Journal of
Mathematical
Inequalities
Volume 18, Number 1 (2024), 329–353
doi:10.7153/jmi-2024-18-19

B_σ TYPE MIXED MORREY SPACES AND THEIR APPLICATIONS

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(*Communicated by T. Burić*)

Abstract. This paper defines and studies the B_σ type mixed Morrey spaces. Furthermore, we introduce the predual spaces of B_σ type mixed Morrey spaces to establish the extrapolation theorem. As applications of extrapolation theory, we character the BMO space in different ways. On the one hand, we character the BMO spaces by the new John-Nirenberg inequality in terms of B_σ type mixed Morrey spaces. On the other hand, we establish the characterization of BMO spaces by the boundedness of commutators of the singular integral operator on B_σ type mixed Morrey spaces.

1. Introduction

The study on B_σ type spaces can be traced back to the work of Matsuoka and Nakai [29], who introduced function spaces $B^{p,\lambda}$ with Morrey-Campanato norms, which unify some central Morrey spaces and usual Morrey-Campanato spaces. After that, Komori and Matsuoka et al. [24] introduced B_σ type spaces by improving the function spaces $B^{p,\lambda}$ with Morrey-Campanato norms.

Let $r>0$, $\sigma\in[0,\infty)$. The B_σ type spaces $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$ are defined as the sets of all functions f on $\mathbb{R}^n$ such that

$$\|f\|_{B_\sigma(E)}=\sup_{r\geqslant 1}\frac{1}{r^\sigma}\|f\|_{E(Q_r)}<\infty \quad \text{and} \quad \|f\|_{\dot{B}_\sigma(E)}=\sup_{r>0}\frac{1}{r^\sigma}\|f\|_{E(Q_r)}<\infty.$$

Throughout this paper, $E(Q_r)$ is a function space with semi-norm $\|\cdot\|_E$ on Q_r, where Q_r is an open cube centered at the origin and side length $2r$, or an open ball centered at the origin and radius r.

In recent decades, B_σ spaces have attracted wide attention as a natural generalization of some classical spaces [22, 23, 28, 33]. In fact, B_σ type space is closely related to the base space E. For example, $E=L^p$, $B_\sigma(E)(\mathbb{R}^n)$ can unify Lebesgue spaces $L^p(\mathbb{R}^n)$, $B^p(\mathbb{R}^n)$ spaces [2], central mean oscillation spaces $\mathrm{CMO}^p(\mathbb{R}^n)$ and $\mathrm{CBMO}^p(\mathbb{R}^n)$ [5,8,25] and Morrey spaces $B^{p,\lambda}(\mathbb{R}^n)$ [29]. Therefore, the uniformity of B_σ type spaces can be used to establish the boundedness of operators on different classical spaces. For instance, Sawano and Yoshida introduced the B_σ type Lebesgue space

Mathematics subject classification (2020): 42B20, 42B25, 42B35.

Keywords and phrases: Mixed Morrey space, duality, extrapolation, John-Nirenberg inequality, characterization.

The research was supported by Natural Science Foundation of China (Grant No. 12061069).

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$B_\sigma(L^p)(\mathbb{R}^n)$ and B_σ type Morrey space $B_\sigma(\mathscr{M}_q^p)(\mathbb{R}^n)$ and established the boundedness of some operators and their commutators by researching the predual of B_σ type spaces. For more details, we can refer to [38].

Morrey spaces were introduced by Morrey in 1938 [30]. They are valuable in the regularity theory of partial differential equations, harmonic analysis, and mathematical physics. For this reason, Morrey spaces were generalized to more general forms, such as generalized Morrey space, local Morrey space, Morrey type space and so on [3, 12–14, 32]. Moreover, the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$, as a generalization of the Lebesgue space $L^p(\mathbb{R}^n)$, was established by Benedek and Panzone [1] in 1961, which can be traced back to the work of Hömander [17]. Influenced by the pioneering work on mixed-norm spaces, Nogyama introduced the mixed Morrey space in [35–37]. Recently, Wei established a series of results, including embedding properties, duality, and the boundedness of operators on some mixed-norm Morrey type spaces [41–43]. Zhou et al. studied the mixed local and global Morrey space and obtained some properties of spaces and boundedness of some operators [27, 39, 47]. As the most profound application of weighted theory, extrapolation can concisely solve the problem of the boundedness of operators on a certain space, but it depends heavily on the duality of this space and the boundedness of the maximal operator on the dual or predual space. Rubio de Francia [7] first introduced the classical extrapolation theory, which has been extended to mixed-norm Morrey spaces [15,41], general Banach spaces [16, 34], weighted function spaces [21, 44], and so on. Hence, this paper's primary purpose is to establish the extrapolation theory on B_σ type mixed Morrey spaces and character BMO spaces via these extrapolation theorems.

The paper is organized as follows. In Section 2, we first give some preliminaries, define the B_σ type mixed Morrey space and B_σ type mixed Lebesgue space and then obtain some basic properties. In Section 3, We gain the predual space of B_σ type mixed Morrey space is H_σ type block space. In Section 4, we first establish the boundedness of maximal operators on H_σ type block spaces. Then, the extrapolation theory on B_σ type mixed Morrey spaces is obtained. In Section 5, by using the extrapolation theory in Section 4, we character the equivalence of BMO spaces in terms of B_σ type Morrey space and classical BMO spaces. We solve the boundedness of some operators and their commutators. Moreover, another new characterization of BMO space via the boundedness of the commutator of the singular integral operator is also obtained.

Finally, we make some conventions on notation. We always denote by C a positive constant, which is independent of the main parameters, but it may vary from line to line. The notation $A \lesssim B$ means that $A \leqslant CB$ with some positive constant C independent of appropriate quantities, and, if $A \lesssim B \lesssim A$, then we write $A \sim B$. For a measurable set E, we denote by χ_E the characteristic function of E and by $|E|$ its n-dimensional Lebesgue measure. Moreover, the letter $\vec{q}$ will denote n-tuples of the numbers in $(0,\infty]$ $(n \geqslant 1)$. For any $q \in [1,\infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. In addition, if $\vec{q} \in [1,\infty]^n$, we denote by $\vec{q}' = (q_1', q_2', \ldots, q_n')$, by $t\vec{q} = (tq_1, tq_2, \ldots, tq_n)$ for any $t \in \mathbb{R}$. By definition, the inequality $0 < \vec{q} < \infty$ means that $0 < q_i < \infty$ for all i. In what follows, Q_r be an open cube centered at the origin and side length $2r$, or an open ball centered at the origin and radius r. We also denote by $\mathscr{Q}(\mathbb{R}^n)$ the set of all cubes whose edges are parallel to the coordinate axes.

2. B_σ-mixed Morrey spaces

In this section, we will define the B_σ type mixed Morrey space. To begin this, let us first recall the mixed Lebesgue space and mixed Morrey space.

DEFINITION 2.1. (Mixed Lebesgue space) ([1]) Let $\vec{p}=(p_1,p_2,\ldots,p_n)\in(0,\infty]^n$. Then, the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined by the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}}:=\left(\int_{\mathbb{R}}\cdots\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x_1,x_2,\ldots,x_n)|^{p_1}\,dx_1\right)^{\frac{p_2}{p_1}}dx_2\right)^{\frac{p_3}{p_2}}\ldots dx_n\right)^{\frac{1}{p_n}}<\infty,$$

If $p_j=\infty$, then we have to make appropriate modifications.

We now define the B_σ type mixed-norm Lebesgue space used in this paper.

DEFINITION 2.2. Let $0\leqslant\sigma$, $1<\vec{p}<\infty$. We define B_σ-mixed Lebesgue spaces $\dot{B}_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)$ and $B_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)$ as the set of all measurable functions f, such that $\|f\|_{\dot{B}_\sigma(L^{\vec{p}})}<\infty$ and $\|f\|_{B_\sigma(L^{\vec{p}})}<\infty$ respectively, where

$$\|f\|_{\dot{B}_\sigma(L^{\vec{p}})}:=\sup_{r>0}\frac{1}{r^\sigma}\|f\chi_{Q_r}\|_{L^{\vec{p}}}\quad\text{and}\quad\|f\|_{B_\sigma(L^{\vec{p}})}:=\sup_{r\geqslant1}\frac{1}{r^\sigma}\|f\chi_{Q_r}\|_{L^{\vec{p}}}.$$

REMARK 2.1.

(1) $\dot{B}_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)$ and $B_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)$ are called the homogeneous B_σ-mixed Lebesgue space and the nonhomogeneous B_σ-mixed Lebesgue space, respectively.

(2) If $\vec{p}=\{p,p,\ldots,p\}$, then spaces $\dot{B}_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)$ and $B_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)$ are B_σ type Lebesgue spaces, introduced by Sawano and Yoshida in [38].

