WEIGHTED MONTGOMERY IDENTITY AND
WEIGHTED HADAMARD INEQUALITIES

SANJA KOVAČ, JOSIP PEČARIĆ AND MIHAELA RIBIČIĆ PENAVA

(*Communicated by J. Matkowski*)

Abstract. In this paper the extension of the weighted Montgomery identity is established by using the integral formula of Pečarić, Matić and Ujević. Further, by using this extended weighted Montgomery identity for functions whose derivatives of order $n-1$ are absolutely continunous functions, new inequalities of the weighted Hermite-Hadamard type are obtained. Also, applications of these results are given for various types of weight function.

1. Introduction

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $P_w(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a} & \text{for } t \in [a,x], \\ \frac{t-b}{b-a} & \text{for } t \in (x,b], \end{cases} \tag{1.1}$$

the Montgomery identity ([6]) states

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt. \tag{1.2}$$

If $w : [a,b] \to [0,\infty)$ is some nonnegative integrable weight function, the weighted Montgomery identity ([7]) states

$$f(x) = \frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt + \int_a^b P_w(x,t)f'(t)dt \tag{1.3}$$

where $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)} & \text{for } t \in [a,x], \\ \frac{W(t)}{W(b)} - 1 & \text{for } t \in (x,b] \end{cases} \tag{1.4}$$

and $W(t) = \int_a^t w(s)ds$ for $t \in [a,b]$, $W(t) = 0$, for $t < a$ and $W(t) = W(b)$, for $t > b$. For the uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ the weighted Montgomery idenity reduces to the Montgomery identity.

Let $g : [a,b] \to \mathbb{R}$ be some function and $x \in [a,b]$. Let $w : [a,b] \to \mathbb{R}$ be some integrable function. The approximation of the integral $\int_a^b w(t)g(t)dt$ will involve the values of the higher order derivatives of g in the node x. We consider subdivision $\sigma = \{x_0 < x_1 < x_2\}$ of the interval $[a,b]$, where $x_0 = a$, $x_1 = x$ and $x_2 = b$. Further, let $\{w_{kj}\}_{j=0,1,\ldots,n}$ be $w-$harmonic sequences on each subinterval $[x_{k-1},x_k]$, $k = 1,2$, such that $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1,\ldots,n$.

In [4] the following theorem has been proved.

THEOREM 1. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and $x \in [a,b]$. Further, let us suppose $\{w_{kj}\}_{j=1\ldots,n}$ are $w-$harmonic sequences of functions on $[x_{k-1},x_k]$, for $k = 1,2$ and some $n \in \mathbb{N}$, defined by the following relations:*

$$w_{1j}(t) := \frac{1}{(j-1)!} \int_a^t (t-s)^{j-1} w(s)ds, \quad t \in [a,x]$$

$$w_{2j}(t) := \frac{1}{(j-1)!} \int_b^t (t-s)^{j-1} w(s)ds, \quad t \in (x,b],$$

for $j = 1,\ldots,n$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous function, then we have

$$\int_a^b w(t)g(t)dt = \sum_{j=1}^n A_j(x)g^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}(t,x)g^{(n)}(t)dt,$$

(1.5)

where for $j = 1,\ldots,n$

$$A_j(x) = \frac{(-1)^{j-1}}{(j-1)!} \int_a^b (x-s)^{j-1} w(s)ds$$

(1.6)

and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} w(s)ds \ for \ t \in [a,x], \\ w_{2n}(t) = \frac{1}{(n-1)!} \int_b^t (t-s)^{n-1} w(s)ds \ for \ t \in (x,b]. \end{cases}$$

(1.7)

REMARK 1. The identity (1.5) was also obtained in [5] and we call it an integral formula of Matić, Pečarić and Ujević.

In [8] and [9] the weighted Hermite-Hadamard inequality for convex functions is given.

THEOREM 2. *Let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I, then we have*

$$f(\lambda) \leqslant \frac{1}{P(b)} \int_a^b p(x)f(x)dx \leqslant \frac{b-\lambda}{b-a}f(a) + \frac{\lambda-a}{b-a}f(b)$$

(1.8)

or

$$P(b)f(\lambda) \leqslant \int_a^b p(x)f(x)dx \leqslant P(b)\left[\frac{b-\lambda}{b-a}f(a) + \frac{\lambda-a}{b-a}f(b)\right] \qquad (1.9)$$

where

$$P(t) = \int_a^t p(x)dx \quad and \quad \lambda = \frac{1}{P(b)}\int_a^b xp(x)dx. \qquad (1.10)$$

THEOREM 3. (The Fejér inequalities) *Let* $p:[a,b] \to \mathbb{R}$ *be nonnegative, integrable and symmetric about* $\frac{a+b}{2}$. *If* $f:[a,b] \to \mathbb{R}$ *is a convex function, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqslant \int_a^b p(x)f(x)\,dx \leqslant \left[\frac{1}{2}f(a) + \frac{1}{2}f(b)\right]\int_a^b p(x)\,dx. \quad (1.11)$$

If f is a concave function, then the inequalities in (1.11) *are reversed.*

Some recent results on Hermite-Hadamard and Fejér inequalities can be found in [2] and [3]. The aim of this paper is to give the extension of the weighted Montgomery identity and to obtain certain Hermite-Hadamard type and Fejér type inequalities. Further, some applications for special cases of weight functions are given.

