m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty}\frac{(n-\eta_2)^{s_2-1}}{(m+n-\eta)^s} \quad (m\in\mathbf{N}). \tag{11}$$

We have the following inequalities:

$$0<k_s(s_2)\left[1-O_1\left(\frac{1}{(m-\eta_1)^{s_2}}\right)\right]<\varpi_s(s_2,m)<k_s(s_2) \quad (m\in\mathbf{N}), \tag{12}$$

where, we indicate

$$O_1\left(\frac{1}{(m-\eta_1)^{s_2}}\right) := \frac{1}{k_s(s_2)}\int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s}du > 0.$$

Proof. For fixed $m \in \mathbf{N}$, we set the following real function: $g(m,t) := \frac{(t-\eta_2)^{s_2-1}}{(m-\eta+t)^s}$ $(t > \eta_2)$. In the following we divide two cases of $s_2 \in (0,1] \cap (0,s)$ and $s_2 \in (1,\frac{3}{2}] \cap (0,s)$ to prove (12).

(i) For $s_2 \in (0,1] \cap (0,s)$, since $(-1)^i g^{(i)}(m,t) > 0$ $(t > 0;\ i = 0,1,2)$, by Hermite-Hadamard's inequality (cf. [6]), setting $u = \frac{t-\eta_2}{m-\eta_1}$, we have

$$\begin{aligned}
\varpi_s(s_2,m) &= (m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty} g(m,n) < (m-\eta_1)^{s-s_2}\int_{\frac{1}{2}}^{\infty} g(m,t)dt \\
&= (m-\eta_1)^{s-s_2}\int_{\frac{1}{2}}^{\infty} \frac{(t-\eta_2)^{s_2-1}}{[(m-\eta_1)+(t-\eta_2)]^s}dt \\
&= \int_{\frac{\frac{1}{2}-\eta_2}{m-\eta_1}}^{\infty} \frac{u^{s_2-1}}{(1+u)^s}du \leqslant \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = k_s(s_2).
\end{aligned}$$

On the other hand, in view of the decreasingness property of series, setting $u = \frac{t-\eta_2}{m-\eta_1}$, we obtain

$$\begin{aligned}
\varpi_s(s_2,m) &= (m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty} g(m,n) > (m-\eta_1)^{s-s_2}\int_1^{\infty} g(m,t)dt \\
&= \int_{\frac{1-\eta_2}{m-\eta_1}}^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = k_s(s_2) - \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}du}{(1+u)^s} \\
&= k_s(s_2)\left[1 - O_1\left(\frac{1}{(m-\eta_1)^{s_2}}\right)\right] > 0,
\end{aligned}$$

where, $O_1\left(\frac{1}{(m-\eta_1)^{s_2}}\right) = \frac{1}{k_s(s_2)}\int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s}du > 0$, satisfying

$$0 < \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}du}{(1+u)^s} < \int_0^{\frac{1-\eta_2}{m-\eta_1}} u^{s_2-1}du = \frac{1}{s_2}\left(\frac{1-\eta_2}{m-\eta_1}\right)^{s_2} \quad (m \in \mathbf{N}).$$

Hence, we obtain (12).

(ii) For $s_2 \in (1,\frac{3}{2}] \cap (0,s)$, by (9), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt \\
&= \int_{\eta_2}^{\infty} g(m,t)dt - h(m),
\end{aligned}$$

where, $h(m)$ is indicated as

$$h(m) := \int_{\eta_2}^{1} g(m,t)dt - \frac{1}{2}g(m,1) - \int_{1}^{\infty} P_1(t)g'(m,t)dt.$$

We obtain $-\frac{1}{2}g(m.t) = \frac{-(1-\eta_2)^{s_2-1}}{2(m-\eta+1)^s}$, and by integrating by parts, it follows that

$$\begin{aligned}
\int_{\eta_2}^{1} g(m,t)dt &= \int_{\eta_2}^{1} \frac{(t-\eta_2)^{s_2-1}dt}{(m-\eta+t)^s} \\
&= \frac{1}{s_2}\int_{\eta_2}^{1} \frac{d(t-\eta_2)^{s_2}}{(m-\eta+t)^s} \\
&= \frac{1}{s_2}\frac{(t-\eta_2)^{s_2}}{(m-\eta+t)^s}\Big|_{\eta_2}^{1} + \frac{s}{s_2}\int_{\eta_2}^{1} \frac{(t-\eta_2)^{s_2}dt}{(m-\eta+t)^{s+1}} \\
&> \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)}\frac{(t-\eta_2)^{s_2+1}}{(m-\eta+t)^{s+1}}\Big|_{\eta_2}^{1} \\
&\quad + \frac{s(s+1)}{s_2(s_2+1)(m-\eta+1)^{s+2}}\int_{\eta_2}^{1} (t-\eta_2)^{s_2+1}dt \\
&= \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)}\frac{(1-\eta_2)^{s_2+1}}{(m-\eta+1)^{s+1}} \\
&\quad + \frac{s(s+1)(1-\eta_2)^{s_2+2}}{s_2(s_2+1)(s_2+2)(m-\eta+1)^{s+2}}.
\end{aligned}$$