(3) In view of above definitions, when $\sigma=0$, we have $\dot{B}_0(L^{\vec{p}})(\mathbb{R}^n)=B_0(L^{\vec{p}})(\mathbb{R}^n)=L^{\vec{p}}(\mathbb{R}^n)$.

(4) Let $1<\vec{p}<\infty$ and $0\leqslant\sigma_1\leqslant\sigma_2<\infty$. It is obviously that $B_{\sigma_1}\left(L^{\vec{p}}\right)(\mathbb{R}^n)\hookrightarrow B_{\sigma_2}\left(L^{\vec{p}}\right)(\mathbb{R}^n)$.

(5) Let $1<\vec{p}<\infty$ and $0<\sigma$. Then, the following equivalence of norms is obtained

$$\dot{B}_\sigma\left(L^{\vec{p}}\right)(\mathbb{R}^n)=\dot{K}^{-\sigma}_{\infty,\vec{p}}(\mathbb{R}^n)=\left\{f\in L^{\vec{p}}_{loc}:\|f\|_{\dot{K}^{-\sigma}_{\infty,\vec{p}}}=\sup_{j\in\mathbb{Z}}2^{-j\sigma}\|f\chi_j\|_{L^{\vec{p}}}<\infty\right\},$$

where $\dot{K}^{-\sigma}_{\infty,\vec{p}}(\mathbb{R}^n)$ are special case of mixed Herz spaces $\dot{K}^{\alpha}_{p,\vec{q}}$, introduced by Wei in [45]. The corresponding classical Herz spaces were defined by Lu and Yang [26], more details can refer to [40,46].

Recall the definition of mixed Morrey space $\mathscr{M}_{\vec{q}}^{p}(\mathbb{R}^n)$. Let $0<p<\infty$, $0<\vec{q}<\infty$ and $\frac{n}{p}\leqslant\sum_{j=1}^{n}\frac{1}{q_j}$, then, mixed Morrey space $\mathscr{M}_{\vec{q}}^{p}(\mathbb{R}^n)$ is the collection of all locally integrable functions on $\mathbb{R}^n$ with norms

$$\|f\|_{\mathscr{M}_{\vec{q}}^{p}}:=\sup_{Q\in\mathscr{Q}(\mathbb{R}^n)}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\|f\chi_Q\|_{L^{\vec{q}}}<\infty.$$

DEFINITION 2.3. Let $0\leqslant\sigma$, $1<p,\vec{q}<\infty$ and $\frac{n}{p}\leqslant\sum_{j=1}^{n}\frac{1}{q_j}$. The homogeneous B_σ type mixed Morrey space $\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ is defined to be the set of all measurable functions f with

$$\|f\|_{\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})}:=\sup_{r>0}\frac{1}{r^\sigma}\|f\chi_{Q_r}\|_{\mathscr{M}_{\vec{q}}^{p}}=\sup_{r>0}\sup_{Q\in\mathscr{Q}}\frac{1}{r^\sigma}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\|f\chi_{Q_r\cap Q}\|_{L^{\vec{q}}}<\infty.$$

Similarly, the nonhomogeneous B_σ type mixed Morrey space $B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ consists of all measurable functions f defined on $\mathbb{R}^n$ for which

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}:=\sup_{r\geqslant1}\frac{1}{r^\sigma}\|f\chi_{Q_r}\|_{\mathscr{M}_{\vec{q}}^{p}}=\sup_{r\geqslant1}\sup_{Q\in\mathscr{Q}}\frac{1}{r^\sigma}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\|f\chi_{Q_r\cap Q}\|_{L^{\vec{q}}}<\infty.$$

REMARK 2.2.

(1) If $\vec{p}=\{p,p,\dots,p\}$, then spaces $\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ and $B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ are B_σ type Morrey spaces introduced by Sawano and Yoshida in [38].

(2) We note that $\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)=B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)=\mathscr{M}_{\vec{q}}^{p}(\mathbb{R}^n)$. By the direct observation, we know that $B_{\sigma_1}\left(\mathscr{M}_{\vec{p}}^{p}\right)(\mathbb{R}^n)\hookrightarrow B_{\sigma_2}\left(\mathscr{M}_{\vec{p}}^{p}\right)(\mathbb{R}^n)$ for $0\leqslant\sigma_1\leqslant\sigma_2<\infty$.

(3) If $n/p=\sum_{j=1}^{n}1/q_i$, then B_σ type mixed Morrey spaces are B_σ type mixed Lebesgue spaces in Definition 2.2. Hence, the results in this paper also hold for B_σ type mixed Lebesgue spaces.

Let $p_0>0$. By the p_0-convexification of Lebesgue, we conclude that

$$\begin{aligned}\|f^{p_0}\|^{1/p_0}_{B_{\sigma p_0}\left(\mathscr{M}_{\vec{q}/p_0}^{p/p_0}\right)}&=\sup_{r>0}\sup_{Q\in\mathscr{Q}}\left[\frac{1}{r^{\sigma p_0}}|Q|^{\frac{p_0}{p}-\frac{1}{n}\sum_{j=1}^{n}\frac{p_0}{q_j}}\right]^{1/p_0}\|f^{p_0}\chi_{Q_r\cap Q}\|^{1/p_0}_{L^{\vec{q}/p_0}}\\&=\sup_{r>0}\sup_{Q\in\mathscr{Q}}\frac{1}{r^\sigma}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\|f\chi_{Q_r\cap Q}\|_{L^{\vec{q}}}=\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}.\end{aligned}$$

That is, the p_0-convexification of $B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ is $B_{\sigma p_0}\left(\mathscr{M}_{\vec{q}/p_0}^{p/p_0}\right)(\mathbb{R}^n)$.

PROPOSITION 2.1. *Let $0\leqslant\sigma$, $1<p,\vec{q}<\infty$ and $\frac{n}{p}\leqslant\sum_{j=1}^{n}\frac{1}{q_j}$. Then, $\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ is continuous embedding into $B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$, that is, for all $f\in\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$, we have $\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\leqslant\|f\|_{\dot{B}_\sigma(\mathscr{M}_{\vec{q}}^{p})}$.*

The proof of Proposition 2.1 can be quickly shown by the Definition 2.3. Hence, we omit the details. One of the main results in this paper is to prove that B_σ type mixed spaces admit a predual respectively. Therefore, we introduce the following definitions of H_σ type block spaces.

DEFINITION 2.4. Let $0 \leqslant \sigma$, $1 < p, \vec{q} < \infty$ and $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$.

(1) If an $L^{\vec{q}'}(\mathbb{R}^n)$ function A satisfies $\operatorname{supp}(A) \subset Q_r \cap Q$ and $\|A\|_{L^{\vec{q}'}} \leqslant r^{-\sigma} |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n} \frac{1}{q_j}}$ for some $r > 0$ and a cube $Q \in \mathscr{Q}$. Then, the function A is called a $(p', \vec{q}', \sigma, r)$-block.

(2) Let $\dot{\mathscr{A}}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$ be the collection of all sequences $\left\{(A_j, r_j, Q_j)\right\}_{j=1}^{\infty}$ for which each A_j is a $(p', \vec{q}', \sigma, r_j)$-block. The homogeneous H_σ-block space $\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ is the set of all measurable functions f that can be represented as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j A_j(x) \quad \text{a.e. } x \in \mathbb{R}^n, \tag{2.1}$$

for some $\left\{\lambda_j\right\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ and $\left\{(A_j, r_j, Q_j)\right\}_{j=1}^{\infty} \in \dot{\mathscr{A}}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$. For any function f, its norm $\|f\|_{\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}$ can be defined by

$$\|f\|_{\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} := \inf\Bigg\{ \sum_{j=1}^{\infty} |\lambda_j| : \left\{\lambda_j\right\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}), \left\{(A_j, r_j, Q_j)\right\}_{j=1}^{\infty} \in \dot{\mathscr{A}}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right), \text{(2.1) holds} \Bigg\}.$$

(3) Denote by $\mathscr{A}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$ the set of all $\left\{(A_j, r_j, Q_j)\right\}_{j=1}^{\infty} \in \dot{\mathscr{A}}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$ such that $r_j \geqslant 1$ for each $j \in \mathbb{N}$ and any $Q_j \in \mathscr{Q}$. The nonhomogeneous H_σ-block space $\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ is the set of all measurable functions f that can be represented as (2.1) and equips with the following norms

$$\|f\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} := \inf\Bigg\{ \sum_{j=1}^{\infty} |\lambda_j| : \left\{\lambda_j\right\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}), \left\{(A_j, r_j, Q_j)\right\}_{j=1}^{\infty} \in \mathscr{A}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right), \text{(2.1) holds} \Bigg\}.$$

It is easy to see that $\dot{H}_0(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n) = H_0(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n) = \mathscr{H}_{\vec{q}'}^{p'}(\mathbb{R}^n)$. That is to say, the H_σ type block can recover to classical Lebesgue space and Block space.