2. Main result

For integrable function $w:[a,b] \to \mathbb{R}$ we put $W_0 = \int_a^b w(t)dt$ and $\lambda = \frac{1}{W_0}\int_a^b tw(t)dt$. For $n \geqslant 2$ we put

$$T_{n,w}(x) = \sum_{j=2}^n A_j(x)g^{(j-1)}(x).$$

Let $g:[a,b] \to \mathbb{R}$ be such that $g^{(n-1)}$ is absolutely continuous function, then the identity (1.5) states:

$$g(x) = \frac{1}{W_0}\int_a^b w(t)g(t)dt - \frac{T_{n,w}(x)}{W_0} - \frac{(-1)^n}{W_0}\int_a^b W_{n,w}(t,x)g^{(n)}(t)dt, \ \ x \in [a,b]. \quad (2.1)$$

REMARK 2. If we put $n = 1$ in (2.1), then we obtain the weighted Montgomery identity (1.3).

REMARK 3. If we put $n = 2$ in (2.1), then we get the following identiy:

$$g(x) = \frac{1}{W_0}\int_a^b w(t)g(t)dt - (\lambda - x)g'(x) - \frac{1}{W_0}\int_a^b W_{2,w}(t,x)g''(t)dt. \qquad (2.2)$$

THEOREM 4. *Let* $g:[a,b] \to \mathbb{R}$ *be such that* $g^{(n-1)}$ *is absolutely continuous function, for* $n \in \mathbb{N}$, *and let* $w:[a,b] \to [0,\infty)$ *be an integrable function. Then we have the following identity:*

$$\frac{1}{W_0}\int_a^b w(t)g(t)dt - g(\lambda) = \frac{T_{n,w}(\lambda)}{W_0} + \frac{(-1)^n}{W_0}\int_a^b W_{n,w}(t,\lambda)g^{(n)}(t)dt. \qquad (2.3)$$

Proof. We apply $x = \lambda$ in (2.1). $\quad\square$

Now, let us consider the special case for $n = 2$.

COROLLARY 1. *Let $g : [a,b] \to \mathbb{R}$ be such that g' is absolutely continuous function and let $w : [a,b] \to [0,\infty)$ be an integrable function. Then we have the following identity:*

$$\frac{1}{W_0}\int_a^b w(t)g(t)dt - g(\lambda) = \frac{1}{W_0}\int_a^b W_{2,w}(t,\lambda)g''(t)dt. \tag{2.4}$$

If, additionally, g is convex function and g'' exists on (a,b), then the following inequality holds

$$\frac{1}{W_0}\int_a^b w(t)g(t)dt - g\left(\frac{1}{W_0}\int_a^b tw(t)dt\right) \geqslant 0 \tag{2.5}$$

and the identity in (2.5) holds if g is linear function.

Proof. We apply Theorem 4 for $n = 2$. Then we have $T_{2,w}(\lambda) = 0$, so the identity (2.4) follows.

If g is convex function and $g''(x)$ exists, then $g''(x) \geqslant 0$ on $[a,b]$ (see [9]). Since w is nonnegative function on $[a,b]$, then it is easy to show that $W_{2,w}(t,\lambda) \geqslant 0$, so the inequality (2.5) is valid.

If g is linear function, then $g(x) = kx + l$ for some $k,l \in \mathbb{R}$. Now we have

$$\frac{1}{W_0}\int_a^b w(t)g(t)dt - g\left(\frac{1}{W_0}\int_a^b tw(t)dt\right)$$

$$= \frac{1}{W_0}\int_a^b w(t)(kt+l)dt - \left(k\cdot\frac{1}{W_0}\int_a^b tw(t)dt + l\right)$$

$$= \frac{k}{W_0}\int_a^b tw(t)dt + \frac{l}{W_0}\int_a^b w(t)dt - \frac{k}{W_0}\int_a^b tw(t)dt - l$$

$$= \frac{l}{W_0}\cdot W_0 - l = l - l = 0. \quad\square$$

REMARK 4. The inequality (2.5) is the special case of the lefthand side of the weighted Hermite-Hadamard inequality for the case where g is convex function whose the first derivative is absolutely continuous function and the second derivative exists.

THEOREM 5. *Let $g : [a,b] \to \mathbb{R}$ be such that $g^{(n-1)}$ is absolutely continuous function, for $n \in \mathbb{N}$, and let $w : [a,b] \to [0,\infty)$ be an integrable function. Then we have the following identity:*

$$\frac{b-\lambda}{b-a}g(a) + \frac{\lambda-a}{b-a}g(b) - \frac{1}{W_0}\int_a^b w(t)g(t)dt$$

$$= -\frac{(b-\lambda)T_{n,w}(a) + (\lambda-a)T_{n,w}(b)}{W_0(b-a)}$$

$$- \frac{(-1)^n}{W_0}\int_a^b\left(\frac{b-\lambda}{b-a}W_{n,w}(t,a) + \frac{\lambda-a}{b-a}W_{n,w}(t,b)\right)g^{(n)}(t)dt. \tag{2.6}$$