We find

$$\begin{aligned}
-g'(m,t) &= -\frac{(s_2-1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-1}}{(m-\eta+t)^{s+1}} \\
&= -\frac{(s_2-1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-2}[(m-\eta+t)-(m-\eta_1)]}{(m-\eta+t)^{s+1}} \\
&= -\frac{(s_2-1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} - \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}} \\
&= \frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} - \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}},
\end{aligned}$$

and for $s_2 \in (1,\frac{3}{2}] \cap (0,s)$, it follows that

$$\begin{aligned}
(-1)^i \frac{d^i}{dt^i}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}\right] &> 0, \\
(-1)^i \frac{d^i}{dt^i}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\right] &> 0 \quad (t > \eta_2;\ i = 0,1,2,3).
\end{aligned}$$

By (8), (9) and (10), for $a := 1-\eta_2 (\in [\frac{3}{4},1])$, we obtain

$$\begin{aligned}
&(s+1-s_2)\int_1^{\infty} P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}dt \\
&> -\frac{(s+1-s_2)a^{s_2-2}}{12(m-\eta+1)^s} - (m-\eta_1)s\int_1^{\infty} P_1(t)dt \\
&> \frac{(m-\eta_1)sa^{s_2-2}}{12(m-\eta+1)^{s+1}} - \frac{(m-\eta_1)s}{720}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\right]''_{t=1} \\
&> \frac{sa^{s_2-2}}{12(m-\eta+1)^s} - \frac{sa^{s_2-1}}{12(m-\eta+1)^{s+1}} \\
&\quad -\frac{s}{720}\left[\frac{(s+1)(s+2)a^{s_2-2}}{(m-\eta+1)^{s+2}} + \frac{2(s+1)(2-s_2)a^{s_2-3}}{(m-\eta+1)^{s+1}} + \frac{(2-s_2)(3-s_2)a^{s_2-4}}{(m-\eta+1)^s}\right],
\end{aligned}$$

and then we have

$$h(m) > \frac{a^{s_2-4}h_1}{(m-\eta+1)^s} + \frac{sa^{s_2-3}h_2}{(m-\eta+1)^{s+1}} + \frac{s(s+1)a^{s_2-2}h_3}{(m-\eta+1)^{s+2}},$$

where, $h_i (i=1,2,3)$ are indicated as

$$\begin{aligned}
h_1 &:= \frac{a^4}{s_2} - \frac{a^3}{2} - \frac{(1-s_2)a^2}{12} - \frac{s(2-s_2)(3-s_2)}{720}, \\
h_2 &:= \frac{a^4}{s_2(s_2+1)} - \frac{a^2}{12} - \frac{(s+1)(2-s_2)}{360}, \quad \text{and} \\
h_3 &:= \frac{a^4}{s_2(s_2+1)(s_2+2)} - \frac{s+2}{720}.
\end{aligned}$$

For $s \in (0,3]$, $s_2 \in [1,\frac{3}{2}] \cap (0,s)$, $a \in [\frac{3}{4},1]$, we find

$$h_1 > \frac{a^2}{12s_2}[s_2^2 - (6a+1)s_2 + 12a^2] - \frac{1}{120}.$$

In view of

$$\begin{aligned}
&\frac{\partial}{\partial a}[s_2^2 - (6a+1)s_2 + 12a^2] \\
&= 6(4a-s_2) \geqslant 6\left(4\cdot\frac{3}{4} - \frac{3}{2}\right) > 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial s_2}[s_2^2 - (6a+1)s_2 + 12a^2] = 2s_2 - (6a+1) \\
&\leqslant 2\cdot\frac{3}{2} - \left(6\cdot\frac{3}{4}+1\right) = 3 - \frac{11}{2} < 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
h_1 &> \frac{(3/4)^2}{12(3/2)}\left[\left(\frac{3}{2}\right)^2 - \left(6\cdot\frac{3}{4}+1\right)\frac{3}{2} + 12\left(\frac{3}{2}\right)^2\right] - \frac{1}{120} \\
&= \frac{3}{128} - \frac{1}{120} > 0, \\
h_2 &> a^2\left(\frac{4a^2}{15} - \frac{1}{12}\right) - \frac{1}{90} \\
&\geqslant \left(\frac{3}{4}\right)^2\left[\frac{4(3/4)^2}{15} - \frac{1}{12}\right] - \frac{1}{90} = \frac{3}{80} - \frac{1}{90} > 0, \\
h_3 &\geqslant \frac{8a^4}{105} - \frac{5}{720} \geqslant \frac{8(3/4)^4}{105} - \frac{1}{144} > 0,
\end{aligned}$$

and then $h(m) > 0$.

