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COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS IN RADEMACHER TYPE p BANACH SPACES

YONGFENG WU AND XIN DENG*

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Abstract. The authors investigate the complete moment convergence for arrays of rowwise independent random elements in Rademacher p Banach spaces. The results obtained in this paper improve the corresponding theorems of Hu et al. (Hu, T.-C., Rosalsky, A., Volodin, A., Zhang, S., 2021. A complete convergence theorem for row sums from arrays of rowwise independent random elements in Rademacher type p Banach spaces. II, Stochastic Anal. Appl., **39** (1), 177–193). Some corollaries and examples are also presented.

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{X}$ be a real separable Banach space with norm $||\cdot||$. The reader may refer to Hu et al. (2012) for more details on the concepts of Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space, random element V, EV, and rowwise independence. In this article, all random elements are defined on the space $(\Omega, \mathscr{F}, P)$ and take values in the space $\mathscr{X}$.

A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

This notion was given firstly by Hsu and Robbins (1947). This of course implies by the Borel-Cantelli lemma that $U_n \to a$ almost surely (a.s.).

Chow (1988) introduced a more general concept of the complete convergence. Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \text{ for some or all } \varepsilon > 0,$$

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* Corresponding author.

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then the above result was called the complete moment convergence. It is worthy to point out that the complete moment convergence is the more general version of the complete convergence, which will been shown in Remark 2.1.

Recently some scholars studied the limit property concerned a Banach space setting (see, [1, 2, 7–11]) and parts of them investigated the complete convergence. However, according to our knowledge, few articles discuss the complete moment convergence for sums of arrays of Banach space valued random elements. Since the complete moment convergence is more general than the complete convergence, it is very significant to study the complete moment convergence for arrays of rowwise independent random elements in Rademacher p Banach spaces.

Hu et al. (2012) obtained the following complete convergence theorem for row sums from arrays of rowwise independent random elements in Rademacher type p Banach spaces.

THEOREM A. (Hu et al., 2012, Theorem 3.1) *Let* $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise independent random elements taking values in a real separable Rademacher type* p $(1 \leqslant p \leqslant 2)$ *Banach space and let* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants. Suppose for some* $J > 0$ *and some* $\delta_1, \delta_2 > 0$ *that*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\left(\left\|V_{n,k}\right\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0, \tag{1.1}$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left\|V_{n,k}\right\|^p I\left(\left\|V_{n,k}\right\| \leqslant \delta_1\right)\right)^J < \infty \tag{1.2}$$

and

$$\sum_{k=1}^{k_n} EV_{n,k} I\left(\left\|V_{n,k}\right\| \leqslant \delta_2\right) \to 0 \text{ as } n \to \infty. \tag{1.3}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\left\|\sum_{k=1}^{k_n} V_{n,k}\right\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.4}$$

Hu et al. (2021) improved partially Theorem A by replacing the condition (1.3) to a stronger one and presented the following result.

THEOREM B. (Hu et al., 2021, Theorem 3.1) *Let* $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise independent random elements taking values in a real separable Rademacher type* p $(1 \leqslant p \leqslant 2)$ *Banach space and let* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants. Suppose for some* $J > 0$ *and some* $\delta_1, \delta_2 > 0$ *that* (1.1), (1.2) *and*

$$\max_{1 \leqslant k \leqslant k_n} \left\|\sum_{i=1}^{k} EV_{n,i} I\left(\left\|V_{n,i}\right\| \leqslant \delta_2\right)\right\| \to 0 \text{ as } n \to \infty. \tag{1.5}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1 \leqslant k \leqslant k_n} \left\|\sum_{i=1}^{k} V_{n,i}\right\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.6}$$

It is clear that (1.6) is more stronger than (1.4). Hu et al. (2021) also presented an example, which shows that Theorem B can fail if (1.5) is weakened to (1.3), that is, under the conditions of Theorem A, the conclusion (1.6) of Theorem B does not necessarily hold.

In this work, the authors shall study the complete moment convergence for row sums from arrays of rowwise independent random elements in Rademacher type p Banach spaces. The authors replace the condition (1.1) to a stronger one and obtain a much stronger result which improves partially Theorem B.

It is obvious that (1.4) and (1.6) are true if $\sum_{n=1}^{\infty} c_n < \infty$. Therefore, in this paper, $\{c_n, n \geqslant 1\}$ is assumed to be a sequence of positive constants such that $\sum_{n=1}^{\infty} c_n = \infty$. In addition, as with Hu et al. (2021), we also assume that $\{k_n, n \geqslant 1\}$ is a sequence of positive integers with $k_n \to \infty$ as $n \to \infty$.

Throughout this paper, the symbol C always stands for a generic positive constant which may differ from one place to another. The symbol $I(A)$ denotes the indicator function of the event A.

2. Lemmas and main result

To prove our main result, we need the following technical lemmas.

LEMMA 2.1. (Hu et al., 2021) *For all integers* $j \geqslant 0$*, there exists a constant* $0 < C_j < \infty$ *depending only on* j *such that for all* $n \geqslant 1$*,* $t > 0$ *and every set* $\{V_k, 1 \leqslant k \leqslant n\}$ *of* n *independent random elements taking values in a real separable Banach space,*

$$P\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k} V_i\right\| > (3^{j+1}-1)t\right)$$
$$\leqslant C_j P\left(\max_{1\leqslant k\leqslant n}\left\|V_k\right\| > 2t\right) + \left(P\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k} V_i\right\| > 2t\right)\right)^{2^j}. \tag{2.1}$$

LEMMA 2.2. (Rosalsky and Van Thanh, 2007) *Let* $\mathscr{X}$ *be a real separable Rademacher type* p $(1 \leqslant p \leqslant 2)$ *Banach space. Then there exists a constant* $0 < A_p < \infty$ *depending only on* p *such that for every sequence* $\{V_k, 1 \leqslant k \leqslant n\}$ *of independent mean 0 random elements taking values in* $\mathscr{X}$*,*

$$E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k} V_i\right\|\right)^p \leqslant A_p \sum_{i=1}^{n} E\left\|V_i\right\|^p, \ \ n \geqslant 1. \tag{2.2}$$

Now we state our main result and the proof.