Proof. We apply $x = a$ and $x = b$ in (2.1) and multiply it by $\frac{b-\lambda}{b-a}$ and $\frac{\lambda-a}{b-a}$ respectively, and add those two identities. $\square$

COROLLARY 2. *Let $g : [a,b] \to \mathbb{R}$ be such that g' is absolutely continuous function and let $w : [a,b] \to [0,\infty)$ be an integrable function. Then we have the following identity*

$$\frac{b-\lambda}{b-a}g(a) + \frac{\lambda-a}{b-a}g(b) - \frac{1}{W_0}\int_a^b w(t)g(t)dt = \frac{1}{W_0(b-a)}\int_a^b y(t)g''(t)dt, \quad (2.7)$$

where

$$y(t) = (\lambda - a)\int_a^t (b - \lambda - t + s)w(s)ds + (b - \lambda)\int_t^b (\lambda - a - s + t)w(s)ds,$$

for $t \in [a,b]$.

If, additionally, g is convex function and g'' exists on (a,b), then we have

$$\frac{1}{W_0}\int_a^b w(t)g(t)dt \leqslant \frac{b-\lambda}{b-a}g(a) + \frac{\lambda-a}{b-a}g(b) \quad (2.8)$$

and the identity in (2.8) holds if g is linear function.

Proof. We apply Theorem 5 for $n = 2$. Then we have

$$-\frac{(b-\lambda)T_{2,w}(a) + (\lambda-a)T_{2,w}(b)}{W_0(b-a)} = \frac{(b-\lambda)(\lambda-a)}{b-a}\left(g'(b) - g'(a)\right)$$

$$= \frac{(b-\lambda)(\lambda-a)}{b-a}\int_a^b g''(t)dt$$

and

$$-\frac{1}{W_0}\int_a^b \left(\frac{b-\lambda}{b-a}W_{2,w}(t,a) + \frac{\lambda-a}{b-a}W_{2,w}(t,b)\right)g''(t)dt \quad (2.9)$$

$$= -\frac{1}{W_0(b-a)}\int_a^b \left[(b-\lambda)\int_b^t (t-s)w(s)ds + (\lambda-a)\int_a^t (t-s)w(s)ds\right]g''(t)dt$$

$$= -\frac{1}{W_0(b-a)}\int_a^b \left[(b-\lambda)\int_t^b (s-t)w(s)ds + (\lambda-a)\int_a^t (t-s)w(s)ds\right]g''(t)dt.$$

Now, we add last two identities to obtain

$$-\frac{(b-\lambda)T_{2,w}(a) + (\lambda-a)T_{2,w}(b)}{W_0(b-a)}$$

$$-\frac{1}{W_0}\int_a^b \left(\frac{b-\lambda}{b-a}W_{2,w}(t,a) + \frac{\lambda-a}{b-a}W_{2,w}(t,b)\right)g''(t)dt$$

$$= \frac{1}{W_0(b-a)}\int_a^b y(t)g''(t)dt \quad (2.10)$$

so the identity (2.7) holds.

If g is convex function and $g''(x)$ exists, then $g''(x) \geqslant 0$ on $[a,b]$ (see [9]). Now, we have to prove that $y(t) \geqslant 0$ for $t \in [a,b]$. After some calculation, we get

$$y(a) = y(b) = 0.$$

The first derivative y' equals:

$$y'(t) = (b - \lambda) \int_t^b w(s)ds - (\lambda - a) \int_a^t w(s)ds$$

and the second derivative $y''(t)$ equals

$$y''(t) = -(b-a)w(t).$$

Since $y'(a) \geqslant 0$ and $y'(b) \leqslant 0$ and the function y is concave on $[a,b]$, we conclude that $y(t) \geqslant 0$, for every $t \in [a,b]$.

If g is linear function, then there exists $k,l \in \mathbb{R}$ such that $g(x) = kx + l$. Then the lefthand side and the righthand side of (2.8) equals to $g(\lambda)$, so the identity in (2.8) holds. $\square$

REMARK 5. The inequality (2.8) is the special case of the righthand side of the weighted Hermite-Hadamard inequality for the case where g is convex function whose the first derivative is absolutely continuous function and the second derivative exists.

Let us consider the case for symmetric function w.

THEOREM 6. *Let* $g : [a,b] \to \mathbb{R}$ *be such that* $g^{(n-1)}$ *is absolutely continuous function, for* $n \in \mathbb{N}$, *and let* $w : [a,b] \to [0,\infty)$ *be an integrable function such that* $w(a+b-t) = w(t)$ *for every* $t \in [a,b]$. *Then we have the following identities:*

$$\frac{1}{W_0} \int_a^b w(t)g(t)dt - g\left(\frac{a+b}{2}\right)$$
$$= \frac{T_{n,w}(\frac{a+b}{2})}{W_0} + \frac{(-1)^n}{W_0} \int_a^b W_{n,w}\left(t, \frac{a+b}{2}\right) g^{(n)}(t)dt \qquad (2.11)$$

and

$$\frac{g(a) + g(b)}{2} - \frac{1}{W_0} \int_a^b w(t)g(t)dt$$
$$= -\frac{T_{n,w}(a) + T_{n,w}(b)}{2W_0} - \frac{(-1)^n}{2W_0(n-1)!} \int_a^b z(t)g^{(n)}(t)dt, \qquad (2.12)$$

where

$$z(t) = \begin{cases} \int_a^b |t-s|^{n-1} w(s)ds, & \text{for } n \text{ even} \\ \int_a^t |t-s|^{n-1} w(s)ds - \int_t^b |t-s|^{n-1} w(s)ds, & \text{for } n \text{ odd.} \end{cases} \qquad (2.13)$$