On the other hand, we also have

$$\begin{aligned}
\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt \\
&= \int_1^{\infty} g(m,t)dt + H(m),
\end{aligned}$$

where, $H(m)$ is indicated as

$$H(m) = \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt.$$

We have obtained that $\frac{1}{2}g(m,t) = \frac{a^{s_2-1}}{2(m-\eta+1)^s}$ and

$$g'(m,t) = \frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}.$$

For $s_2 \in [1, \frac{3}{2}] \cap (0,s)$, $0 < s \leqslant 3$, by (7), we obtain

$$\begin{aligned}
&-(s+1-s_2)\int_1^{\infty} P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}dt > 0, \\
&s(m-\eta_1)\int_1^{\infty} P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}dt \\
&> \frac{-s(m-\eta_1)a^{s_2-2}}{12(m-\eta+1)^{s+1}} = \frac{-(m-\eta+1)s+as}{12(m-\eta+1)^{s+1}}a^{s_2-2} \\
&= \frac{-sa^{s_2-2}}{12(m-\eta+1)^s} + \frac{sa^{s_2-1}}{12(m-\eta+1)^{s+1}} > \frac{-sa^{s_2-2}}{12(m-\eta+1)^s}.
\end{aligned}$$

Hence, we have

$$\begin{aligned} H(m) &> \frac{(1-\eta_2)^{s_2-1}}{2(m-\eta+1)^s} - \frac{sa^{s_2-2}}{12(m-\eta+1)^s} \\ &= \frac{(\frac{a}{2}-\frac{s}{12})a^{s_2-2}}{(m-\eta+1)^s} \\ &\geqslant \frac{(\frac{3/4}{2}-\frac{s}{12})a^{s_2-2}}{(m-\eta+1)^s} = \frac{(\frac{3}{8}-\frac{3}{12})a^{s_2-2}}{(m-\eta+1)^s} > 0. \end{aligned}$$

Therefore, we obtain the following inequalities:

$$\int_1^\infty g(m,t)dt < \sum_{n=1}^\infty g(m,n) < \int_{\eta_2}^\infty g(m,t)dt.$$

In view of the the results in the case (i), we still can obtain (12).
The lemma is proved. □

LEMMA 3. *We have the following more accurate reverse Hardy-Hilbert's inequality:*

$$\begin{aligned} I_\lambda &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n-\eta)^\lambda} \\ &> (k_\lambda(\lambda_2))^{\frac{1}{p}} (k_\lambda(\lambda_1))^{\frac{1}{q}} \\ &\quad \times \left[\sum_{m=1}^\infty \left[1 - O_1\left(\frac{1}{(m-\eta_1)^{\lambda_2}} \right) \right] (m-\eta_1)^{p(1-\widehat{\lambda}_1)-1} a_m^p \right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{n=1}^\infty (n-\eta_2)^{q(1-\widehat{\lambda}_2)-1} b_n^q \right]^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Proof. By the symmetry, for $s \in (0,3]$, $s_1 \in (0,\frac{3}{2}] \cap (0,s)$, we obtain the following inequalities for the next weight coefficient:

$$\begin{aligned} 0 &< k_s(s_1)\left[1 - O_2\left(\frac{1}{(n-\eta_2)^{s_1}} \right) \right] < \omega_s(s_1,n) \\ &:= (n-\eta_2)^{s-s_1} \sum_{m=1}^\infty \frac{(m-\eta_1)^{s_1-1}}{(m+n-\eta)^s} < k_s(s_1) \quad (n \in \mathbf{N}), \end{aligned} \tag{14}$$

where,

$$O_2\left(\frac{1}{(n-\eta_2)^{s_1}} \right) := \frac{1}{k_s(s_1)} \int_0^{\frac{1-\eta_1}{n-\eta_2}} \frac{u^{s_1-1}}{(1+u)^s} du > 0.$$