THEOREM 2.1. *Let* $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise independent random elements taking values in a real separable Rademacher type* p $(1 \leqslant p \leqslant 2)$ *Banach space and let* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants. Suppose for some* $J > 1$ *and some* $\delta_1, \delta_2 > 0$ *that*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E\left\|V_{n,k}\right\|^q I\left(\left\|V_{n,k}\right\| > \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0 \textit{ and } 0 < q \leqslant p, \tag{2.3}$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\big|\big|V_{n,k}\big|\big|^p I\big(\big|\big|V_{n,k}\big|\big| \leqslant \delta_1\big) \right)^J < \infty, \tag{2.4}$$

$$\sum_{k=1}^{k_n} E\big|\big|V_{n,k}\big|\big|^q I\big(\big|\big|V_{n,k}\big|\big| > \min\{\delta_1,\delta_2\}\big) \to 0 \ \textit{as} \ n \to \infty \tag{2.5}$$

and

$$\max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} EV_{n,i} I\big(\big|\big|V_{n,i}\big|\big| \leqslant \delta_2\big) \right|\right| \to 0 \ \textit{as} \ n \to \infty. \tag{2.6}$$

Then

$$\sum_{n=1}^{\infty} c_n E\left\{ \max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| - \varepsilon \right\}_+^q < \infty \quad \textit{for all } \varepsilon > 0. \tag{2.7}$$

Proof. Choose a positive integer j such that $2^j > J > 1$. Let $\varepsilon > 0$ be arbitrary and $\delta = \max\{\delta_1,\delta_2\}$ and $t_0 = \big(2\delta(3^{j+1}-1)\big)^q$. Without loss of generality, we may assume $0 < \varepsilon < \min\{\delta_1,\delta_2\}$. For any fixed $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n E\left\{ \max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| - \varepsilon \right\}_+^q \\
=\ & \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| - \varepsilon > t^{1/q} \right) \mathrm{d}t \\
=\ & \sum_{n=1}^{\infty} c_n \int_0^{t_0} P\left(\max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| > \varepsilon + t^{1/q} \right) \mathrm{d}t \\
& + \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| > \varepsilon + t^{1/q} \right) \mathrm{d}t \\
\leqslant\ & t_0 \sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| > \varepsilon \right) + \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n} \left|\left| \sum_{i=1}^{k} V_{n,i} \right|\right| > t^{1/q} \right) \mathrm{d}t \\
=:\ & I_1 + I_2.
\end{aligned}$$

Noting that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\big(\big|\big|V_{n,k}\big|\big| > \varepsilon\big) \leqslant \varepsilon^q \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E\big|\big|V_{n,k}\big|\big|^q I\big(\big|\big|V_{n,k}\big|\big| > \varepsilon\big) < \infty,$$

which indicates (2.3) implies (1.1). By Theorem B, we have $I_1 < \infty$. To prove (2.7), it is enough to prove $I_2 < \infty$.

For $n \geqslant 1$, $1 \leqslant k \leqslant k_n$ and $t \geqslant t_0$, let

$$V_{n,k}^{'} = V_{n,k} I\big(\big|\big|V_{n,k}\big|\big| \leqslant t^{1/q}/(2(3^{j+1}-1))\big),$$

$$V_{n,k}^{''} = V_{n,k} I\big(\big|\big|V_{n,k}\big|\big| > t^{1/q}/(2(3^{j+1}-1))\big).$$

Then

$$
\begin{aligned}
I_2 \leqslant\ & \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{'}-EV_{n,i}^{'})\right|\right| > t^{1/q}/2\right)\mathrm{d}t \\
& + \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{''}+EV_{n,i}^{'})\right|\right| > t^{1/q}/2\right)\mathrm{d}t \\
=:\ & I_3+I_4.
\end{aligned}
$$

It follows from the definition of $V_{n,k}^{'}$ that

$$
\max_{1\leqslant k\leqslant k_n}||V_{n,k}^{'}-EV_{n,k}^{'}|| \leqslant \frac{t^{1/q}}{3^{j+1}-1} \text{ almost surely (a.s.).} \tag{2.8}
$$