Proof. If w is symmetric function on $[a,b]$, then is easy to show that

$$W_0 = 2 \cdot \int_a^{\frac{a+b}{2}} w(t)dt$$

and

$$
\begin{aligned}
\lambda &= \frac{1}{W_0} \int_a^b tw(t)dt = \frac{1}{W_0} \int_a^{\frac{a+b}{2}} tw(t)dt + \frac{1}{W_0} \int_{\frac{a+b}{2}}^b tw(t)dt \\
&= \frac{1}{W_0} \int_a^{\frac{a+b}{2}} tw(t)dt + \frac{1}{W_0} \int_a^{\frac{a+b}{2}} (a+b-s)w(s)ds \\
&= \frac{1}{W_0} \int_a^{\frac{a+b}{2}} tw(t)dt + \frac{a+b}{W_0} \int_a^{\frac{a+b}{2}} w(s)ds - \frac{1}{W_0} \int_a^{\frac{a+b}{2}} sw(s)ds \\
&= \frac{a+b}{2}.
\end{aligned}
$$

The first identity follows from (2.3) and the second identity follows from (2.6) by applying $x = \lambda = \frac{a+b}{2}$. $\quad\square$

Then for $n = 2$ we get the Fejér inequalities:

COROLLARY 3. *Let* $g : [a,b] \to \mathbb{R}$ *be such that* g' *is absolutely continuous function and let* $w : [a,b] \to [0,\infty)$ *be an integrable function such that* $w(a+b-t) = w(t)$ *for every* $t \in [a,b]$. *Then the following identities hold:*

$$\frac{1}{W_0} \int_a^b w(t)g(t)dt - g\left(\frac{a+b}{2}\right) = \frac{1}{W_0} \int_a^b W_{2,w}\left(t, \frac{a+b}{2}\right) g''(t)dt \tag{2.14}$$

and

$$\frac{g(a)+g(b)}{2} - \frac{1}{W_0} \int_a^b w(t)g(t)dt = \int_a^b y(t)g''(t)dt, \tag{2.15}$$

where

$$y(t) = \frac{b-a}{4} - \frac{1}{2W_0} \int_a^b |t-s| w(s)ds.$$

If g *is convex function and* g'' *exists, then the following inequalities hold:*

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{W_0} \int_a^b w(t)g(t)dt \leqslant \frac{g(a)+g(b)}{2}. \tag{2.16}$$

If g *is concave function and* g'' *exists, then the inequalities in* (2.16) *are reversed.*

Proof. First, since w is symmetric function, by applying $n = 2$ to the identity (2.11) we get $T_{2,w}\left(\frac{a+b}{2}\right) = 0$, so the identity (2.14) is valid. Further, we apply $n = 2$ to

the righthand side of the identity (2.12) and get:

$$-\frac{T_{2,w}(a) + T_{2,w}(b)}{2W_0} - \frac{1}{2W_0}\int_a^b z(t)g''(t)dt$$

$$= -\frac{W_0\frac{b-a}{2}g'(a) - W_0\frac{b-a}{2}g'(b)}{2W_0} - \frac{1}{2W_0}\int_a^b \left(\int_a^b |t-s|w(s)ds\right)g''(t)dt$$

$$= \frac{b-a}{4}\left(g'(b) - g'(a)\right) - \frac{1}{2W_0}\int_a^b \left(\int_a^b |t-s|w(s)ds\right)g''(t)dt$$

$$= \frac{b-a}{4}\int_a^b g''(s)ds - \frac{1}{2W_0}\int_a^b \left(\int_a^b |t-s|w(s)ds\right)g''(t)dt$$

$$= \int_a^b \left(\frac{b-a}{4} - \frac{1}{2W_0}\int_a^b |t-s|w(s)ds\right)g''(t)dt \tag{2.17}$$

so the identity (2.15) holds.

The inequalities (2.16) holds from the inequalities (2.5) and (2.8) with $\lambda = \frac{a+b}{2}$. □

3. Examples

Using general identities (2.3), (2.6), (2.11) and (2.12) obtained in the previous section we shall give some special cases for various types of weight functions. Further, we obtain the Hermite-Hadamard type and Fejér type inequalities for these weight functions, as special cases of Corollary 1, 2 and 3.