3. The duality of B_σ type function spaces

In this section, we mainly establish the predual spaces of B_σ type mixed Morrey space. Before giving the key theorem in this section, we first recall the definition of predual space. The spaces B are called the predual space of Banach space A, if $B^* = A$.

THEOREM 3.1. *Let* $0 \leqslant \sigma$, $1 < p, \vec{q} < \infty$ *and* $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$. *Then the spaces* $\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ *and* $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ *are preduals of* $\dot{B}_\sigma\left(\mathscr{M}_{\vec{q}}^{p}\right)(\mathbb{R}^n)$ *and* $B_\sigma\left(\mathscr{M}_{\vec{q}}^{p}\right)(\mathbb{R}^n)$, *respectively.*

By the Remark 2.1 and 2.2, Theorem 3.1 imply that space $L^{\vec{p}'}(\mathbb{R}^n)$ is predual of $L^{\vec{p}}(\mathbb{R}^n)$, which is shown in [1] by Benedek and Panzone. At the same time, Theorem 3.1 implies that space $\mathscr{H}_{\vec{q}'}^{p'}(\mathbb{R}^n)$ is predual of $\mathscr{M}_{\vec{q}}^{p}(\mathbb{R}^n)$, which is shown in [36] by Nogayama. To begin the proof, we first give the following key estimation and lemmas of density.

LEMMA 3.1. *Let* $0 \leqslant \sigma$, $1 < p, \vec{q} < \infty$ *and* $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$. *If function* $g \in L^{\vec{q}'}(\mathbb{R}^n)$, $Q \in \mathscr{Q}$, *then, for any* $0 < r$ *and* $1 \leqslant r$, *we have the following estimations respectively.*

$$\left\| r^{-\sigma} |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n} \frac{1}{q_j}} \chi_{Q_r \cap Q}\, g \right\|_{\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \leqslant \|g\|_{L^{\vec{q}'}}$$

and

$$\left\| r^{-\sigma} |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n} \frac{1}{q_j}} \chi_{Q_r \cap Q}\, g \right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \leqslant \|g\|_{L^{\vec{q}'}}.$$

The Lemma 3.1 can be seen as boundedness of linear operators $T_1 : g \in L^{\vec{q}'}(\mathbb{R}^n) \to \chi_{Q_r} g \in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n) \hookrightarrow \dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. In particular, the norms of operators are less than or equal to $r^{\sigma} |Q|^{\frac{1}{n}\sum_{j=1}^{n} \frac{1}{q_j} - \frac{1}{p}}$.

Proof of Lemma 3.1. We first claim that for any $Q \in \mathscr{Q}$ and $1 \leqslant r$, the function $B = \frac{r^{-\sigma}|Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \chi_{Q_r \cap Q}\, g}{\|g\|_{L^{\vec{q}'}}}$ is a $(p', \vec{q}', \sigma, r)$ block. By definition of the function B, we know that $\operatorname{supp} B \subset Q_r \cap Q$. In the next, we will check that

$$\|B\|_{L^{\vec{q}'}} = \frac{r^{-\sigma}|Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \|g\chi_{Q_r \cap Q}\|_{L^{\vec{q}'}}}{\|g\|_{L^{\vec{q}'}}} \leqslant r^{-\sigma}|Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}.$$

It implies that B is a $(p', \vec{q}', \sigma, r)$ block. So, we have

$$r^{-\sigma}|Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \chi_{Q_r \cap Q}\, g = \|g\|_{L^{\vec{q}'}} \cdot \frac{r^{-\sigma}|Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \chi_{Q_r \cap Q}\, g}{\|g\|_{L^{\vec{q}'}}} = \sum_{j=1}^{\infty} \lambda_j A_j,$$

where $\lambda_1 = \|g\|_{L^{\vec{q}'}}$, $\lambda_2 = \lambda_3 = \ldots = 0$ and A_j are $(p', \vec{q}, \sigma, r)$ blocks. Furthermore, we directly conclude that

$$\left\| r^{-\sigma} |Q|^{\frac{1}{p} - \frac{1}{n} \sum_{j=1}^n \frac{1}{q_j}} \chi_{Q_r \cap Q} g \right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \leqslant \|g\|_{L^{\vec{q}'}}.$$

The proof is finished. $\square$

DEFINITION 3.1. Let $0 \leqslant \sigma$, $1 < p, \vec{q}, \vec{p} < \infty$ and $E = \mathscr{H}_{\vec{q}'}^{p'}(\mathbb{R}^n)$.

(1) The B_σ block space $\dot{\mathscr{H}}_\sigma(E)(\mathbb{R}^n)$, linear subspace of $\dot{H}_\sigma(E)(\mathbb{R}^n)$, is defined to be the set of all functions f satisfying $\operatorname{supp}(f) \subset Q_R \backslash Q_{R^{-1}}$ for some $0 < R$.

(2) The B_σ block space $\mathscr{H}_\sigma(E)(\mathbb{R}^n)$, linear subspace of $H_\sigma(E)(\mathbb{R}^n)$, is defined to be the set of all functions f satisfying $\operatorname{supp}(f) \subset Q_R$ for some $1 \leqslant R$.

LEMMA 3.2. *Let $0 \leqslant \sigma$, $1 < p, \vec{q} < \infty$ and $\frac{n}{p} \leqslant \sum_{j=1}^n \frac{1}{q_j}$. The spaces $\dot{\mathscr{H}}_\sigma(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n)$ and $\mathscr{H}_\sigma(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n)$ are dense in $\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ and $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, respectively.*

Proof of Lemma 3.2. It suffice to prove that for any $f \in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$, there exist $g \in \mathscr{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$, such that $\|f - g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \to 0$. Let $f \in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. Then function f can be represented as $f = \sum_{j=1}^\infty \lambda_j A_j$ and

$$\inf \left\{ \sum_{j=1}^\infty |\lambda_j| : \{\lambda_j\}_{j=1}^\infty \in \ell^1(\mathbb{N}), \{(A_j, r_j, Q_j)\}_{j=1}^\infty \in \mathscr{A}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right), (2.1) \text{ holds} \right\} < \infty.$$

Consequently, we evidently see that

$$\sum_{j=1}^J \lambda_j A_j \in \mathscr{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n).$$

Hence, one just to prove $\left\| f - \sum_{j=1}^J \lambda_j A_j \right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \to 0$ when $J \to \infty$.

$$\left\| f - \sum_{j=1}^J \lambda_j A_j \right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} = \left\| \sum_{j=J+1}^\infty \lambda_j A_j \right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \leqslant \sum_{j=J+1}^\infty \lambda_j,$$

then, $\left\| f - \sum_{j=1}^J \lambda_j A_j \right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \to 0$ when $J \to \infty$.

It is implies that spaces $\mathscr{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ is dense in $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$.

We now turn to prove $\dot{\mathscr{H}}_\sigma(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n)$ is dense in $\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. Let $f \in \dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. By the definition of spaces $\dot{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, we see that f can be represented as

$$f(x)=\sum_{j=1}^{\infty}\lambda_j A_j(x) \quad \text{a.e. } x\in\mathbb{R}^n,$$

where A_j is a $(p',\vec{q}',\sigma,r_j)$-block. From this, we have $\operatorname{supp}(\chi_{R^{-1}\leqslant|x|\leqslant R}A_j)\subset Q_R\backslash Q_{R^{-1}}$ and $\operatorname{supp}(A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j)\subset Q_j\cap Q$ for any block A_j and some $Q\in\mathscr{Q}$.

Let $g_J=\sum_{j=1}^{J}\lambda_j\chi_{R^{-1}\leqslant|x|\leqslant R}A_j(x)$. Then, $\operatorname{supp}(g)\subset Q_R\backslash Q_{R^{-1}}$, hence, $g\in\mathscr{H}_\sigma(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n)$. If we prove that $\lim\limits_{J\to\infty}\|f-g_J\|_{\mathscr{H}_{\vec{q}'}^{p'}}=0$, the assertion follows.