Thus by Lemma 2.1, we have

$$
\begin{aligned}
I_3 =\ & \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{'}-EV_{n,i}^{'})\right|\right| > (3^{j+1}-1)\frac{t^{1/q}}{2(3^{j+1}-1)}\right)\mathrm{d}t \\
\leqslant\ & C_j \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}||V_{n,k}^{'}-EV_{n,k}^{'}|| > \frac{t^{1/q}}{3^{j+1}-1}\right)\mathrm{d}t \\
& + \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} \left\{P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{'}-EV_{n,i}^{'})\right|\right| > \frac{t^{1/q}}{3^{j+1}-1}\right)\right\}^{2^j}\mathrm{d}t \\
\leqslant\ & 0+\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} \left\{P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{'}-EV_{n,i}^{'})\right|\right| > \frac{t^{1/q}}{3^{j+1}-1}\right)\right\}^{J}\mathrm{d}t \qquad (\text{by } (2.8)) \\
\leqslant\ & C\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} \left\{t^{-p/q}E\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{'}-EV_{n,i}^{'})\right|\right|\right)^{p}\right\}^{J}\mathrm{d}t \\
& \qquad (\text{by the Markov inequality}) \\
\leqslant\ & C\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} \left\{t^{-p/q}\sum_{k=1}^{k_n}E||(V_{n,k}^{'}-EV_{n,k}^{'})||^{p}\right\}^{J}\mathrm{d}t \qquad (\text{by Lemma 2.2}) \\
\leqslant\ & C\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} \left\{t^{-p/q}\sum_{k=1}^{k_n}E||V_{n,k}^{'}||^{p}\right\}^{J}\mathrm{d}t \\
& \qquad (\text{by the } C_r\text{-inequality and Jensen's inequality}) \\
=\ & C\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} \left\{t^{-p/q}\sum_{k=1}^{k_n}E||V_{n,k}||^{p}I(||V_{n,k}||\leqslant\delta_1)\right. \\
& \left. +t^{-p/q}\sum_{k=1}^{k_n}E||V_{n,k}||^{p}I\left(\delta_1<||V_{n,k}||\leqslant\frac{t^{1/q}}{2(3^{j+1}-1)}\right)\right\}^{J}\mathrm{d}t
\end{aligned}
$$

$$\leqslant C\sum_{n=1}^{\infty}c_n\int_{t_0}^{\infty}\left\{t^{-p/q}\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^pI\big(\big|\big|V_{n,k}\big|\big|\leqslant\delta_1\big)\big)\right\}^J\mathrm{d}t$$
$$+C\sum_{n=1}^{\infty}c_n\int_{t_0}^{\infty}\left\{t^{-p/q}\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^pI\Big(\delta_1<\big|\big|V_{n,k}\big|\big|\leqslant\frac{t^{1/q}}{2(3^{j+1}-1)}\Big)\right\}^J\mathrm{d}t$$
$$=: I_5+I_6.$$

By $0<q\leqslant p$, $J>1$ and (2.4), we have

$$I_5=C\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^pI\big(\big|\big|V_{n,k}\big|\big|\leqslant\delta_1\big)\big)\right)^J\int_{t_0}^{\infty}t^{-pJ/q}\mathrm{d}t$$
$$\leqslant C\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^pI\big(\big|\big|V_{n,k}\big|\big|\leqslant\delta_1\big)\big)\right)^J<\infty.$$

By $0<q\leqslant p$, $J>1$ and (2.3), we have

$$I_6\leqslant\big(2(3^{j+1}-1)\big)^{(q-p)J}C$$
$$\times\sum_{n=1}^{\infty}c_n\int_{t_0}^{\infty}\left\{t^{-1}\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\Big(\delta_1<\big|\big|V_{n,k}\big|\big|\leqslant\frac{t^{1/q}}{2(3^{j+1}-1)}\Big)\right\}^J\mathrm{d}t$$
$$\leqslant\big(2(3^{j+1}-1)\big)^{(q-p)J}C\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)\right)^J\int_{t_0}^{\infty}t^{-J}\mathrm{d}t$$
$$\leqslant C\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)\right)^J.$$

By (2.5), we have

$$\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)\to 0\quad\text{as } n\to\infty.$$

Therefore, there exists a positive integer N_1 such that

$$\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)<1$$

holds uniformly for all $n>N_1$ and $t\geqslant t_0$. Then by (2.3), we can obtain

$$I_6\leqslant C\sum_{n=1}^{N_1}c_n\left(\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)\right)^J$$
$$+C\sum_{n=N_1+1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)\right)^J$$
$$\leqslant C+C\sum_{n=N_1+1}^{\infty}c_n\sum_{k=1}^{k_n}E\big|\big|V_{n,k}\big|\big|^qI\big(\big|\big|V_{n,k}\big|\big|>\delta_1\big)<\infty.$$

Finally we will show $I_4 < \infty$. Observing that

$$
\begin{aligned}
&\max_{t\geqslant t_0}\max_{1\leqslant k\leqslant k_n} 2t^{-1/q}\left\|\sum_{i=1}^{k} EV'_{n,i}\right\| \\
=\ & 2\max_{t\geqslant t_0}\max_{1\leqslant k\leqslant k_n} t^{-1/q}\left\|\sum_{i=1}^{k} EV_{n,i}I\big(||V_{n,i}|| \leqslant \frac{t^{1/q}}{2(3^{j+1}-1)}\big)\right\| \\
\leqslant\ & 2\max_{t\geqslant t_0}\max_{1\leqslant k\leqslant k_n} t^{-1/q}\left\|\sum_{i=1}^{k} EV_{n,i}I\Big(||V_{n,i}|| \leqslant \delta_2\Big)\right\| \\
&+2\max_{t\geqslant t_0}\max_{1\leqslant k\leqslant k_n} t^{-1/q}\left\|\sum_{i=1}^{k} EV_{n,i}I\Big(\delta_2 < ||V_{n,i}|| \leqslant \frac{t^{1/q}}{2(3^{j+1}-1)}\Big)\right\| \\
=:\ & J_1 + J_2.
\end{aligned}
$$

From the (2.6), we have

$$
J_1 \leqslant 2t_0^{-1/q}\max_{1\leqslant k\leqslant k_n}\left\|\sum_{i=1}^{k} EV_{n,i}I\big(||V_{n,i}|| \leqslant \delta_2\big)\right\| \to 0 \quad \text{as } n\to\infty.
$$