EXAMPLE 1. $w(t) = 1$, for $t \in [a,b]$.
Since w is symmetric function, we can apply Theorem 6 to get identities:

$$\frac{1}{b-a}\int_a^b g(t)dt - g\left(\frac{a+b}{2}\right)$$

$$= \frac{T_{n,w}(\frac{a+b}{2})}{b-a} + \frac{(-1)^n}{b-a}\int_a^b W_{n,w}\left(t, \frac{a+b}{2}\right)g^{(n)}(t)dt \tag{3.1}$$

and

$$\frac{g(a) + g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)dt$$

$$= -\frac{T_{n,w}(a) + T_{n,w}(b)}{2(b-a)} - \frac{(-1)^n}{2(b-a)n!}\int_a^b \left((t-b)^n + (t-a)^n\right)g^{(n)}(t)dt,$$

where

$$T_{n,w}\left(\frac{a+b}{2}\right) = \sum_{\substack{j=2 \\ j \text{ odd}}}^n \frac{(b-a)^j}{2^{j-1}(j-1)!}g^{(j-1)}\left(\frac{a+b}{2}\right), \tag{3.2}$$

$$\frac{T_{n,w}(a) + T_{n,w}(b)}{2(b-a)} = \sum_{j=2}^n \frac{(-1)^{j-1}(b-a)^{j-1}}{2j!}\cdot\left(g^{(j-1)}(a) + g^{(j-1)}(b)\right)$$

and

$$W_{n,w}\left(t,\frac{a+b}{2}\right) = \begin{cases} w_{1n}(t) = \frac{(t-a)^n}{n!} & \text{for } t \in \left[a,\frac{a+b}{2}\right], \\ w_{2n}(t) = \frac{(t-b)^n}{n!} & \text{for } t \in \left(\frac{a+b}{2},b\right]. \end{cases} \tag{3.3}$$

Specially, for $n = 2$ we get the identities from the Corollary 3

$$\frac{1}{b-a}\int_a^b g(t)dt - g\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b W_{2,w}\left(t,\frac{a+b}{2}\right)g''(t)dt \tag{3.4}$$

and

$$\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)dt = \int_a^b y(t)g''(t)dt, \tag{3.5}$$

where

$$y(t) = \frac{(b-t)(t-a)}{2(b-a)}.$$

It is easy to check that $W_{2,w}(t,\frac{a+b}{2}) \geqslant 0$ and $y(t) \geqslant 0$, for every $t \in [a,b]$. Therefore, if g is a convex function and g'' exists, then the following inequalities hold:

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b g(t)dt \leqslant \frac{g(a)+g(b)}{2},$$

which are the classical Hermite-Hadamard inequalities.

EXAMPLE 2. $w(t) = \frac{1}{\sqrt{1-t^2}}$, for $t \in (-1,1)$.

Since w is symmetric function, we can apply Theorem 6 to get identities:

$$\frac{1}{\pi}\int_{-1}^1 \frac{g(t)}{\sqrt{1-t^2}}dt - g(0) = \frac{T_{n,w}(0)}{\pi} + \frac{(-1)^n}{\pi}\int_{-1}^1 W_{n,w}(t,0)g^{(n)}(t)dt \tag{3.6}$$

and

$$\frac{g(-1)+g(1)}{2} - \frac{1}{\pi}\int_{-1}^1 \frac{g(t)}{\sqrt{1-t^2}}dt$$
$$= -\frac{T_{n,w}(-1)+T_{n,w}(1)}{2\pi} - \frac{(-1)^n}{2\pi(n-1)!}\int_{-1}^1 z(t)g^{(n)}(t)dt,$$

where

$$T_{n,w}(x) = \sum_{j=2}^n A_j(x)g^{(j-1)}(x) \tag{3.7}$$

and

$$A_j(x) = \begin{cases} \frac{(-1-x)^{j-1}\pi}{(j-1)!}F(1-j,\frac{1}{2},1;\frac{2}{x+1}), & x \neq -1, \\ \frac{2^{j-1}}{(j-1)!}B(\frac{1}{2},j-\frac{1}{2}), & x = -1 \end{cases}$$

and

$$z(t) = \begin{cases} \int_{-1}^{1} \frac{|t-s|^{n-1}}{\sqrt{1-s^2}} ds, & \text{for } n \text{ even} \\ \int_{-1}^{t} \frac{|t-s|^{n-1}}{\sqrt{1-s^2}} ds - \int_{t}^{1} \frac{|t-s|^{n-1}}{\sqrt{1-s^2}} ds, & \text{for } n \text{ odd.} \end{cases} \tag{3.8}$$

Here B is Beta function

$$B(p,q) = \int_{0}^{1} t^{p-1}(1-t)^{q-1} dt, \quad Re(p), Re(q) > 0$$

and F is hypergeometric function

$$F(a,b,c;x) = \frac{1}{B(b,c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt,$$

for $c > b > 0$.

Specially, for $n = 2$ we get the identities from the Corollary 3:

$$\frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}} dt - g(0) = \frac{1}{\pi} \int_{-1}^{1} W_{2,w}(t,0)g''(t) dt \tag{3.9}$$

and

$$\frac{g(-1)+g(1)}{2} - \frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}} dt = \int_{-1}^{1} y(t)g''(t) dt, \tag{3.10}$$

where

$$y(t) = \frac{1}{2} - \frac{1}{2\pi} \int_{-1}^{1} \frac{|t-s|}{\sqrt{1-s^2}} ds.$$

It is easy to check that $W_{2,w}(t,0) \geqslant 0$ and $y(t) \geqslant 0$, for every $t \in (-1,1)$. So if g is a convex function and g'' exists, then the following inequalities hold:

$$g(0) \leqslant \frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}} dt \leqslant \frac{g(-1)+g(1)}{2},$$

which are the Fejér inequalities for special weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$.