We proceed to show that

$$\|f-g_J\|_{\mathscr{H}_{\vec{q}'}^{p'}}\leqslant\sum_{j=1}^{J}\lambda_j\left\|\left(A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right)\right\|_{\mathscr{H}_{\vec{q}'}^{p'}}+\left\|\sum_{j=J+1}^{\infty}\lambda_jA_j\right\|_{\mathscr{H}_{\vec{q}'}^{p'}}=I+II.$$

Hence, by definition of space $\mathscr{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ and Lemma 3.1, for any $j\in\mathbb{N}$, we just estimate that

$$\begin{aligned}
&\left\|A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\\
&=r_j^\sigma|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{p}}\left\|A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right\|_{L^{\vec{q}'}}\\
&\quad\times\left\|\frac{A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j}{\left\|r_j^\sigma|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{p}}\left(A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right)\right\|_{L^{\vec{q}'}}}\right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\\
&\leqslant r_j^\sigma|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{p}}\left\|A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right\|_{L^{\vec{q}'}}.
\end{aligned}$$

Thus, the following result can be obtained

$$\left\|A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant\lim_{R\to\infty}r_j^\sigma|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{p}}\left\|A_j-\chi_{R^{-1}\leqslant|x|\leqslant R}A_j\right\|_{L^{\vec{q}'}}=0. \tag{3.1}$$

Since $f\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, show that

$$\lim_{J\to\infty}\left\|\sum_{j=J+1}^{\infty}\lambda_jA_j\right\|_{\mathscr{H}_{\vec{q}'}^{p'}}\to 0. \tag{3.2}$$

Combining (3.1) and (3.2), yields

$$\lim_{J\to\infty}\lim_{R\to\infty}\sum_{j=1}^{J}\lambda_j\chi_{R^{-1}\leqslant|x|\leqslant R}A_j=\sum_{j=1}^{\infty}\lambda_jA_j=f\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n).$$

Since $\sum_{j=1}^{J}\lambda_j\chi_{R^{-1}\leqslant|x|\leqslant R}A_j(x)\in\mathscr{H}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, hence, we deduce this lemma. □

Proof of Theorem 3.1. Let $f\in B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ and $g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. By definition of space $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, for any $\varepsilon>0$, there exist a decomposition

$$g=\sum_{j=1}^{\infty}\lambda_jA_j \tag{3.3}$$

such that

$$\left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{\ell^1}\leqslant(1+\varepsilon)\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}, \tag{3.4}$$

where $\{\lambda_j\}_{j=1}^{\infty}\in\ell^1(\mathbb{N})$ and $\{(A_j,r_j,Q_j)\}_{j=1}^{\infty}\in\mathscr{A}_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$.

Futhermore, by decomposition (3.3), the Hölder inequality on mixed-norm spaces and Definition 2.4 yield

$$\begin{aligned}\|f\cdot g\|_{L^1}&\leqslant\sum_{j=1}^{\infty}|\lambda_j|\int_{\mathbb{R}^n}\left|f(x)\chi_{Q_{r_j}\cap Q_j}(x)A_j(x)\right|dx\\&\leqslant\sum_{j=1}^{\infty}|\lambda_j|\left\|f\chi_{Q_{r_j}\cap Q_j}\right\|_{L^{\vec{q}}}\|A_j\|_{L^{\vec{q}'}}\\&\leqslant\sum_{j=1}^{\infty}|\lambda_j|\left\|f\chi_{Q_{r_j}\cap Q_j}\right\|_{L^{\vec{q}}}r_j^{-\sigma}|Q|^{\frac{1}{p}-\frac{1}{q}}\\&\leqslant\left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{\ell^1}\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}.\end{aligned} \tag{3.5}$$

Using the control relation of (3.4), we conclude that

$$\|f\cdot g\|_{L^1}\leqslant(1+\varepsilon)\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\cdot\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}.$$

Since $\varepsilon>0$ is arbitrary, letting $\varepsilon\to0$, we have

$$\|f\cdot g\|_{L^1}\leqslant\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\cdot\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}<\infty. \tag{3.6}$$

It implies that $f\cdot g\in L^1(\mathbb{R}^n)$ for all $f\in B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ and $g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. Moreover, for $f\in B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$, we define the functional $L_f:H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)\to\mathbb{C}$ by

$$L_f(g)=\int_{\mathbb{R}^n}f(x)g(x)dx,\quad g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n).$$

In view of the definition of norm of operator L_f on $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, we obtain

$$\|L_f\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}=\sup_{g\neq 0}\frac{1}{\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}}\left|\int_{\mathbb{R}^n}f(x)g(x)dx\right|.$$

By the inequality (3.6), we deduce that

$$\left|\int_{\mathbb{R}^n}f(x)g(x)dx\right|\leqslant\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}.$$

Hence,

$$\|L_f\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}\leqslant\sup_{g\neq 0}\frac{\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}}{\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}}\leqslant\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}. \tag{3.7}$$

We now turn to prove $\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\leqslant\|L_f\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}$. Let $L:H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)\to\mathbb{C}$ is a bounded linear functional. Furthermore, for $1\leqslant r$ and $Q\in\mathscr{Q}$, the functional $L_{r,Q}:L^{\vec{q}'}(\mathbb{R}^n)\to\mathbb{C}$ is defined by

$$L_{r,Q}(g)=L\left(r^{-\sigma}\cdot|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}\chi_{Q_r\cap Q}g\right),\quad g\in L^{\vec{q}'}(\mathbb{R}^n).$$

From the Riesz representation theorem, for each cube Q and $r\geqslant 1$ there exists $f_{r,Q}\in L^{\vec{q}}(\mathbb{R}^n)$ such that

$$L_{r,Q}(g)=\int_{\mathbb{R}^n}f_{r,Q}(x)g(x)dx,$$

for all $g\in L^{\vec{q}'}(\mathbb{R}^n)$ and that $\|f_{r,Q}\|_{L^{\vec{q}}}=\|L_{r,Q}\|_{(L^{\vec{q}'})^*}$.

According to the definition of norm of operator $L_{r,Q}$ on $L^{\vec{q}'}(\mathbb{R}^n)$, we obtain

$$\|L_{r,Q}\|_{(L^{\vec{q}'})^*}=\sup_{g\neq 0}\frac{\left|L\left(r^{-\sigma}\cdot|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}\chi_{Q_r\cap Q}g\right)\right|}{\|g\|_{L^{\vec{q}'}}}.$$

Using the functional L is bounded and Lemma 3.1, we deduce that

$$\begin{aligned}\left|L\left(r^{-\sigma}\cdot\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}\chi_{Q_r\cap Q}g\right)\right|&\leqslant\|L\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}\left\|r^{-\sigma}\cdot|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}\chi_{Q_r\cap Q}g\right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\\&\leqslant\|L\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}\|g\|_{\vec{q}'}.\end{aligned}$$

Hence,

$$\|L_{r,Q}\|_{(L^{\vec{q}'})^*}\leqslant\|L\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}.$$

Since $\|f_{r,Q}\|_{L^{\vec{q}}} = \|L_{r,Q}\|_{\left(L^{\vec{q}'}\right)^*}$, we deduce

$$\|f_{r,Q}\|_{L^{\vec{q}}} \leqslant \|L\|_{\left(H_\sigma\left(\mathscr{H}^{p'}_{\vec{q}'}\right)\right)^*}. \tag{3.8}$$

We will prove $\|f\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})} \leqslant \|f_{r,Q}\|_{L^{\vec{q}}}$. By the definition of functional L, then, for any $1 \leqslant r$ and $Q \in \mathscr{Q}$, we get

$$L(\chi_{Q_r \cap Q} g) = \int_{\mathbb{R}^n} r^\sigma \cdot |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{p}} f_{r,Q}(x) g dx.$$

Thus, when $1 \leqslant r_1 \leqslant r_2$ and $Q_1 \subset Q_2$, one deduces that

$$\begin{aligned}
L\left(\chi_{Q_{r_1} \cap Q_1} g\right) &= r_1^\sigma \cdot |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{p}} \int_{\mathbb{R}^n} f_{r_1,Q_1}(x) \chi_{Q_{r_1} \cap Q_1}(x) g(x) dx \\
&= L\left(\chi_{Q_{r_1} \cap Q_1} \chi_{Q_{r_1} \cap Q_1} g\right) = L\left(\chi_{Q_{r_1} \cap Q_1} \chi_{Q_{r_2} \cap Q_2} g\right) \\
&= r_2^\sigma \cdot |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{p}} \int_{\mathbb{R}^n} f_{r_2,Q_2}(x) \chi_{Q_{r_1} \cap Q_1}(x) g(x) dx.
\end{aligned}$$

As a consequence, one concludes that

$$r_1^\sigma \cdot |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{p}} f_{r_1,Q_1}(x) = r_2^\sigma \cdot |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{p}} f_{r_2,Q_2}(x) \quad \text{a.e. } x \in Q_{r_1} \cap Q_1. \tag{3.9}$$

The inequality (3.9) implies that the definition of function f independent to r and Q,

$$f(x) = r^\sigma \cdot |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{p}} f_{r,Q}(x) \quad \text{a.e. } x \in Q_r \cap Q.$$

Moreover, by the inequation (3.8), we see that

$$r^{-\sigma} \cdot |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} \|f \chi_{Q_r \cap Q}\|_{L^{\vec{q}}} = \|f_{r,Q} \chi_{Q_r \cap Q}\|_{L^{\vec{q}}} \leqslant \|f_{r,Q}\|_{L^{\vec{q}}} \leqslant \|L\|_{\left(H_\sigma\left(\mathscr{H}^{p'}_{\vec{q}'}\right)\right)^*}. \tag{3.10}$$

Hence, taking the supremum on both side of (3.10), we get

$$\|f\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})} \leqslant \|L\|_{\left(H_\sigma\left(\mathscr{H}^{p'}_{\vec{q}'}\right)\right)^*}. \tag{3.11}$$

By direct observe, one obtains

$$\int_{\mathbb{R}^n} f_{r,Q}(x) g(x) dx = r^{-\sigma} \cdot |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} L_f(\chi_{Q_r \cap Q} g) \qquad \text{for all } g \in L^{\vec{q}'}$$

and

$$\int_{\mathbb{R}^n} f_{r,Q}(x) g(x) dx = L\left(r^{-\sigma} \cdot |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} \chi_{Q_r \cap Q} g\right) = r^{-\sigma} \cdot |Q|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} L(\chi_{Q_r \cap Q} g).$$

Next, we show that g is a function in the set of finite linear combinations of $(p',\vec{q}',\sigma,r_j)$-block. Since the L are identical on the $(p',\vec{q}',\sigma,r_j)$-block, and the set of finite linear combinations of $(p',\vec{q}',\sigma,r_j)$-block are $\mathscr{H}_\sigma(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n)$.