From the (2.5), we have

$$
\begin{aligned}
J_2 &\leqslant 2\max_{t\geqslant t_0} t^{-1/q}\sum_{i=1}^{k_n} E||V_{n,i}||I\Big(\delta_2 < ||V_{n,i}|| \leqslant \frac{t^{1/q}}{2(3^{j+1}-1)}\Big) \\
&\leqslant \frac{1}{3^{j+1}-1}\max_{t\geqslant t_0}\sum_{i=1}^{k_n} EI\Big(\delta_2 < ||V_{n,i}|| \leqslant \frac{t^{1/q}}{2(3^{j+1}-1)}\Big) \\
&\leqslant \frac{1}{3^{j+1}-1}\sum_{i=1}^{k_n} P\big(||V_{n,i}|| > \delta_2\big) \\
&\leqslant \frac{1}{3^{j+1}-1}\delta_2^{-q}\sum_{k=1}^{k_n} E||V_{n,k}||^q I\big(||V_{n,k}|| > \delta_2\big) \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

It follows by $J_1 \to 0$ and $J_2 \to 0$ as $n\to\infty$ that

$$
\max_{t\geqslant t_0}\max_{1\leqslant k\leqslant k_n} 2t^{-1/q}\left\|\sum_{i=1}^{k} EV'_{n,i}\right\| \to 0 \quad \text{as } n\to\infty
$$

and so there exists a positive integer N_2 such that

$$
\max_{1\leqslant k\leqslant k_n}\left\|\sum_{i=1}^{k} EV'_{n,i}\right\| < t^{1/q}/4
$$

holds uniformly for all $n > N_2$ and $t \geqslant t_0$. Hence

$$
\begin{aligned}
I_4 &\leqslant \sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} V''_{n,i}\right|\right| + \max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} EV'_{n,i}\right|\right| > t^{1/q}/2\right)\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} V''_{n,i}\right|\right| > t^{1/q}/4\right)\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} c_n \int_{t_0}^{\infty} P\left(\bigcup_{k=1}^{k_n}\left[||V_{n,k}|| > \frac{t^{1/q}}{2(3^{j+1}-1)}\right]\right)\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n}\int_{t_0}^{\infty} P\left(||V_{n,k}|| > \frac{t^{1/q}}{2(3^{j+1}-1)}\right)\mathrm{d}t.
\end{aligned}
$$

Noting that

$$\int_a^{\infty} P(|Y| > t^{1/\theta})\mathrm{d}t \leqslant E|Y|^{\theta} I(|Y| > a^{1/\theta}),$$

then we have by (2.3)

$$I_4 \leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E||V_{n,k}||^q I\big(||V_{n,k}|| > \delta\big) < \infty.$$

The proof is completed. $\square$

REMARK 2.1. Noting that

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} c_n E\left\{\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} V_{n,i}\right|\right| - \varepsilon\right\}_+^q \\
&= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} V_{n,i}\right|\right| > \varepsilon + t^{1/q}\right)\mathrm{d}t \\
&\geqslant \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon^q} P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} V_{n,i}\right|\right| > \varepsilon + t^{1/q}\right)\mathrm{d}t \\
&\geqslant \varepsilon^q \sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} V_{n,i}\right|\right| > 2\varepsilon\right),
\end{aligned}
$$

hence (2.7) is much stronger than (1.6) and Theorem 2.1 improves partially Theorem B.

OPEN PROBLEM. Since (2.3) is stronger than (1.1) and we add the condition (2.5), it is worthy to point out that whether Theorem 2.1 remains true under the conditions of Theorem B. It is an interesting and challenging work. Despite our efforts to solve this problem, it is still an open problem.

3. Corollaries

Take $c_n = 1/n$, $k_n = n$ in Theorem 2.1, we can obtain directly the following corollary.

COROLLARY 3.1. *Let $\{V_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space. Suppose for some $J > 0$ and some $\delta_1, \delta_2 > 0$ that*

$$\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} E\|V_{n,k}\|^q I(\|V_{n,k}\| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0 \text{ and } 0 < q \leqslant p, \tag{3.1}$$

$$\sum_{n=1}^{\infty} n^{-1} \left(\sum_{k=1}^{n} E\|V_{n,k}\|^p I(\|V_{n,k}\| \leqslant \delta_1) \right)^J < \infty, \tag{3.2}$$

$$\sum_{k=1}^{n} E\|V_{n,k}\|^q I(\|V_{n,k}\| > \min\{\delta_1, \delta_2\}) \to 0 \text{ as } n \to \infty \tag{3.3}$$

and

$$\max_{1 \leqslant k \leqslant n} \left\| \sum_{i=1}^{k} EV_{n,i} I(\|V_{n,i}\| \leqslant \delta_2) \right\| \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Then

$$\sum_{n=1}^{\infty} n^{-1} E\left\{ \max_{1 \leqslant k \leqslant n} \left\| \sum_{i=1}^{k} V_{n,i} \right\| - \varepsilon \right\}_+^q < \infty \quad \text{for all } \varepsilon > 0. \tag{3.5}$$

COROLLARY 3.2. *Let $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent mean 0 random elements taking values in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space and let $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that (2.3),*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\|V_{n,k}\|^q \right)^J < \infty \quad \text{for some } 0 < q \leqslant p \text{ and } J > 1 \tag{3.6}$$

and

$$\sum_{k=1}^{k_n} E\|V_{n,k}\|^{\max\{q,1\}} I(\|V_{n,k}\| > \delta) \to 0 \quad \text{as } n \to \infty \quad \text{for some } \delta > 0. \tag{3.7}$$

Then (2.7) holds.