EXAMPLE 3. $w(t) = \sqrt{1-t^2}$, for $t \in [-1,1]$.
Since w is symmetric function, we can apply Theorem 6 to get identities:

$$\frac{2}{\pi} \int_{-1}^{1} g(t)\sqrt{1-t^2} dt - g(0)$$
$$= \frac{2 \cdot T_{n,w}(0)}{\pi} + \frac{(-1)^n \cdot 2}{\pi} \int_{-1}^{1} W_{n,w}(t,0)g^{(n)}(t) dt \tag{3.11}$$

and

$$\frac{g(-1)+g(1)}{2} - \frac{2}{\pi} \int_{-1}^{1} g(t)\sqrt{1-t^2} dt$$
$$= -\frac{T_{n,w}(-1)+T_{n,w}(1)}{\pi} - \frac{(-1)^n}{\pi(n-1)!} \int_{-1}^{1} z(t)g^{(n)}(t) dt,$$

where

$$T_{n,w}(x) = \sum_{j=2}^{n} A_j(x) g^{(j-1)}(x), \tag{3.12}$$

$$A_j(x) = \begin{cases} \frac{4(-1-x)^{j-1}}{(j-1)!} B(\frac{3}{2},\frac{3}{2}) F(1-j,\frac{3}{2},3;\frac{2}{x+1}), & x \neq -1, \\ \frac{2^{j+1}}{(j-1)!} B(\frac{3}{2},j+\frac{1}{2}), & x = -1 \end{cases}$$

and

$$z(t) = \begin{cases} \int_{-1}^{1} |t-s|^{n-1}\sqrt{1-s^2}ds, & \text{for } n \text{ even} \\ \int_{-1}^{1} |t-s|^{n-1}\sqrt{1-s^2}ds - \int_t^1 |t-s|^{n-1}\sqrt{1-s^2}ds, & \text{for } n \text{ odd}. \end{cases} \tag{3.13}$$

Specially, for $n=2$ get the identities from the Corollary 3:

$$\frac{2}{\pi}\int_{-1}^{1} g(t)\sqrt{1-t^2}dt - g(0) = \frac{2}{\pi}\int_{-1}^{1} W_{2,w}(t,0)g''(t)dt \tag{3.14}$$

and

$$\frac{g(-1)+g(1)}{2} - \frac{2}{\pi}\int_{-1}^{1} g(t)\sqrt{1-t^2}dt = \int_{-1}^{1} y(t)g''(t)dt, \tag{3.15}$$

where

$$y(t) = \frac{1}{2} - \frac{1}{\pi}\int_{-1}^{1} |t-s|\cdot\sqrt{1-s^2}ds.$$

Since $w(t)$ is nonnegative function on $[-1,1]$, it is easy to check that $W_{2,w}(t,0) \geqslant 0$ and $y(t) \geqslant 0$, so if g is a convex function and g'' exists, then the following inequalities hold:

$$g(0) \leqslant \frac{2}{\pi}\int_{-1}^{1} g(t)\sqrt{1-t^2}dt \leqslant \frac{g(-1)+g(1)}{2},$$

which are the Fejér inequalities for special weight function $w(t) = \sqrt{1-t^2}$.

EXAMPLE 4. $w(t) = \sqrt{t}$, for $t \in [0,1]$.
First we apply Theorem 4 to get identity:

$$\frac{3}{2}\int_0^1 g(t)\sqrt{t}dt - g\left(\frac{3}{5}\right)$$
$$= \frac{3T_{n,w}(3/5)}{2} + \frac{(-1)^n \cdot 3}{2}\int_0^1 W_{n,w}\left(t,\frac{3}{5}\right)g^{(n)}(t)dt \tag{3.16}$$

and after applying Theorem 5 we get:

$$\frac{2}{5}g(0) + \frac{3}{5}g(1) - \frac{3}{2}\int_0^1 g(t)\sqrt{t}dt$$
$$= -\frac{6T_{n,w}(0)+9T_{n,w}(1)}{10} - \frac{3}{2}\int_0^1 \left(\frac{2}{5}W_{n,w}(t,0) + \frac{3}{5}W_{n,w}(t,1)\right)g^{(n)}(t)dt,$$

where

$$W_{n,w}(t,x) = \begin{cases} \frac{t^{n+1/2}}{(n-1)!}B(n,\frac{3}{2}) & \text{for } t \in [0,x], \\ \frac{(t-1)^n}{n!}F(-\frac{1}{2},1,n+1;1-t) & \text{for } t \in (x,1], \end{cases}$$

$$T_{n,w}(x) = \sum_{j=2}^n A_j(x)g^{(j-1)}(x) \tag{3.17}$$

and

$$A_j(x) = \begin{cases} \frac{(-x)^j}{(j-1)!}B(1,\frac{3}{2})F(1-j,\frac{3}{2},\frac{5}{2},\frac{1}{x}), & x \neq 0 \\ \frac{1}{(j-1)!}B(1,j+\frac{1}{2}), & x = 0. \end{cases}$$