From $\mathscr{H}_\sigma(\mathscr{H}_{\vec{q}'}^{p'})(\mathbb{R}^n)$ are dense $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, hence, we can obtain $L=L_f$.

According to the inequality (3.11), we have

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\leqslant\|L_f\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}. \tag{3.12}$$

From (3.7) and (3.12), one deduces that $\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}=\left\|L_f\right\|_{\left(H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\right)^*}$. □

In fact, Theorem 3.1 also implies the following characterization for the B_σ type spaces.

COROLLARY 3.1. *If $f\in B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ and for all $g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}f(x)g(x)dx<\infty$. Then, for all $g\in B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$, the following equivalence hold*

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\sim\sup\left\{\int_{\mathbb{R}^n}f(x)g(x)dx:\ g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)\ \text{and}\ \|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant 1\right\}.$$

Proof. By the Definition of space $B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$, then, there exist a Q and Q_r such that

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)}\leqslant C\frac{1}{r^\sigma}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}\|f\|_{L^{\vec{q}}(Q_r\cap Q)}.$$

Furthermore, there exist $h\in L^{\vec{q}'}(Q_r\cap Q)$ with $\|h\|_{L^{\vec{q}'}(Q_r\cap Q)}\leqslant 1$ such that

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)}\leqslant C\frac{1}{r^\sigma}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}\int_{(Q_r\cap Q)}|f(x)h(x)|\,dx.$$

Let $g(x)=\frac{1}{r^\sigma}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}\cdot h(x)$. We know that $g(x)$ is a $(p',\vec{q}',\sigma,r)$-block. Then, $g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ and

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)}\leqslant C\int_{\mathbb{R}^n}|f(x)g(x)|\,dx.$$

Thus, combining the (3.6), we complete the proof. □

4. Extrapolation

In this section, we extend the extrapolation theory, firstly established by Rubio de Francia, to B_σ type space. The proof of the extrapolation relies on the iteration algorithm and the Muckenhoupt weight function. Especially the iteration algorithm generated by the Hardy-Littlewood maximal operator. Hence. We recall the definitions of the maximal operator and the Muckenhoupt weight.

Let any function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator $M(f)$ is defined by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(x)| dx.$$

DEFINITION 4.1.

(1) Let $1 < p < \infty$. We say that a locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ belongs to A_p weight if

$$[\omega]_{A_p} = \sup_{Q \in \mathscr{Q}} \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty.$$

(2) We say that a locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ belongs to A_1 weight if for any $Q \in \mathscr{Q}$

$$\frac{1}{|Q|} \int_Q \omega(y) dy \leqslant C\omega(x), \text{ a.e. } x \in Q,$$

for some constants $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \cup_{p \geqslant 1} A_p$.

The key ingredient of extrapolation is boundedness of the Hardy-Littlewood maximal operator on H_σ type block spaces. Thus, we first give the following lemma.

LEMMA 4.1. *Let $0 \leqslant \sigma < n - \sum_{j=1}^n \frac{1}{q_j}$, $1 < p, \vec{q} < \infty$ and $\frac{n}{p'} \geqslant \sum_{j=1}^n \frac{1}{q_j'}$. Then, the Hardy-Littlewood maximal operator is bounded on $\dot{H}_\sigma \left(\mathscr{H}^p_{\vec{q}} \right)(\mathbb{R}^n)$ and $H_\sigma \left(\mathscr{H}^p_{\vec{q}} \right)(\mathbb{R}^n)$.*

Proof. Let $f \in H_\sigma \left(\mathscr{H}^p_{\vec{q}} \right)(\mathbb{R}^n)$. Then function f can be represented as $f = \sum_{j=1}^\infty \lambda_j A_j$. From this reason, we just prove for any $(p, \vec{q}, \sigma, r)$ block A, we can establish $\|MA\| \leqslant C$.

Let A is a $(p, \vec{q}, \sigma, r)$ block. Then, we have the following decomposition

$$MA = \chi_{Q_{2r}} MA + \sum_{j=1}^\infty \chi_{Q_{2^{j+1}r} \backslash Q_{2^j r}} MA := I_1 + I_2.$$

We first estimate I_2, for any $x \in \mathbb{R}^n$, we get

$$\begin{aligned}
\chi_{Q_{2^{j+1}r} \backslash Q_{2^j r}} MA &\leqslant \frac{1}{(2^j r)^n} \int_{Q_r \cap Q} |A(y)| dy \\
&\leqslant \frac{1}{(2^j r)^n} \|\chi_{Q_r}\|_{L^{\vec{q}'}} \|A\|_{L^{\vec{q}}} \leqslant 2^{-jn} r^{-\sum_{j=1}^n \frac{1}{q_j}} \|A\|_{L^{\vec{q}}}.
\end{aligned}$$

Hence, we know that

$$\left\|\chi_{Q_{2^{j+1}r}\setminus Q_{2^j r}} MA\right\|_{L^{\vec{q}}} \leqslant 2^{-j\sum_{j=1}^n \frac{1}{q_j}} \|A\|_{L^{\vec{q}}}.$$

By a direct compute, one obtains that $2^{j(\sum_{j=1}^n \frac{1}{q_j}-\sigma)}\chi_{Q_{2^{j+1}r}\setminus Q_{2^j r}} MA$ is a $(p,\vec{q},\sigma,2^j r)$ block, it is easy to see

$$\left\|2^{j(\sum_{j=1}^n \frac{1}{q'_j}-\sigma)}\chi_{Q_{2^{j+1}r}\setminus Q_{2^j r}} MA\right\|_{L^{\vec{q}}} \lesssim (2^j r)^{-\sigma}|Q|^{\frac{1}{p}-\sum_{j=1}^n \frac{1}{q'_j}},$$

which implies that

$$\left\|\chi_{Q_{2^{j+1}r}\setminus Q_{2^j r}} MA\right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}}^{p}\right)} \lesssim 2^{j(\sigma-\sum_{j=1}^n \frac{1}{q'_j})}.$$

Using the same way, we conclude that

$$\|\chi_{Q_{2r}} MA\|_{H_\sigma\left(\mathscr{H}_{\vec{q}}^{p}\right)} \leqslant C.$$

Hence, we deduce that $MA \in H_\sigma\left(\mathscr{H}_{\vec{q}}^{p}\right)(\mathbb{R}^n)$. Furthermore, by the Lemma 3.2, we know that M is also bounded on $\mathscr{H}_\sigma\left(\mathscr{H}_{\vec{q}}^{p}\right)(\mathbb{R}^n)$. $\square$

Based on the boundedness of the Hardy-Littlewood maximal operator, the following iteration algorithm generated by the maximal operator is established.

Let $1 < p,\vec{q} < \infty$, $0 < p_0 < p,\vec{q} < \infty$, $0 < \sigma < \sum_{j=1}^n \frac{1}{q_j}$ and $\frac{n}{p} \leqslant \sum_{j=1}^n \frac{1}{q_j}$. By Lemma 4.1, then, we know M is bounded on $H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)(\mathbb{R}^n)$. Let B be the operator norm of M on $H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)(\mathbb{R}^n)$, that is

$$B = \|M\|_{H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)\to H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)}.$$

For any non-negative locally integral function h, the iteration algorithm is defined by

$$\mathscr{R}h := \sum_{k=0}^{\infty}\frac{M^k h}{2^k B^k},$$

where M^k is the k-th iterations of M and we denote by $M^0 h = h$.