By the reverse Hölder's inequality (cf. [18]), we obtain

$$
\begin{aligned}
I_\lambda &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-\eta)^\lambda}\left[\frac{(m-\eta_1)^{(1-\lambda_1)/q}a_m}{(n-\eta_2)^{(1-\lambda_2)/p}}\right]\left[\frac{(n-\eta_2)^{(1-\lambda_2)/p}b_n}{(m-\eta_1)^{(1-\lambda_1)/q}}\right] \\
&\geqslant \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{(m+n-\eta)^\lambda}\frac{(m-\eta_1)^{(1-\lambda_1)(p-1)}}{(n-\eta_2)^{1-\lambda_2}}\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^q}{(m+n-\eta)^\lambda}\frac{(n-\eta_2)^{(1-\lambda_2)(q-1)}}{(m-\eta_1)^{1-\lambda_1}}\right]^{\frac{1}{q}} \\
&= \left[\sum_{m=1}^{\infty}\varpi_\lambda(\lambda_2,m)(m-\eta_1)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty}\omega_\lambda(\lambda_1,n)(n-\eta_2)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

Then by (12) and (14) (for $s=\lambda$, $s_i=\lambda_i\ \ (i=1,2)$), in view of $0<p<1$, $q<0$, we obtain (13).

The lemma is proved. □

LEMMA 4. *For $t>0$, we have the following inequalities:*

$$\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}A_m \geqslant t^{-1}\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}a_m, \tag{15}$$

$$\sum_{n=1}^{\infty}e^{-t(n-\eta_2)}B_n \geqslant t^{-1}\sum_{n=1}^{\infty}e^{-t(n-\eta_2)}b_n. \tag{16}$$

Proof. In view of $e^{-t(m-\eta_1)}A_m=o(1)\ \ (m\to\infty)$, by Abel's summation by parts formula, we find

$$
\begin{aligned}
&\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}a_m \\
&= \lim_{m\to\infty}e^{-t(m-\eta_1)}A_m+\sum_{m=1}^{\infty}[e^{-t(m-\eta_1)}-e^{-t(m-\eta_1+1)}]A_m \\
&= \sum_{m=1}^{\infty}[e^{-t(m-\eta_1)}-e^{-t(m-\eta_1+1)}]A_m=(1-e^{-1})\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}A_m.
\end{aligned}
$$

Since $1-e^{-1}<t\ \ (t>0)$, we have the following inequality:

$$\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}a_m \leqslant t\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}A_m,$$

namely, (15) follows. In the same way, we have (16).

The lemma is proved. □

3. Main results and a few particular inequalities

THEOREM 1. *We have the following more accurate reverse Hardy-Hilbert's inequality with two partial sums:*

$$\begin{aligned} I &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_mB_n}{(m+n-\eta)^{\lambda}} \\ &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}(k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}} \\ &\quad \times\left[\sum_{m=1}^{\infty}\left[1-O_1\left(\frac{1}{(m-\eta_1)^{\lambda_2}}\right)\right](m-\eta_1)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} \\ &\quad \times\left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{17}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, *we have* $k_{\lambda}(\lambda_1)=B(\lambda_1,\lambda_2)$,

$$0<\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\lambda_1)-1}a_m^p<\infty, \quad 0<\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\lambda_2)-1}b_n^q<\infty.$$

and the following inequality:

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_mB_n}{(m+n-\eta)^{\lambda}} \\ &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\left[1-O_1\left(\frac{1}{(m-\eta_1)^{\lambda_2}}\right)\right](m-\eta_1)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}} \\ &\quad \times\left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{18}$$

Proof. In view of the expression that

$$\frac{1}{(m+n-\eta)^{\lambda+2}}=\frac{1}{\Gamma(\lambda+2)}\int_0^{\infty}t^{(\lambda+2)-1}e^{-(m+n-\eta)t}dt,$$

by (15) and (16), it follows that

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+2)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_mB_n\int_0^{\infty}t^{\lambda+1}e^{-(m+n-\eta)t}dt \\ &= \frac{1}{\Gamma(\lambda+2)}\int_0^{\infty}t^{\lambda+1}\left[\sum_{m=1}^{\infty}e^{-(m-\eta_1)t}A_m\right]\left[\sum_{n=1}^{\infty}e^{-(n-\eta_2)t}B_n\right]dt \\ &\geqslant \frac{1}{\Gamma(\lambda+2)}\int_0^{\infty}t^{\lambda+1}\left[t^{-1}\sum_{m=1}^{\infty}e^{-(m-\eta_1)t}a_m\right]\left[t^{-1}\sum_{n=1}^{\infty}e^{-(n-\eta_2)t}b_n\right]dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\lambda+2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \int_0^{\infty} t^{\lambda-1} e^{-(m+n-\eta)t} dt \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}}.
\end{aligned}
$$

Then by (13), we have (17).