Proof. In view of Theorem 2.1 for the case $\delta_1 = \delta_2 = \delta$, we need only to verify

(2.4), (2.5) and (2.6). We first verify (2.4).

$$\begin{aligned}\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E\big\|V_{n,k}\big\|^p I\big(\big\|V_{n,k}\big\| \leqslant \delta_1\big)\right)^J &= \delta_1^{pJ}\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E\frac{\big\|V_{n,k}\big\|^p}{\delta_1^p} I\big(\big\|V_{n,k}\big\| \leqslant \delta_1\big)\right)^J\\ &\leqslant \delta_1^{pJ}\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E\frac{\big\|V_{n,k}\big\|^q}{\delta_1^q} I\big(\big\|V_{n,k}\big\| \leqslant \delta_1\big)\right)^J\\ &\leqslant \delta_1^{(p-q)J}\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E\big\|V_{n,k}\big\|^q\right)^J < \infty.\end{aligned}$$

On the other hand, we can easily verify (2.5) for the case $\delta_1 = \delta_2 = \delta$ from the condition (3.7). Finally, since the $V_{n,i}$ all have mean 0, we can obtain

$$\begin{aligned}\max_{1\leqslant k\leqslant k_n}\left\|\sum_{i=1}^{k} EV_{n,i}I\big(\big\|V_{n,i}\big\| \leqslant \delta_2\big)\right\| &= \max_{1\leqslant k\leqslant k_n}\left\|\sum_{i=1}^{k} EV_{n,i}I\big(\big\|V_{n,i}\big\| > \delta_2\big)\right\|\\ &\leqslant \sum_{k=1}^{k_n} E\big\|V_{n,k}\big\| I\big(\big\|V_{n,k}\big\| > \delta_2\big)\\ &\leqslant \sum_{k=1}^{k_n} E\big\|V_{n,k}\big\|^{\max\{q,1\}} I\big(\big\|V_{n,k}\big\| > \delta\big) \to 0 \text{ as } n\to\infty,\end{aligned}$$

which shows (2.6) holds and completes the proof. □

An array of random elements $\{V_{n,k}, 1\leqslant k\leqslant k_n, n\geqslant 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant $0<C<\infty$ such that

$$P(\big\|V_{n,k}\big\| > x) \leqslant CP(|X|>x) \tag{3.8}$$

for all $x\geqslant 0$ and all $1\leqslant k\leqslant k_n$ and all $n\geqslant 1$.

The above concept of stochastic domination is a generalization of the concept of identical distributions. Stochastic dominance of $\{V_{n,k}, 1\leqslant k\leqslant k_n, n\geqslant 1\}$ by the random variable X implies $E\big\|V_{n,k}\big\|^p \leqslant CE|X|^p$ if the p-moment of $|X|$ exists, i. e., if $E|X|^p<\infty$.

COROLLARY 3.3. *Let $\{V_{n,k}, 1\leqslant k\leqslant k_n, n\geqslant 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p $(1\leqslant p\leqslant 2)$ Banach space. Suppose that $\{V_{n,k}, 1\leqslant k\leqslant k_n, n\geqslant 1\}$ is stochastically dominated by a random variable X. Let $\{a_{n,k}, 1\leqslant k\leqslant k_n, n\geqslant 1\}$ be an array of constants such that*

$$\sum_{k=1}^{k_n} |a_{n,k}|^p = \mathscr{O}(n^{-\alpha}) \quad \textit{for some } \alpha>0. \tag{3.9}$$

Suppose that

$$k_n = o(n^{\alpha/(p-1)}) \quad \textit{for } 1<p\leqslant 2 \tag{3.10}$$

and

$$E|X|^p < \infty. \tag{3.11}$$

Then for all $\beta < \alpha - 1$,

$$\sum_{n=1}^{\infty} n^{\beta} E\left\{\max_{1\leqslant k\leqslant k_n}\left|\left|\sum_{i=1}^{k} a_{n,i}V_{n,i}\right|\right| - \varepsilon\right\}_+^q < \infty \quad \textit{for all } \varepsilon > 0 \textit{ and } 0 < q \leqslant p. \tag{3.12}$$

Proof. Taking $c_n = n^{\beta}$ and replacing $V_{n,k}$ with $a_{n,k}V_{n,k}$ in Theorem 2.1, we need only to verify (2.3), (2.4), (2.5) and (2.6).

Firstly, note that for $0 < q \leqslant p$, (3.9) and $\beta < \alpha - 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{k_n} E||a_{n,k}V_{n,k}||^q I\big(||a_{n,k}V_{n,k}|| > \varepsilon\big) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{k_n} |a_{n,k}|^q E|X|^q I\big(|a_{n,k}X| > \varepsilon\big) \\
&\leqslant C\varepsilon^{q-p} \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{k_n} |a_{n,k}|^p E|X|^p I\big(|a_{n,k}X| > \varepsilon\big) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{k_n} |a_{n,k}|^p E|X|^p \\
&\leqslant C \sum_{n=1}^{\infty} n^{\beta-\alpha} < \infty.
\end{aligned}$$

Secondly, by $\beta < \alpha - 1$ and $J > 1$, we have $J > 1 > \frac{\beta+1}{\alpha}$, then $\beta - \alpha J < -1$. It follows by (3.9) and $\beta - \alpha J < -1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{k=1}^{k_n} E||a_{n,k}V_{n,k}||^p I\big(||a_{n,k}V_{n,k}|| \leqslant \delta_1\big)\right)^J &\leqslant \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{k=1}^{k_n} |a_{n,k}|^p E||V_{n,k}||^p\right)^J \\
&\leqslant C \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{k=1}^{k_n} |a_{n,k}|^p E|X|^p\right)^J \\
&\leqslant C \sum_{n=1}^{\infty} n^{\beta-\alpha J} < \infty.
\end{aligned}$$