Next, we will check that the indices $\sigma, p_0, p, \vec{q}$ meet the conditions of the Lemma 4.1. Since $(\vec{q}/p_0)' = ((q_1/p_0)', (q_2/p_0)', \ldots, (q_n/p_0)')$ and $(q_j/p_0)' = q_j/(q_j - p_0)$, then

$$\sigma p_0 \leqslant n - \sum_{j=1}^n 1/(q_j/p_0)' = n - \sum_{j=1}^n \frac{q_j - p_0}{q_j} = p_0\sum_{j=1}^n \frac{1}{q_j}.$$

Form this, we conclude that $\sigma \leqslant \sum_{j=1}^n \frac{1}{q_j}$. Similarly, $(p/p_0)' = p/(p-p_0)$, by the direct calculation, $n(p/p_0)' \geqslant n - \sum_{j=1}^n 1/(q_j/p_0)'$ shows that $1 - p_0/p \geqslant 1 - 1/n\sum_{j=1}^n p_0/q_j$. It implies that $\frac{n}{p} \leqslant \sum_{j=1}^n \frac{1}{q_j}$.

PROPOSITION 4.1. *Let* $1 < p, \vec{q} < \infty$, $0 < p_0 < p, \vec{q} < \infty$, $0 < \sigma < \sum_{j=1}^n \frac{1}{q_j}$ *and* $\frac{n}{p} \leqslant \sum_{j=1}^n \frac{1}{q_j}$. *For any* $h \in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$, *the operator* $\mathscr{R}$ *has the following properties:*

$$h(x) \leqslant \mathscr{R}h(x), \tag{R1}$$

$$\|\mathscr{R}h\|_{H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)} \leqslant 2\|h\|_{H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)}, \tag{R2}$$

$$[\mathscr{R}h]_{A_1} \leqslant 2B. \tag{R3}$$

These properites above can be deduced from the definition of operator $\mathscr{R}$ and the boundedness of the Hardy-Littlewood maximal operator M on space $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ (Lemma 4.1).

THEOREM 4.1. *Let* $1 < p, \vec{q} < \infty$, $0 < p_0 < p, \vec{q} < \infty$, $0 < \sigma < \sum_{j=1}^n \frac{1}{q_j}$ *and* $\frac{n}{p} \leqslant \sum_{j=1}^n \frac{1}{q_j}$. *Assume that for some family* $\mathscr{F}$ *of pairs* (f,g) *of nonnegative functions* f, g *such that for every*

$$\omega \in \left\{\mathscr{R}h : h \in H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right) \text{ with } \|h\|_{H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)} \leqslant 1\right\} \tag{4.1}$$

we have

$$\int_{\mathbb{R}^n} |g(x)|^{p_0}\omega(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^{p_0}\omega(x)dx < \infty. \tag{4.2}$$

Then, for any $(f,g) \in \mathscr{F}$ *and* $g \in B_\sigma(\mathscr{M}_{\vec{q}}^p)(\mathbb{R}^n)$, *we get*

$$\|g\|_{B_\sigma(\mathscr{M}_{\vec{q}}^p)} \leqslant C\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^p)}.$$

Proof. Let $(f,g) \in \mathscr{F}$ and $g \in B_\sigma(\mathscr{M}_{\vec{q}}^p)(\mathbb{R}^n)$. For any $h \in H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)(\mathbb{R}^n)$ with $\|h\|_{H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)} \leqslant 1$, the p_0-convexification of $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$ and the Corollary 3.1 deduce that

$$\|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^p)}^{p_0} = \|f^{p_0}\|_{B_{\sigma p_0}\left(\mathscr{M}_{\vec{q}/p_0}^{p/p_0}\right)} = \sup_{\|h\|_{H_{\sigma p_0}\left(\mathscr{H}_{(\vec{q}/p_0)'}^{(p/p_0)'}\right)} \leqslant 1} \left|\int_{\mathbb{R}^n} f^{p_0}(x)h(x)dx\right|.$$

Since $g \in B_\sigma(\mathscr{M}_{\vec{q}}^p)(\mathbb{R}^n)$, by the property (R1) and (R2) of k-iterations operators, the

Hölder inequality and the boundedness of Hardy-Littlewood maximal operator, then

$$\begin{aligned}\int_{\mathbb{R}^n}|f(x)|^{p_0}|h(x)|dx &\leqslant \int_{\mathbb{R}^n}|f(x)|^{p_0}\mathscr{R}h(x)dx\\ &\leqslant C\int_{\mathbb{R}^n}|g(x)|^{p_0}|\mathscr{R}h(x)dx\\ &\leqslant C\left\|g^{p_0}\right\|_{B_{\sigma p_0}\left(\mathscr{M}^{p/p_0}_{\vec{q}/p_0}\right)}\|h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}\\ &\leqslant C\left\|g\right\|^{p_0}_{B_\sigma(\mathscr{M}^p_{\vec{q}})},\end{aligned}$$

which implies that

$$\|g\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}\leqslant C\|f\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}. \quad \square$$

COROLLARY 4.1. *Let* $1<p,\vec{q}<\infty,\ 0<p_0<p,\vec{q}<\infty,\ 0<\sigma<\sum_{j=1}^n\frac{1}{q_j}$ *and* $\frac{n}{p}\leqslant\sum_{j=1}^n\frac{1}{q_j}$. *Assume that for every*

$$\omega\in\left\{\mathscr{R}h : h\in H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)\ with\ \|h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}\leqslant 1\right\},$$

the operator $T:\ L^{p_0}_w(\mathbb{R}^n)\to L^{p_0}_w(\mathbb{R}^n)$ *satisfies*

$$\int_{\mathbb{R}^n}|Tf(x)|^{p_0}w(x)dx\lesssim\int_{\mathbb{R}^n}|f(x)|^{p_0}w(x)dx. \tag{4.3}$$

Then, for every $f\in B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)$, *we obtain*

$$\|Tf\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}\leqslant\|f\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}.$$

Proof. We just need to check the conditions in this theorem satisfy the Theorem 4.1. For any $f\in B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n), h\in H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)(\mathbb{R}^n)$ and $\|h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}\leqslant 1$. Using Proposition 4.1 and (3.6), one deduces that

$$\begin{aligned}\int_{\mathbb{R}^n}|f(x)|^{p_0}|h(x)|dx &\lesssim \left\|g^{p_0}\right\|_{B_{\sigma p_0}\left(\mathscr{M}^{p/p_0}_{\vec{q}/p_0}\right)}\|\mathscr{R}h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}\\ &\lesssim \|g\|^{p_0}_{B_\sigma(\mathscr{M}^p_{\vec{q}})}\|h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}.\end{aligned}$$

Thus, we have

$$B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)\hookrightarrow\bigcap_{h\in H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)(\mathbb{R}^n),\|h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}\leqslant 1} L^{p_0}_{\mathscr{R}h}(\mathbb{R}^n). \tag{4.4}$$

Then, let $\mathscr{F}=\left\{(|Tf|,|f|): f\in B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)\right\}$. For any

$$\omega\in\left\{\mathscr{R}h: h\in H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)\text{ with }\|h\|_{H_{\sigma p_0}\left(\mathscr{H}^{(p/p_0)'}_{(\vec{q}/p_0)'}\right)}\leqslant 1\right\},$$

the inclusion (4.4) deduces that $B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)\hookrightarrow L^{p_0}_\omega(\mathbb{R}^n)$. Hence, by (4.3), we assure (4.2) in Theorem 4.1 is valid for the pairs $\mathscr{F}$. As a corollary of Theorem 4.1, we conclude that

$$\|Tf\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}\lesssim\|f\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})},\qquad\forall f\in B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n).$$

This proof is completed. $\square$

REMARK 4.1. In this section, although we only proved the extrapolation theory on space $B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)$, the extrapolation theory on space $\dot{B}_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)$ can be established because the proof is similar. Significantly, the Remark 2.2 and the Theorem 4.1 assure that the extrapolation theorems on B_σ type mixed Lebesgue space hold, but we omit the details.

5. Applications

In this section, we give some applications to illustrate the advantages and effects of extrapolation theory. On the one hand, we character the BMO space via John-Nirenberg inequality in terms of the B_σ type mixed Morrey space $B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)$ and the boundedness of commutator generated by BMO space. On the other hand, we solve the boundedness of some classical operators.

The bounded mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ introduced by John and Nirenberge [20] can be seen as a natural generalization of essentially bounded function space $L^\infty(\mathbb{R}^n)$. We first recall the definition of $\mathrm{BMO}(\mathbb{R}^n)$ space.