The theorem is proved. □

THEOREM 2. *For $\lambda \in (0,1]$, $\lambda_1 \in (0,\frac{1}{2}] \cap (0,\lambda)$, $\lambda_2 \in (0,\lambda)$, if $\lambda_1+\lambda_2=\lambda$, then the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (17) is the best possible.*

Proof. We now prove that the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}B(\lambda_1, \lambda_2)(=\frac{B(\lambda_1,\lambda_2)}{\lambda(\lambda+1)})$ in (18) is the best possible by the condition.

For any $0<\varepsilon<\lambda_1 \min\{p,|q|\}$, we set

$$
\widetilde{a}_m := m^{\lambda_1-\frac{\varepsilon}{p}-1}, \ \widetilde{b}_n := m^{\lambda_2-\frac{\varepsilon}{q}-1} \ \ (m,n \in \mathbf{N}).
$$

Since $0<\lambda_1-\frac{\varepsilon}{p}<1$, by the decreasingness property of series, we have

$$
\widetilde{A}_m := \sum_{i=1}^{m} \widetilde{a}_i = \sum_{i=1}^{m} i^{\lambda_1-\frac{\varepsilon}{p}-1} < \int_0^m t^{\lambda_1-\frac{\varepsilon}{p}-1} dt = \frac{m^{\lambda_1-\frac{\varepsilon}{p}}}{\lambda_1-\frac{\varepsilon}{p}}.
$$

In the same way, for $0<\lambda_2-\frac{\varepsilon}{q}<\lambda \leqslant 1$, we obtain

$$
\widetilde{B}_n := \sum_{k=1}^{m} \widetilde{b}_k < \frac{n^{\lambda_2-\frac{\varepsilon}{q}}}{\lambda_2-\frac{\varepsilon}{q}} \ \ (n \in \mathbf{N}).
$$

If there exists a positive constant $M \geqslant \frac{B(\lambda_1,\lambda_2)}{\lambda(\lambda+1)}$, such that (18) is valid when we replace $\frac{B(\lambda_1,\lambda_2)}{\lambda(\lambda+1)}$ by M, then in particular, for $\eta_1=\eta_2=\eta=0$, substitution of $a_m=\widetilde{a}_m$, $b_n=\widetilde{b}_n$, $A_m=\widetilde{A}_m$ and $B_n=\widetilde{B}_n$ in (18), we have

$$
\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\widetilde{A}_m \widetilde{B}_n}{(m+n)^{\lambda}} \\
&> M \left[\sum_{m=1}^{\infty} \left[1-O_1\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p(1-\lambda_1)-1} \widetilde{a}_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} \widetilde{b}_n^q \right]^{\frac{1}{q}}. \qquad (19)
\end{aligned}
$$

In the following, we obtain that $M \leqslant \frac{B(\lambda_1,\lambda_2)}{\lambda(\lambda+1)}$, which follows that $M=\frac{B(\lambda_1,\lambda_2)}{\lambda(\lambda+1)}$ is the best possible constant factor in (18).

By (19) and the decreasingness property of series, we obtain

$$\begin{aligned}
\widetilde{I} &> M\left[\sum_{m=1}^{\infty}\left[1-O_1\left(\frac{1}{m^{\lambda_2}}\right)\right]m^{p(1-\lambda_1)-1}m^{p\lambda_1-\varepsilon-p}\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}n^{q\lambda_2-\varepsilon-q}\right]^{\frac{1}{q}} \\
&= M\left[\sum_{m=1}^{\infty}m^{-\varepsilon-1}-\sum_{m=1}^{\infty}m^{-\varepsilon-1}O_1\left(\frac{1}{m^{\lambda_2}}\right)\right]^{\frac{1}{p}}\left(1+\sum_{n=2}^{\infty}n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\
&> M\left(\int_1^{\infty}x^{-\varepsilon-1}dx-O(1)\right)^{\frac{1}{p}}\left(1+\int_1^{\infty}y^{-\varepsilon-1}dy\right)^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon}(1-\varepsilon O(1))^{\frac{1}{p}}(\varepsilon+1)^{\frac{1}{q}}.
\end{aligned}$$