Thirdly, we let $\delta_0 = \min\{\delta_1, \delta_2\}$. Then we have by (3.9) and (3.11) that

$$\begin{aligned}
\sum_{k=1}^{k_n} E||a_{n,k}V_{n,k}||^q I\big(||a_{n,k}V_{n,k}|| > \delta_0\big) &= \sum_{k=1}^{k_n} |a_{n,k}|^q E|X|^q I\big(|a_{n,k}X| > \delta_0\big) \\
&\leqslant \delta_0^{q-p} \sum_{k=1}^{k_n} |a_{n,k}|^p E|X|^p I\big(|a_{n,k}X| > \delta_0\big) \\
&\leqslant C \sum_{k=1}^{k_n} |a_{n,k}|^p \leqslant Cn^{-\alpha} \quad \text{as } n \to \infty,
\end{aligned}$$

which shows (2.5) holds.

Finally, we will verify (2.6). We can obtain by Jensen's inequality and (3.9)–(3.11) that

$$
\begin{aligned}
&\max_{1\leqslant k\leqslant k_n}\left\|\sum_{i=1}^{k}E(a_{n,i}V_{n,i})I\big(\|a_{n,i}V_{n,i}\|\leqslant \delta_2\big)\right\| \\
&\leqslant \sum_{i=1}^{k_n}|a_{n,i}|E\|V_{n,i}\| \leqslant CE|X|\sum_{k=1}^{k_n}|a_{n,k}| \\
&\leqslant Ck_n^{1-1/p}\left(\sum_{k=1}^{k_n}|a_{n,k}|^p\right)^{1/p} \\
&\leqslant C\frac{k_n^{1-1/p}}{n^{\alpha/p}} \\
&=\begin{cases} Cn^{-\alpha}\to 0 \quad \text{as } n\to\infty, & p=1,\\ C\big(\frac{k_n}{n^{\alpha/(p-1)}}\big)^{1-1/p}\to 0 \quad \text{as } n\to\infty, & 1<p\leqslant 2.\end{cases}
\end{aligned}
$$

The proof is completed. □

REMARK 3.2. The condition (3.10) is weaker than (4.8) in Corollary 4.5 by Hu et al. (2021), and the rest of the conditions are same, but (3.12) is much stronger than (4.10). Therefore, Corollary 3.3 improves Corollary 4.5 in Hu et al. (2021).

4. Examples

The following example is a modification of Example 5.3 of Hu et al. (2021). We will discuss in two cases. Case I illustrates Theorem 2.1 and Case II shows that Theorem 2.1 can fail if the condition (2.6) is not met.

EXAMPLE 4.1. Take $k_n = 2n$ and $c_n = n^{-1}$ for $n \geqslant 1$ and let $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent random variables such that

$$P\left(V_{n,k}=-\frac{1}{(n+2)^\gamma}\right)=\frac{n^2+2n}{(n+1)^2} \quad\text{and}\quad P\big(V_{n,k}=n^\theta\big)=\frac{1}{(n+1)^2},\quad 1\leqslant k\leqslant n$$

and

$$P\left(V_{n,k}=\frac{1}{(n+2)^\gamma}\right)=\frac{n^2+2n}{(n+1)^2} \quad\text{and}\quad P\big(V_{n,k}=-n^\theta\big)=\frac{1}{(n+1)^2},\quad n+1\leqslant k\leqslant 2n,$$

where $\gamma \geqslant 1$ and $0 < \theta < 1/q$.

Case I: $\gamma > 1$

Firstly, for arbitrary $\varepsilon > 0$ and all large n, we have by $\theta q < 1$ that

$$\begin{aligned}\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|V_{n,k}|^q I(|V_{n,k}| > \varepsilon) &= \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{2n} E|V_{n,k}|^q I(|V_{n,k}| > \varepsilon)\\ &= \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{2n} n^{\theta q} P(|V_{n,k}| = n^{\theta})\\ &= 2\sum_{n=1}^{\infty} \frac{n^{\theta q}}{(n+1)^2} < \infty,\end{aligned}$$

then (2.3) is verified.

Secondly, let $p = 2$, $J > 1$ and $\delta_1 > 0$. For all large n, it follows by $\gamma > 1$ that

$$\begin{aligned}&\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EV_{n,k}^2 I(|V_{n,k}| \leqslant \delta_1)\right)^J\\ &= \sum_{n=1}^{\infty} n^{-1} \left(\sum_{k=1}^{2n} EV_{n,k}^2 I(|V_{n,k}| \leqslant \delta_1)\right)^J\\ &= \sum_{n=1}^{\infty} n^{-1} \left(n\left(-\frac{1}{(n+2)^{\gamma}}\right)^2 \frac{n^2+2n}{(n+1)^2} + n\left(\frac{1}{(n+2)^{\gamma}}\right)^2 \frac{n^2+2n}{(n+1)^2}\right)^J\\ &< 2^J \sum_{n=1}^{\infty} \frac{n^{J-1}}{(n+2)^{2\gamma J}}\\ &< 2^J \sum_{n=1}^{\infty} \frac{1}{(n+2)^{(2\gamma-1)J+1}} < \infty,\end{aligned}$$

then (2.4) is also verified.