Let $f\in L_{\mathrm{loc}}(\mathbb{R}^n)$ and $f_Q=\frac{1}{|Q|}\int_Q f(x)dx$. The mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ is defined by

$$\mathrm{BMO}(\mathbb{R}^n):=\left\{f\in L_{\mathrm{loc}}(\mathbb{R}^n):\|f\|_{\mathrm{BMO}}=\frac{1}{|Q|}\int_Q|f(x)-f_Q|\,dx<\infty\right\}.$$

This generalization can be used to solve the endpoint estimation of classical singular integral operators and commutators. For this reason, BMO space plays a vital role in harmonic analysis. Therefore, the characterization of BMO space has gradually become a topic of concern to researchers. On the one hand, $\mathrm{BMO}(\mathbb{R}^n)$ space can be characterized via John-Nirenberg inequality as follows.

$$|\{x\in Q:|f(x)-f_Q|>t\}|\leqslant ce^{-c_1t/\|f\|_{\mathrm{BMO}}}|Q|, \tag{5.1}$$

where c and c_1 are positive constants. This John-Nirenberg inequality for $\mathrm{BMO}(\mathbb{R}^n)$ tells us that the following norms are equivalent for any $1 \leqslant p < \infty$,

$$\|f\|_{\mathrm{BMO}_p} = \sup_{Q} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{\frac{1}{p}},$$

where the supremum is taken over all balls Q in $\mathscr{Q}$.

5.1. John-Nirenberg Inequality and a new Characterization of BMO

The John-Nirenberg inequality can be extended to some other function spaces, such as on variable one exponents spaces [11], Morrey spaces [12] and ball Banach function spaces [18]. The following theorem further generalizes the John-Nirenberg inequalities in terms of B_σ type mixed Morrey spaces.

THEOREM 5.1. *Let $1 < p, \vec{q} < \infty$, $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$ and $0 < \sigma < \sum_{j=1}^{n} \frac{1}{q_j}$. Then, there exist constants $C, C_1 > 0$ such that for any $\gamma > 0, f \in \mathrm{BMO} \backslash \mathscr{C}$, where $\mathscr{C}$ denotes the set of constant functions and $Q \in \mathbb{R}^n$, we have*

$$\left\| \chi_{\{x \in Q : |f(x) - f_Q| \geqslant \gamma\}} \right\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})} \leqslant C e^{-\frac{C_1 \gamma}{\|f\|_{BMO}}} \|\chi_Q\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}.$$

Proof. $\omega \in A_\infty$ assure that there exist an $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{\omega(E)}{\omega(Q)} \leqslant C \left(\frac{|E|}{|Q|} \right)^{\varepsilon} \tag{5.2}$$

for any $Q \in \mathscr{Q}$ and all measurable subsets E of B, where C depends on n and $[\omega]_{A_\infty}$.

On the other hand, the classical John-Nirenberg inequalities (5.1) and characterization of A_∞ (5.2) yield that

$$\int \chi_{\{x \in Q : |f(x) - f_Q| > \gamma\}}(x) \omega(x) dx \leqslant C_2 e^{-\frac{C_3 \gamma}{\|f\|_{\mathrm{BMO}}}} \int \chi_Q(x) \omega(x) dx,$$

where C_2 depends on $[\omega]_{A_\infty}$. By [9, section 7.3.2], $[\omega]_{A_\infty} \leqslant [\omega]_{A_1}$, we find that C_2 depends on $[\omega]_{A_1}$.

For any $\omega \in A_1$, in view of the fact $A_1 \subset A_\infty$, the conditions (4.1) given in Theorem 4.1 are fulfilled. According to Theorem 4.1 and

$$\mathscr{F} = \left\{ \left(\chi_{\{x \in Q : |f(x) - f_Q| > \gamma\}}, e^{-\frac{C_3 \varepsilon \gamma}{\|f\|_{\mathrm{BMO}}}} \chi_Q \right) : Q \in \mathbb{R}^n \right\}$$

yield

$$\left\| \chi_{\{x \in Q : |f(x) - f_Q| \geqslant \gamma\}} \right\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})} \leqslant C e^{-\frac{C_1 \gamma}{\|f\|_{\mathrm{BMO}}}} \|\chi_Q\|_{B_\sigma(\mathscr{M}^p_{\vec{q}})}.$$

This proof is completed. $\square$

In the following, we define a new BMO space in terms of B_σ type mixed Morrey space. Let $0 \leqslant \sigma$, $1 < p, \vec{q} < \infty$ and $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$. The $\mathrm{BMO}_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}(\mathbb{R}^n)$ space be defined by

$$\mathrm{BMO}_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}(\mathbb{R}^n) = \left\{ f \in L_{\mathrm{loc}} : \|f\|_{\mathrm{BMO}_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}} = \sup_{Q \in \mathscr{Q}} \frac{\|(f - f_Q)\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}{\|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}} < \infty \right\}.$$

To ensure that $\|\cdot\|_{BMO_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}$ is a well-defined. We claim that

$$\|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})} > C, \quad \forall Q \in \mathscr{Q}.$$

Proof. It just to show that for any $0 < R$ and $r \geqslant 1$, there exist $C > 0$,such that $\chi_{Q(R)} \in B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$ because $|g| \leqslant |f|$ $a.e.$, then ,$\|g\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})} \leqslant \|f\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}$. Hence, when $1 \leqslant r \leqslant R$, by the direct compute, we have

$$\begin{aligned} \|\chi_{Q_R}\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})} &= \sup_{r \geqslant 1} \sup_{Q} r^{-\sigma} |Q|^{\frac{1}{p}\sum_{j=1}^{n}\frac{1}{q_i}} \|\chi_{Q_R}\chi_{Q_r \cap Q}\|_{L^{\vec{q}}} \\ &\geqslant R^{-\sigma} |Q_R|^{\frac{1}{p}\sum_{j=1}^{n}\frac{1}{q_i}} \|\chi_{Q_R}\|_{L^{\vec{q}}} = R^{n/p-\sigma} = C > 0. \end{aligned}$$

Similarly, when $r \geqslant R > 0$, we have

$$\begin{aligned} \|\chi_{Q_R}\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})} &= \sup_{r \geqslant 1} \sup_{Q} r^{-\sigma} |Q|^{\frac{1}{p}\sum_{j=1}^{n}\frac{1}{q_i}} \|\chi_{Q_R}\chi_{Q_r \cap Q}\|_{L^{\vec{q}}} \\ &\geqslant r^{-\sigma} |Q(r)|^{\frac{1}{p}\sum_{j=1}^{n}\frac{1}{q_i}} \|\chi_{Q_r}\|_{L^{\vec{q}}} = r^{n/p-\sigma} = C > 0, \end{aligned}$$

which implies the proof is finished. $\square$

THEOREM 5.2. *Let* $1 < p, \vec{q} < \infty$, $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$ *and* $0 < \sigma < \sum_{j=1}^{n} \frac{1}{q_j}$. *Then the norms* $\|\cdot\|_{\mathrm{BMO}_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}$ *and* $\|\cdot\|_{\mathrm{BMO}}$ *are mutually equivalent.*

To begin the proof of Theorem 5.2, we first establish the below lemma.

LEMMA 5.1. *Let* $1 < p, \vec{q} < \infty$, $\frac{n}{p} \leqslant \sum_{j=1}^{n} \frac{1}{q_j}$ *and* $0 < \sigma < \sum_{j=1}^{n} \frac{1}{q_j}$. *Then, there is a constant* $C \geqslant 1$ *such that*

$$C^{-1}|Q| \leqslant \|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})} \|\chi_Q\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \leqslant C|Q|, \quad \forall Q \in \mathscr{Q}.$$

Proof. According to inequality (3.6) yields the left inequality. For any $Q \in \mathscr{Q}$, we define the projection

$$(P_Q g)(y) = \left(\frac{1}{|Q|} \int_Q |g(x)| dx \right) \chi_Q(y).$$

There exists a constant $C>0$ such that for any $Q\in\mathscr{Q}, P_Q(f)\leqslant C\mathrm{M}(f)$. Because of (2) of Lemma 4.1, the Hardy-Littlewood maximal operator is bounded on $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)$. Then, for any $Q\in\mathscr{Q}$, one concludes that

$$\|P_Q(f)\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant C\|M(f)\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant C\|f\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}.$$

Then, there exists a constant $C>0$ such that

$$\sup_Q\|P_Q\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)\to H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant C.$$

Furthermore, by Corollary 3.1 yields that

$$\begin{aligned}
&\|\chi_Q\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\\
&=\sup\left\{\left|\int_Q g(x)dx\right|\|\chi_Q\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}:g\in H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right),\|g\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant 1\right\}\\
&\leqslant|Q|\,\|P_Qg\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\leqslant C|Q|.
\end{aligned}$$

It is finished with the proof. $\square$

Proof of Theorem 5.2. From inequality (3.6), we know that

$$\int_Q|f(x)-f_Q|\,dx\leqslant\|(f-f_Q)\,\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\|\chi_Q\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}.$$

By Lemma 5.1, we deduce that

$$\int_Q|f(x)-f_Q|\,dx\leqslant C|Q|\frac{\|(f-f_Q)\,\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}{\|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}.$$

Therefore,

$$\|f\|_{\mathrm{BMO}}\leqslant C\|f\|_{\mathrm{BMO}_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}.$$

For any $j\in\mathbb{N}$, the John-Nirenberg inequality on $B_\sigma(\mathscr{M}_{\vec{q}}^{p})$ ensures that

$$\left\|\chi_{\{x\in Q:2^j<|f(x)-f_Q|\leqslant 2^{j+1}\}}\right\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\leqslant Ce^{\left[-\frac{C_1 2^j}{\|f\|_{\mathrm{BMO}}}\right]}\|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}.$$

Multiplying $2^{(j+1)}$ on both sides and summing over j, then

$$\|(f-f_Q)\,\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\leqslant C\|f\|_{\mathrm{BMO}}\|\chi_Q\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}.$$

Hence, we obtain that

$$\|f\|_{\mathrm{BMO}_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}}\leqslant C\|f\|_{\mathrm{BMO}},\quad\forall\, f\in\mathrm{BMO}.$$

It is finished with this proof. $\square$

On the other hand, the $\mathrm{BMO}(\mathbb{R}^n)$ space can be characterized via the boundedness of commutators, which are generated by the singular integral operator or the fractional integral operator, on different spaces.