By (14), for $\eta_1=\eta_2=\eta=0,\ s=\lambda+2,\ s_1=\lambda_1+1-\frac{\varepsilon}{p}(\in(0,\frac{3}{2}]\ \cap(0,\lambda+2))$, we have

$$\begin{aligned}
\widetilde{I} &< \frac{1}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\sum_{n=1}^{\infty}\left[n^{\lambda_2+1+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{m^{(\lambda_1+1-\frac{\varepsilon}{p})-1}}{(m+n)^{\lambda+2}}\right]n^{-\varepsilon-1} \\
&= \frac{1}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\sum_{n=1}^{\infty}\omega_{\lambda+2}(\lambda_1+1-\frac{\varepsilon}{p},n)n^{-\varepsilon-1} \\
&< \frac{k_{\lambda+2}(\lambda_1+1-\frac{\varepsilon}{p})}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\left(1+\sum_{n=2}^{\infty}n^{-\varepsilon-1}\right) \\
&< \frac{k_{\lambda+2}(\lambda_1+1-\frac{\varepsilon}{p})}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\left(1+\int_1^{\infty}y^{-\varepsilon-1}dy\right) \\
&= \frac{k_{\lambda+2}(\lambda_1+1-\frac{\varepsilon}{p})}{\varepsilon(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}(\varepsilon+1).
\end{aligned}$$

In view of the above results, we have

$$\frac{k_{\lambda+2}(\lambda_1+1-\frac{\varepsilon}{p})}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}(\varepsilon+1) > \varepsilon\widetilde{I} > M(1-\varepsilon O(1))^{\frac{1}{p}}(\varepsilon+1)^{\frac{1}{q}}.$$

Setting $\varepsilon\to 0^+$, in view of the continuity of the beta function, we find

$$\frac{1}{\lambda(\lambda+1)}B(\lambda_1,\lambda_2)=\frac{1}{\lambda_1\lambda_2}B(\lambda_1+1,\lambda_2+1)\geqslant M.$$

Hence, $M=\frac{1}{\lambda(\lambda+1)}B(\lambda_1,\lambda_2)$ is the best possible constant factor in (18).

The theorem is proved. □

THEOREM 3. *If the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$$

in (17) *is the best possible, then for*

$$\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1)) \cap \left[q\left(\frac{3}{2}-\lambda_1\right), p\left(\frac{3}{2}-\lambda_1\right)\right],$$

we have $\lambda_1+\lambda_2=\lambda$.

Proof. For $\widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\frac{\lambda-\lambda_1-\lambda_2}{p}+\lambda_1$, $\widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda-\lambda_1-\lambda_2}{q}+\lambda_2$, we find $\widehat{\lambda}_1+\widehat{\lambda}_2=\lambda$. For $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, we have $\widehat{\lambda}_1 \in (0,\lambda)$ and then $\widehat{\lambda}_2=\lambda-\widehat{\lambda}_1 \in (0,\lambda)$; for

$$\lambda-\lambda_1-\lambda_2 \in \left[q\left(\frac{3}{2}-\lambda_1\right), p\left(\frac{3}{2}-\lambda_1\right)\right],$$

we have $\widehat{\lambda}_1, \widehat{\lambda}_2 \leqslant \frac{3}{2}$.

By (18), we still have

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_m B_n}{(m+n-\eta)^{\lambda}} \\ &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}B(\widehat{\lambda}_1,\widehat{\lambda}_2) \\ &\quad \times \left[\sum_{m=1}^{\infty}\left[1-O_1\left(\frac{1}{(m-\eta_1)^{\widehat{\lambda}_2}}\right)\right](m-\eta_1)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{20}$$

If the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (17) is the best possible, then compare with the constant factors in (17) and (20) , we have the following inequality:

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}} \geqslant \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}B(\widehat{\lambda}_1,\widehat{\lambda}_2)(\in \mathbf{R}_+),$$

namely,

$$B(\widehat{\lambda}_1,\widehat{\lambda}_2) \leqslant (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}. \tag{21}$$

By the reverse Hölder's inequality (cf. [18]), we obtain

$$
\begin{aligned}
B(\widehat{\lambda}_1,\widehat{\lambda}_2) &= k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\
&= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(1+u)^\lambda}du\\
&= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2-1}{p}}}{(1+u)^\lambda}(u^{\frac{\lambda_1-1}{q}})du\\
&\geqslant \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda}du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}du\right]^{\frac{1}{q}}\\
&= \left[\int_0^\infty \frac{v^{\lambda_2-1}dv}{(1+v)^\lambda}\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{u^{\lambda_1-1}du}{(1+u)^\lambda}\right]^{\frac{1}{q}}\\
&= (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}. \qquad (22)
\end{aligned}
$$

Then we have

$$B(\widehat{\lambda}_1,\widehat{\lambda}_2) \geqslant (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}},$$

which follows that (22) protains the form of equality based on inequality (21).