Thirdly, for some $\delta_1, \delta_2 > 0$ and all large n, we have by $\theta q < 1$ that

$$\begin{aligned}\sum_{k=1}^{k_n} E|V_{n,k}|^q I(|V_{n,k}| > \min\{\delta_1, \delta_2\}) &= \sum_{k=1}^{2n} E|V_{n,k}|^q I(|V_{n,k}| > \min\{\delta_1, \delta_2\})\\ &= \sum_{k=1}^{2n} n^{\theta q} P(|V_{n,k}| = n^{\theta})\\ &= \frac{2n^{\theta q+1}}{(n+1)^2} \to 0 \text{ as } n \to \infty,\end{aligned}$$

which shows (2.5) is verified.

Finally, for some $\delta_2 > 0$ and all large n, we have by $\gamma > 1$ that

$$\max_{1\leqslant k\leqslant k_n}\left|\sum_{i=1}^{k} EV_{n,i}I\big(|V_{n,i}|\leqslant\delta_2\big)\right| = \max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k} EV_{n,i}I\left(|V_{n,i}|=\frac{1}{(n+2)^{\gamma}}\right)\right|$$
$$\leqslant \sum_{i=1}^{2n} E|V_{n,i}|I\left(|V_{n,i}|=\frac{1}{(n+2)^{\gamma}}\right)$$
$$= \frac{2n}{(n+2)^{\gamma}}\times\frac{n^2+2n}{(n+1)^2}\to 0 \text{ as } n\to\infty$$

and so (2.6) is verified. Therefore, by Theorem 2.1,

$$\sum_{n=1}^{\infty} n^{-1}E\left\{\max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k}V_{n,i}\right|-\varepsilon\right\}_{+}^{q}<\infty \quad \text{for all } \varepsilon>0 \text{ and } 0<q\leqslant 2.$$

Case II: $\gamma = 1$

(2.3) and (2.5) can be verified by similar discussion in Case I, we need only consider (2.4) and (2.6). For (2.4), we also let $p = 2$, $J > 1$ and $\delta_1 > 0$. For all large n, it follows by $\gamma = 1$ that

$$\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} EV_{n,k}^2 I\big(|V_{n,k}|\leqslant\delta_1\big)\right)^J$$
$$=\sum_{n=1}^{\infty} n^{-1}\left(n\left(-\frac{1}{n+2}\right)^2\frac{n^2+2n}{(n+1)^2}+n\left(\frac{1}{n+2}\right)^2\frac{n^2+2n}{(n+1)^2}\right)^J$$
$$<2^J\sum_{n=1}^{\infty}\frac{1}{(n+2)^{J+1}}<\infty.$$

Next we will verify that (2.6) fails. For all $\delta_2 > 0$ and all large n, we find that

$$\max_{1\leqslant k\leqslant k_n}\left|\sum_{i=1}^{k} EV_{n,i}I\big(|V_{n,i}|\leqslant\delta_2\big)\right| = \max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k} EV_{n,i}I\left(|V_{n,i}|=\frac{1}{n+2}\right)\right|$$
$$\geqslant \left|\sum_{i=1}^{n} EV_{n,i}I\left(|V_{n,i}|=\frac{1}{n+2}\right)\right|$$
$$=\left|n\left(-\frac{1}{n+2}\right)\times\frac{n^2+2n}{(n+1)^2}\right|\to 1 \text{ as } n\to\infty$$

and so (2.6) fails.

Finally, noting that for all $n \geqslant 1$,

$$\left|n\left(-\frac{1}{n+2}\right)\right|>\frac{1}{4}.$$

Hence by similar discussion in Example 5.3 by Hu et al. (2021), we have

$$\begin{aligned}\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leqslant k\leqslant k_n}\left|\sum_{i=1}^{k} V_{n,i}\right|-\frac{1}{8}\right\}_+^q &= \sum_{n=1}^{\infty} n^{-1} E\left\{\max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k} V_{n,i}\right|-\frac{1}{8}\right\}_+^q \\ &= \sum_{n=1}^{\infty} n^{-1}\int_0^{\infty} P\left(\max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k} V_{n,i}\right|>\frac{1}{8}+t^{1/q}\right)\mathrm{d}t \\ &\geqslant \sum_{n=1}^{\infty} n^{-1}\int_0^{8^{-q}} P\left(\max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k} V_{n,i}\right|>\frac{1}{8}+t^{1/q}\right)\mathrm{d}t \\ &\geqslant 8^{-q}\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant k\leqslant 2n}\left|\sum_{i=1}^{k} V_{n,i}\right|>\frac{1}{4}\right) \\ &\geqslant 8^{-q}\sum_{n=1}^{\infty} n^{-1}\big(1+o(1)\big)e^{-1}=\infty,\end{aligned}$$

which shows (2.7) fails. Therefore, Theorem 2.1 can fail if (2.6) is not met.

The following example illustrates Corollary 3.2.

EXAMPLE 4.2. Let $\{V_n, n\geqslant 1\}$ be a sequence of independent and identically distributed random elements taking values in a real separable Rademacher type p $(1\leqslant p\leqslant 2)$ Banach space. Let $\alpha\geqslant 0$, $J>1$, $\lambda>(\alpha+1)(J-1)$ and $E\|V_1\|^{\max\{q,1\}}<\infty$ for some $q\in(0,p]$. Taking

$$V_{n,k}=\frac{V_k}{n^{\frac{\alpha+J+\lambda+1}{qJ}}},\quad 1\leqslant k\leqslant n,\ n\geqslant 1.$$

We will verify that the conditions (2.3), (3.6) and (3.7) of Corollary 3.2 hold with $k_n=n$ and $c_n=n^{\alpha}$.