5.2. Characterization of BMO space via the boundedness of commutator

It is well know that the commutators $[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ are introduced by Coifman et al. [6], who also proved the commutators $[b,T]$ is bounded on L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$, where T is the classical Calderón-Zygmund operator. We next introduce the convolution type Calderón-Zygmund singular integral operator Tf is defined by

$$Tf(x) = p \cdot v \cdot \int_{\mathbb{R}^n} K(x-y)f(y)\mathrm{d}y,$$

where function K is called kernel satisfying (i) the size condition:

$$|K(x)| \leqslant C|x|^{-n}, \quad x \neq 0$$

and (ii) the regularity conditions: for some $\varepsilon > 0$

$$\left|K(x-y) - K\left(x'-y\right)\right| + \left|K(y-x) - K\left(y-x'\right)\right| \leqslant c_1 \left(\frac{|x-x'|}{|x-y|}\right)^{\varepsilon} |x-y|^{-n}$$

whenever $2|x-x'| < |x-y|$.

Let $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$, then the fractional integral operator I_α are defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \ \ 0 < \alpha < n.$$

In particular, Chanillo [4] proved a similar characterization for the fractional integral operator I_α. That is to say, the commutator $[b, I_\alpha]$ is bounded from L^p to L^q, if and only if $b \in \mathrm{BMO}$, when $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. The theory has been extended to more general spaces for the last thirty years. Thus, we extend the characterization of BMO space via commutators on $B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)$ in the following. In order to state the results in this section, we need the following lemma.

LEMMA 5.2. ([10]) *Let the Calderón-Zygmund singular integral operator T be defined as above. Then for all $1 < p < \infty$, $w \in A_p$ and $b \in \mathrm{BMO}$, also have*

$$\|[b,T]f\|_{L^p_w(\mathbb{R}^n)} \lesssim \|f\|_{L^p_w(\mathbb{R}^n)}.$$

THEOREM 5.3. *Let $1 < p, \vec{q} < \infty$, $0 < p, \vec{q} < \infty$, $0 < \sigma < \sum_{j=1}^n \frac{1}{q_j}$ and $\frac{n}{p} \leqslant \sum_{j=1}^n \frac{1}{q_j}$. Then, the follwing statements are equivalent*

(1) $b \in \mathrm{BMO}(\mathbb{R}^n)$*;*

(2) *the commutator $[b,T]$ is bounded from $B_\sigma(\mathscr{M}^p_{\vec{q}})(\mathbb{R}^n)$ into itself,*

Proof. $(1) \Rightarrow (2)$: In view of the Corollary 4.1 and Lemma 5.2 can quickly gain the boundedness of commutator $[b,T]$.

$(2)\Rightarrow(1)$: We follow the idea of Janson [19]. Assume that the commutator $[b,T]$ is bounded on $B_\sigma(\mathscr{M}_{\vec{q}}^{p})(\mathbb{R}^n)$. We choose $z_0\neq 0$ and $\delta>0$ such that $\frac{1}{K(x)}$ can be expressed in the cube $Q(z_0,\sqrt{n}\delta)$ as an absolutely convergent Fourier series,

$$\frac{1}{K(x)}=\sum_{m=0}^{\infty}a_m e^{i\mu_m\cdot x}.$$

Set $z_1=z_0/\delta$. Note that for all z such that $|z-z_1|<\sqrt{n}$, we have

$$\frac{1}{K(x)}=\frac{\delta^{-n}}{K(\delta z)}=\delta^{-n}\sum a_m e^{i\delta_1\nu_m\cdot z}.$$

For given cubes $Q=Q(x_0,r)$ and $Q'=Q_1(y_0,r)$, where $y_0=x_0-z_1r$ if $x\in Q$ and $y\in Q'$, then

$$\left|\frac{x-y}{r}-z_1\right|\leqslant\left|\frac{x-x_0}{r}\right|+\left|\frac{y-y_0}{r}\right|<\sqrt{n}.$$

Let $s(x)=\operatorname{sgn}(b(x)-b_{Q'})$. Then we have the following estimates:

$$\begin{aligned}
&\int_Q\left|b(x)-b_{Q'}\right|dx\\
=&\int_Q\left(b(x)-b_{Q'}\right)\overline{s(x)}dx\\
=&\frac{1}{|Q'|}\int_Q\int_{Q'}(b(x)-b(y))\overline{s(x)}dxdy\\
=&\frac{1}{|Q'|}\int_Q\int_{Q'}(b(x)-b(y))\overline{s(x)}\frac{r^nK(x-y)}{K\left(\frac{x-y}{r}\right)}dxdy\\
\sim&\sum_{m=1}^{\infty}a_m\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x)-b(y))\overline{s(x)}K(x-y)e^{i\frac{\delta}{r}\mu_m\cdot x}e^{i\frac{\delta}{r}\mu_m\cdot y}\chi_Q(x)\chi_{Q'}(y)dxdy.
\end{aligned}$$

Setting

$$\begin{aligned}
g_m(x)&=e^{i\frac{\delta}{r}\mu_m\cdot x}\chi_Q(x)\overline{s(x)},\\
f_m(y)&=e^{i\frac{\delta}{r}\mu_m\cdot y}\chi_{Q'}(y),
\end{aligned}$$

one obtains that

$$\begin{aligned}
\int_Q\left|b(x)-b_{Q'}\right|dx&\leqslant\sum_{m=1}^{\infty}a_m\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x)-b(y))K(x-y)g_m(x)f_m(y)dxdy\\
&\leqslant\sum_{m=1}^{\infty}a_m\int_{\mathbb{R}^n}|[b,T](f_m)(x)||g_m(x)|dx\\
&\leqslant\sum_{m=1}^{\infty}a_m\|[b,T](f_m)\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\|g_m\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\\
&\leqslant\|[b,T]\|\sum_{m=1}^{\infty}a_m\|(f_m)\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\|g_m\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}\\
&\leqslant\|[b,T]\|\sum_{m=1}^{\infty}a_m\|\chi_{Q'}\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})}\|\chi_Q\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)}.
\end{aligned}$$

From the choice of Q', we can find a proper cube $Q_0 \subseteq \mathbb{R}^n$, such that $Q, Q' \subseteq Q_0$ and $|Q_0| \sim |Q|$. According to Lemma 5.1, we get

$$\int_Q \left|b(x) - b_{Q'}\right| dx \leqslant \|[b,T]\| \sum_{m=1}^{\infty} a_m \left\|\chi_{Q_0}\right\|_{B_\sigma(\mathscr{M}_{\vec{q}}^{p})} \left\|\chi_{Q_0}\right\|_{H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)} \lesssim \|[b,T]\| \, |Q|.$$

Furthermore,

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\, dx \leqslant \frac{1}{|Q|}\int_Q \left|b(x) - b_{Q'}\right| dx \lesssim 2\,\|[b,T]\| \leqslant C < \infty.$$

This proof is finished. $\square$

REMARK 5.1. We do not know whether the index σ in the Theorem 5.3 is sharp. By the proof of this theorem, we know that the boundedness of the commutator is obtained through extrapolation theory. However, the extrapolation theorem relies heavily on the boundedness of the Hardy-Littlewood maximal operator on block space $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$. Moreover, the condition of σ is strong when the Hardy-Littlewood maximal operator is bounded on block space $H_\sigma\left(\mathscr{H}_{\vec{q}'}^{p'}\right)(\mathbb{R}^n)$.

Acknowledgements. The authors thank the referees for their careful reading and helpful comments which indeed improved the presentation of this article.

Funding information. The research was supported by Natural Science Foundation of China (Grant No. 12061069).

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(Received October 14, 2023)

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