We observe that (22) protains the form of equality if and only if there exist constants A and B, such that they are not both zero, satisfying (cf. [18])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-1} = \frac{B}{A}u^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$, and then $\lambda-\lambda_1-\lambda_2 = 0$. Hence, we have $\lambda_1+\lambda_2 = \lambda$.

The theorem is proved. □

REMARK 1. For $\lambda = 1$, $\lambda_1 = \lambda_2 = \frac{1}{2}$ in (18), we have the following inequality with the best possible constant factor $\frac{\pi}{2}$:

$$
\begin{aligned}
&\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{A_mB_n}{m+n-\eta}\\
&> \frac{\pi}{2}\left[\sum_{m=1}^\infty\left[1-O_1\left(\frac{1}{(m-\eta_1)^{1/2}}\right)\right](m-\eta_1)^{\frac{p}{2}-1}a_m^p\right]^{\frac{1}{p}}\\
&\times\left[\sum_{n=1}^\infty (n-\eta_2)^{\frac{q}{2}-1}b_n^q\right]^{\frac{1}{q}}. \qquad (23)
\end{aligned}
$$

In particular,

(a) for $\eta_1 = \eta_2 = \eta = 0$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_m B_n}{m+n}$$
$$> \frac{\pi}{2}\left[\sum_{m=1}^{\infty}\left[1-O_1\left(\frac{1}{m^{1/2}}\right)\right]m^{\frac{p}{2}-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right]^{\frac{1}{q}}. \tag{24}$$

(b) for $\eta_1 = \eta_2 = \frac{1}{4}$, $\eta = \frac{1}{2}$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_m B_n}{m+n-\frac{1}{2}}$$
$$> \frac{\pi}{2}\left[\sum_{m=1}^{\infty}\left[1-O_1\left(\frac{1}{(m-\frac{1}{4})^{1/2}}\right)\right]\left(m-\frac{1}{4}\right)^{\frac{p}{2}-1}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{n=1}^{\infty}\left(n-\frac{1}{4}\right)^{\frac{q}{2}-1}b_n^q\right]^{\frac{1}{q}}. \tag{25}$$

4. Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters and the techniques of real analysis, using Hermite-Hadamard's inequality, Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate reverse Hardy-Hilbert's inequality with two partial sums is given in Theorem 1. The equivalent conditions of the best possible constant factor related to several parameters are provided by Theorem 2 and Theorem 3. We also obtain some particular inequalities in Remark 1. The lemmas and theorems provide an extensive account of this type of inequalities.

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REFERENCES

[1] V. ADIYASUREN, T. BATBOLD AND L. E. AZAR, *A new discrete Hilbert-type inequality involving partial sums*, Journal of Inequalities and Applications, 2019: 127, 2019.

[2] V. ADIYASUREN, T. BATBOLD AND M. KRNIĆ, *Hilbert-type inequalities involving differential operators, the best constants and applications*, Math. Inequal. Appl., **18** (2015), 111–124.

[3] L. E. AZAR, *The connection between Hilbert and Hardy inequalities*, Journal of Inequalities and Applications, 2013: 452, 2013.

[4] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[5] B. HE, *A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor*, Journal of Mathematical Analysis and Applications, **431** (2015), 890–902.

[6] B. HE, Y. R. ZHONG AND B. C. YANG, *On a more accurate Hilbert-type inequality involving the partial sums*, Journal of Mathematical Inequalities, **15** (4) (2021), 1647–1662.

[7] Y. HONG, *On the structure character of Hilbert's type integral inequality with homogeneous kernel and application*, Journal of Jilin University (Science Edition), **55** (2) (2017), 189–194.

[8] Y. HONG, B. HE AND B. C. YANG, *Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory*, Journal of Mathematics Inequalities, **12** (3) (2018), 777–788.

[9] Y. HONG, Q. L. HUANG, B. C. YANG AND J. L. LIAO, *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, Journal of Inequalities and Applications (2017), 2017: 316.

[10] Y. HONG, J. Q. LIAO, B. C. YANG AND Q. CHEN, *A class of Hilbert multiple integral inequalities with the kernel of generalized homogeneous function and its applications*, Journal of Inequalities and Applications (2020), 2020: 140.

[11] Y. HONG AND Y. WEN, *A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor*, Annals Mathematica, **37A** (3) (2016), 329–336.

[12] Q. L. HUANG, *A new extension of Hardy-Hilbert-type inequality*, Journal of Inequalities and Applications (2015), 2015: 397.

[13] X. S. HUANG, R. C. LUO AND B. C. YANG, *On a new extended half-discrete Hilbert's inequality involving partial sums*, Journal of Inequalities and Applications (2020) 2020: 16.