Firstly, for all $\varepsilon>0$, we have by $\lambda>(\alpha+1)(J-1)$ that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha}\sum_{k=1}^{n} E\|V_{n,k}\|^q I(\|V_{n,k}\|>\varepsilon) &= \sum_{n=1}^{\infty} n^{\alpha}\sum_{k=1}^{n}\frac{E\|V_1\|^q}{n^{\frac{\alpha+J+\lambda+1}{J}}} I\big(\|V_1\|>\varepsilon n^{\frac{\alpha+J+\lambda+1}{qJ}}\big) \\ &\leqslant CE\|V_1\|^q\sum_{n=1}^{\infty} n^{\alpha+1-\frac{\alpha+J+\lambda+1}{J}} \\ &= CE\|V_1\|^q\sum_{n=1}^{\infty} n^{-1-\frac{\lambda-(\alpha+1)(J-1)}{J}}<\infty\end{aligned}$$

and so (2.3) is verified.

Secondly, we also obtain by $\lambda>(\alpha+1)(J-1)>0$ that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha}\left(\sum_{k=1}^{n} E\|V_{n,k}\|^q\right)^J &= \sum_{n=1}^{\infty} n^{\alpha}\left(\sum_{k=1}^{n}\frac{E\|V_1\|^q}{n^{\frac{\alpha+J+\lambda+1}{J}}}\right)^J \\ &= \big(E\|V_1\|^q\big)^J\sum_{n=1}^{\infty}\frac{1}{n^{\lambda+1}}<\infty\end{aligned}$$

and so (3.6) is verified.

Finally, we will verify (3.7). If $q \leqslant 1$, by $E\big|\big|V_1\big|\big|^{\max\{q,1\}} < \infty$, we have

$$\sum_{k=1}^{n} E\big|\big|V_{n,k}\big|\big|^{\max\{q,1\}} I\big(\big|\big|V_{n,k}\big|\big| > \delta\big) = \sum_{k=1}^{n} \frac{E\big|\big|V_1\big|\big|}{n^{\frac{\alpha+J+\lambda+1}{qJ}}} I\big(\big|\big|V_1\big|\big| > n^{\frac{\alpha+J+\lambda+1}{qJ}}\delta\big)$$
$$\leqslant E\big|\big|V_1\big|\big| n^{\frac{(q-1)J-(\alpha+1)-\lambda}{qJ}} \to 0 \quad \text{as } n \to \infty.$$

If $q > 1$, we also have by $E\big|\big|V_1\big|\big|^{\max\{q,1\}} < \infty$ that

$$\sum_{k=1}^{n} E\big|\big|V_{n,k}\big|\big|^{\max\{q,1\}} I\big(\big|\big|V_{n,k}\big|\big| > \delta\big) = \sum_{k=1}^{n} \frac{E\big|\big|V_1\big|\big|^q}{n^{\frac{\alpha+J+\lambda+1}{J}}} I\big(\big|\big|V_1\big|\big| > n^{\frac{\alpha+J+\lambda+1}{qJ}}\delta\big)$$
$$\leqslant E\big|\big|V_1\big|\big|^q n^{-\frac{\alpha+\lambda+1}{J}} \to 0 \quad \text{as } n \to \infty.$$

Thus, by Corollary 3.2,

$$\sum_{n=1}^{\infty} n^{\alpha} E\left\{ n^{-\frac{\alpha+J+\lambda+1}{qJ}} \max_{1\leqslant k\leqslant n} \left|\left|\sum_{i=1}^{k} V_i\right|\right| - \varepsilon \right\}_+^q < \infty \quad \text{for all } \varepsilon > 0.$$

The last example illustrates Corollary 3.3.

EXAMPLE 4.3. Let $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type 2 Banach space. Suppose that $\{V_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ is stochastically dominated by a random variable X with $EX^2 < \infty$.

For a real number x, the symbol $[x]$ denotes the maximal integer which is not more than x. Let $\alpha > 0$ and $0 < \lambda < 1$. Taking $p = 2$, $k_n = [n^{\lambda\alpha}]$ and $a_{n,k} = n^{-(\lambda+1)\alpha/2}$ in Corollary 3.3, we can verify that

$$k_n = o(n^{\alpha})$$

and

$$\sum_{k=1}^{k_n} |a_{n,k}|^p = \sum_{k=1}^{[n^{\lambda\alpha}]} a_{n,k}^2 = [n^{\lambda\alpha}]\, n^{-(\lambda+1)\alpha} = \mathcal{O}(n^{-\alpha}).$$

Then by Corollary 3.3, for all $\beta < \alpha - 1$,

$$\sum_{n=1}^{\infty} n^{\beta} E\left\{ n^{-(\lambda+1)\alpha/2} \max_{1\leqslant k\leqslant [n^{\lambda\alpha}]} \left|\left|\sum_{i=1}^{k} V_{n,i}\right|\right| - \varepsilon \right\}_+^q < \infty \quad \text{for all } \varepsilon > 0 \text{ and } 0 < q \leqslant 2.$$

Conflict of interest. The authors declare that they have no conflicts of interest.

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Yongfeng Wu
School of Mathematics and Computer Science
Tongling University
Anhui, Tongling, 244000, China
and
School of Mathematics and Finance
Chuzhou University
Anhui, Chuzhou, 239000, China

Xin Deng
School of Mathematics and Finance
Chuzhou University
Anhui, Chuzhou, 239000, China
e-mail: Tzdx0120@163.com