Specially, for $n = 2$ we get the identities from the Corollaries 1 and 2:

$$\frac{3}{2}\int_0^1 g(t)\sqrt{t}dt - g\left(\frac{3}{5}\right) = \frac{3}{2}\int_0^1 W_{2,w}\left(t,\frac{3}{5}\right)g''(t)dt \tag{3.18}$$

and

$$\frac{2}{5}g(0) + \frac{3}{5}g(1) - \frac{3}{2}\int_0^1 g(t)\sqrt{t}dt = \frac{3}{2}\int_0^1 y(t)g''(t)dt,$$

where

$$y(t) = \frac{3}{5}\int_0^t \left(\frac{2}{5}-t+s\right)\sqrt{s}ds + \frac{2}{5}\int_t^1 \left(\frac{3}{5}-s+t\right)\sqrt{s}ds.$$

Since $w(t) \geqslant 0$ on $[0,1]$, we conclude after the Corollaries 1 and 2 that $W_{2,w}(t,\frac{3}{5}) \geqslant 0$ and $y(t) \geqslant 0$. So if g is a convex function and g'' exists, then the following inequalities hold:

$$g\left(\frac{3}{5}\right) \leqslant \frac{3}{2}\int_0^1 g(t)\sqrt{t}dt \leqslant \frac{2g(0)+3g(1)}{5},$$

which are the Fejér inequalities for special weight function $w(t) = \sqrt{t}$.

EXAMPLE 5. $w(t) = \sqrt{\frac{1-t}{1+t}}$, for $t \in (-1,1]$.

First we apply Theorem 4 to get identity:

$$\frac{1}{\pi}\int_{-1}^1 g(t)\sqrt{\frac{1-t}{1+t}}dt - g\left(-\frac{1}{2}\right)$$

$$= \frac{T_{n,w}(-1/2)}{\pi} + \frac{(-1)^n}{\pi}\int_{-1}^1 W_{n,w}\left(t,-\frac{1}{2}\right)g^{(n)}(t)dt \tag{3.19}$$

and after applying Theorem 5 we get:

$$\frac{3}{4}g(-1) + \frac{1}{4}g(1) - \frac{1}{\pi}\int_{-1}^1 g(t)\sqrt{\frac{1-t}{1+t}}dt$$

$$= -\frac{3T_{n,w}(-1) + T_{n,w}(1)}{4\pi} - \frac{1}{\pi}\int_{-1}^1 \left(\frac{3}{4}W_{n,w}(t,-1) + \frac{1}{4}W_{n,w}(t,1)\right)g^{(n)}(t)dt,$$

where

$$W_{n,w}(t,x) = \begin{cases} \frac{2^{\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(-\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}) & \text{for } t \in [-1,x], \\ (-1)^n\frac{2^{-\frac{1}{2}}(1-t)^{n+\frac{1}{2}}}{(n-1)!}B(\frac{3}{2},n)F(\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{1-t}{2}) & \text{for } t \in (x,1], \end{cases}$$

$$T_{n,w}(x) = \sum_{j=2}^{n} A_j(x)g^{(j-1)}(x) \tag{3.20}$$

and

$$A_j(x) = \begin{cases} \frac{2(-1-x)^{j-1}}{(j-1)!}B(\frac{3}{2},\frac{1}{2})F(1-j,\frac{1}{2},2,\frac{2}{x+1}), & x \neq -1 \\ \frac{2^j}{(j-1)!}B(\frac{3}{2},j-\frac{1}{2}), & x = -1. \end{cases}$$

Specially, for $n = 2$ we get the identities from the Corollaries 1 and 2:

$$\frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1-t}{1+t}}dt - g\left(-\frac{1}{2}\right) = \frac{1}{\pi}\int_{-1}^{1}W_{2,w}\left(t,-\frac{1}{2}\right)g''(t)dt \tag{3.21}$$

and

$$\frac{3}{4}g(-1) + \frac{1}{4}g(1) - \frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1-t}{1+t}}dt = \frac{1}{2\pi}\int_{-1}^{1}y(t)g''(t)d,t$$

where

$$y(t) = \frac{1}{2}\int_{-1}^{t}\left(\frac{3}{2}-t+s\right)\sqrt{\frac{1-s}{1+s}}ds + \frac{3}{2}\int_{t}^{1}\left(\frac{1}{2}-s+t\right)\sqrt{\frac{1-s}{1+s}}ds.$$

Since $w(t) \geq 0$ on $[0,1]$, we conclude after the Corollaries 1 and 2 that $W_{2,w}(t,-\frac{1}{2}) \geq 0$ and $y(t) \geq 0$. So if g is a convex function and g'' exists, then the following inequalities hold:

$$g\left(-\frac{1}{2}\right) \leq \frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1-t}{1+t}}dt \leq \frac{3g(-1)+g(1)}{4},$$

which are the Fejér inequalities for special weight function $w(t) = \sqrt{\frac{1-t}{1+t}}$.