[14] Z. X. HUANG AND B. C. YANG, *Equivalent property of a half-discrete Hilbert's inequality with parameters*, Journal of Inequalities and Applications (2018) 2018: 333.

[15] X. S. HUANG AND B. C. YANG, *On a more accurate Hilbert-type inequality in the whole plane with the general homogeneous kernel*, Journal of Inequalities and Applications (2021), 2021: 10.

[16] M. KRNIĆ AND J. PEČARIĆ, *Extension of Hilbert's inequality*, J. Math. Anal., Appl. **324** (1) (2006), 150–160.

[17] M. KRNIĆ AND J. PEČARIĆ, *General Hilbert's and Hardy's inequalities*, Mathematical inequalities & applications, **8** (1) (2005), 29–51.

[18] J. C. KUANG, *Applied inequalities*, Shangdong Science and Technology Press, Jinan, China, 2021.

[19] I. PERIĆ AND P. VUKOVIĆ, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach Journal of Mathematical Analysis, **5** (2) (2011), 33–43.

[20] M. TH. RASSIAS AND B. C. YANG, *A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function*, Applied Mathematics and Computation, **225** (2013), 263–277.

[21] M. TH. RASSIAS AND B. C. YANG, *On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function*, Applied Mathematics and Computation, **242** (2013), 800–813.

[22] M. TH. RASSIAS AND B. C. YANG, *On half-discrete Hilbert's inequality*, Applied Mathematics and Computation, **220** (2013), 75–93.

[23] M. TH. RASSIAS, B. C. YANG AND A. RAIGORODSKII, *On the reverse Hardy-type integral inequalities in the whole plane with the extended Riemann-Zeta function*, Journal of Mathematics Inequalities, **14** (2) (2020), 525–546.

[24] A. Z. WANG AND B. C. YANG, *Equivalent property of a more accurate half-discrete Hilbert's inequality*, Journal of Applied Analysis and Computation, **10** (3) (2020), 920–934.

[25] A. Z. WANG, B. C. YANG AND Q. CHEN, *Equivalent properties of a reverse half-discret Hilbert's inequality*, Journal of Inequalities and Applications (2019), 2019: 279.

[26] Z. T. XIE, Z. ZENG AND Y. F. SUN, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2, Advances and Applications in Mathematical Sciences, **12** (7) (2013), 391–401.

[27] D. M. XIN, *A Hilbert-type integral inequality with the homogeneous kernel of zero degree*, Mathematical Theory and Applications, **30** (2) (2010), 70–74.

[28] D. M. XIN, B. C. YANG AND A. Z. WANG, *Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane*, Journal of Function Spaces, vol. 2018, Article ID2691816, 8 pages.

[29] J. S. XU, *Hardy-Hilbert's inequalities with two parameters*, Advances in Mathematics, **36** (2) (2007), 63–76.

[30] B. C. YANG, *On a generalization of Hilbert double series theorem*, J. Nanjing Univ. Math. Biquarterly, **18** (1) (2001), 145–152.
[31] B. C. YANG, *The norm of operator and Hilbert-type inequalities*, Science Press, Beijing, China, 2009.
[32] B. C. YANG AND L. DEBNATH, *Half-discrete Hilbert-type inequalities*, World Scientific Publishing, Singapore, 2014.
[33] B. C. YANG AND M. KRNIĆ, *A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree* 0, Journal of Mathematical Inequalities, **6** (3) (2012), 401–417.
[34] B. C. YANG, S. H. WU AND Q. CHEN, *A new extension of Hardy-Hilbert's inequality containing kernel of double power functions*, Mathematics, 2020, **8**, 339, doi:10.3390/math8060894.
[35] B. C. YANG, S. H. WU AND J. LIAO, *On a new extended Hardy-Hilbert's inequality with parameters*, Mathematics, 2020, **8**, 73, doi:10.3390/math8010073.
[36] B. C. YANG, S. H. WU AND A. Z. WANG, *On a reverse half-discrete Hardy-Hilbert's inequality with parameters*, Mathematics, 2019, **7**, 1054.
[37] Z. ZHEN, K. RAJA RAMA GANDHI AND Z. T. XIE, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2 *and with the integral*, Bulletin of Mathematical Sciences and Applications, **3** (1) (2014), 11–20.

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Aizhen Wang
School of Mathematics
Guangdong University of Education
Guangzhou, Guangdong 510303, P. R. China
e-mail: ershimath@163.com

Bicheng Yang
School of Mathematics
Guangdong University of Education
Guangzhou, Guangdong 510303, P. R. China
e-mail: bcyang@gdei.edu.cn