EXAMPLE 6. $w(t) = \frac{1}{\sqrt{t}}$, for $t \in (0,1]$.
First we apply Theorem 4 to get identity:

$$\frac{1}{2}\int_{0}^{1}\frac{g(t)}{\sqrt{t}}dt - g\left(\frac{1}{3}\right) = \frac{T_{n,w}(1/3)}{2} + \frac{(-1)^n}{2}\int_{0}^{1}W_{n,w}\left(t,\frac{1}{3}\right)g^{(n)}(t)dt \tag{3.22}$$

and after applying Theorem 5 we get:

$$\frac{2}{3}g(0) + \frac{1}{3}g(1) - \frac{1}{2}\int_{0}^{1}\frac{g(t)}{\sqrt{t}}dt$$
$$= -\frac{2T_{n,w}(0)+T_{n,w}(1)}{6} - \frac{1}{2}\int_{0}^{1}\left(\frac{2}{3}W_{n,w}(t,0)+\frac{1}{3}W_{n,w}(t,1)\right)g^{(n)}(t)dt,$$

where

$$W_{n,w}(t,x) = \begin{cases} \frac{t^{n-1/2}}{(n-1)!}B(n,\frac{1}{2}) & \text{for } t \in [0,x], \\ \frac{(t-1)^n}{n!}F(\frac{1}{2},1,n+1;1-t) & \text{for } t \in (x,1], \end{cases}$$

$$T_{n,w}(x) = \sum_{j=2}^{n} A_j(x)g^{(j-1)}(x) \tag{3.23}$$

and

$$A_j(x) = \begin{cases} \frac{2(-x)^j}{(j-1)!}F(1-j,\frac{1}{2},\frac{3}{2},\frac{1}{x}), & x \neq 0 \\ \frac{2}{(2j-1)(j-1)!}, & x = 0. \end{cases}$$

Specially, for $n = 2$ we get the identities from the Corollaries 1 and 2:

$$\frac{1}{2}\int_0^1 \frac{g(t)}{\sqrt{t}}dt - g\left(\frac{1}{3}\right) = \frac{1}{2}\int_0^1 W_{2,w}\left(t,\frac{1}{3}\right)g''(t)dt \tag{3.24}$$

and

$$\frac{2}{3}g(0) + \frac{1}{3}g(1) - \frac{1}{2}\int_0^1 \frac{g(t)}{\sqrt{t}}dt = \frac{1}{2}\int_0^1 y(t)g''(t)dt,$$

where

$$y(t) = \frac{1}{3}\int_0^t \left(\frac{2}{3}-t+s\right)\frac{1}{s}ds + \frac{2}{3}\int_t^1 \left(\frac{1}{3}-s+t\right)\frac{1}{s}ds.$$

Since $w(t) \geqslant 0$ on $[0,1]$, we conclude after the Corollaries 2 and 1 that $W_{2,w}(t,\frac{1}{3}) \geqslant 0$ and $y(t) \geqslant 0$. So if g is a convex function and g'' exists, then the following inequalities hold:

$$g\left(\frac{1}{3}\right) \leqslant \frac{1}{2}\int_0^1 \frac{g(t)}{\sqrt{t}}dt \leqslant \frac{2g(0)+g(1)}{3},$$

which are the Fejér inequalities for special weight function $w(t) = \frac{1}{\sqrt{t}}$.

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, A. ČIVLJAK, S. KOVAČ, J. PEČARIĆ, M. RIBIČIĆ PENAVA, *General integral identitieas and related inequalities*, Element, Zagreb 2013.

[2] J. BARIĆ, LJ. KVESIĆ, J. PEČARIĆ, M. RIBIČIĆ PENAVA, *Estimates on some quadrature rules via weghted Hermite-Hadamard inequality*, Applicable Analysis and Discrete Mathematics, **16** (2022) pp. 232–245.

[3] J. BARIĆ, LJ. KVESIĆ, J. PEČARIĆ, M. RIBIČIĆ PENAVA, *Fejér type inequalities for higher order convex functions and quadrature formulae*, Aequationes mathematicae **96** (2) (2022), pp. 417–430.

[4] S. KOVAČ, J. PEČARIĆ, *Weighted version of general integral formula of Euler type*, Math. Inequal. Appl., **13** (2010) (3) 579–599.

[5] M. MATIĆ, J. PEČARIĆ, N. UJEVIĆ, *Generalizations of weighted version of Ostrowski's inequality and some related results*, J. of Inequal. Appl., **5** (2000), 639–666.

[6] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Inequalities for functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht 1994.

[7] J. PEČARIĆ, *On the Čebyšev inequality*, Bul. Inst. Politehn. Timisoara, **25** (39) (1980), 10–11.

[8] J. PEČARIĆ, I. PERIĆ, *Refinements of the integral form of Jensen's and the Lah-Ribarić inequalities and applications for Csisyar divergence*, J. Inequal. Appl. **2020** (108) (2020), pp. 16.

[9] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc., San Diego, 1992.

Sanja Kovač
Faculty of Geotechnical Engineering
University of Zagreb
Hallerova aleja 7, 42 000 Varaždin, Croatia
e-mail: sanja.kovac@gfv.unizg.hr

Josip Pečarić
Croatian Academy of Sciences and Arts
Zagreb, Croatia
e-mail: pecaric@hazu.hr

Mihaela Ribičić Penava
School of Applied Mathematics and Informatics
Josip Juraj Strossmayer University of Osijek
Trg Ljudevita Gaja 6, 31000 Osijek, Croatia
e-mail: mihaela@mathos.hr

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com