Journal of
Mathematical
Inequalities
Volume 18, Number 1 (2024), 375–387
doi:10.7153/jmi-2024-18-21

A SHARP MID–POINT TYPE INEQUALITY

MOHSEN ROSTAMIAN DELAVAR*, MOHSEN KIAN AND MANUEL DE LA SEN

(*Communicated by M. Krnić*)

Abstract. This paper deals with a sharp version of mid-point type inequality in connection with fractional integrals of real valued absolutely continuous functions as a generalization and refinement of non-sharp classical mid-point inequality which is presented by the Riemann integrals of differentiable real valued functions whose derivative absolute values are convex. Some special functions, numerical means and a mid-point type formula are considered to discuss about some applications of main results.

1. Introduction and preliminaries

Celebrated Hermite-Hadamard's inequality gives a lower and an upper bound for the mean value ([24, 34]) of a convex function $\mathscr{F}:[a,b]\to\mathbb{R}$ as follows (see also [7, 9, 12, 13, 16, 19, 22, 28, 29, 31, 37]):

$$\mathscr{F}\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b\mathscr{F}(x)\mathrm{d}x\leqslant\frac{\mathscr{F}(a)+\mathscr{F}(b)}{2}. \tag{1}$$

Mid-point inequality related to (1) means how to estimate the difference between left side and middle of (1). One of the most important answers to this question has been presented in the following result which has been used to obtain various applications in mathematical inequalities, operator theory, numerical approximation of integrals, special means and random variables in statistics and special functions such as Euler's beta and gamma (see [6, 7, 15, 30, 33, 36]):

THEOREM 1.1. [15] *Let* $\mathscr{F}:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°, $a,b\in I^\circ$ *with* $a<b$. *If* $|\mathscr{F}'|$ *is convex on* $[a,b]$, *then*

$$\left|\mathscr{F}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\mathscr{F}(x)\mathrm{d}x\right|\leqslant\frac{(b-a)}{8}\Big(|\mathscr{F}'(a)|+|\mathscr{F}'(b)|\Big). \tag{2}$$

Mathematics subject classification (2020): 26A33, 26A51, 26D10, 26D15.
Keywords and phrases: Mid-point type inequality, fractional integrals, Euler's gamma function, special means.

* Corresponding author.

Note that (2) is not sharp ($\frac{1}{8}$ is not the smallest possible constant) and Theorem 1.1 does not work for functions whose derivative absolute values are not convex. Considering these mentioned points, we set some new criterions and work on the class of absolutely continuous functions defined on $[a,b] \subset \mathbb{R}$. In addition, we consider the concept of generalized Riemann-Liouville fractional integrals as a generalized form of Riemann integrals and also generalized Chebyshev functionals.

The concept of *absolutely continuity* is defined as the following [26]:

DEFINITION 1.2. A real function f is absolutely continuous on $[a,b]$ if, corresponding to any $\varepsilon > 0$, we can produce a $\delta > 0$ such that for any collection $\{(a_i,b_i)\}_1^n$ of disjoint open subintervals of $[a,b]$ with $\sum_1^n(b_i - a_i) < \delta$, we get $\sum_1^n |f(b_i) - f(a_i)| < \varepsilon$.

It would be interesting for readers to know that any convex function defined on $[a,b]$, is $\mathscr{M}$-Lipschitz and so absolutely continuous ([23, 26]). So the class of absolutely continuous functions defined on $[a,b]$, includes any convex and $\mathscr{M}$-Lipschitz functions defined on $[a,b]$.

We consider the class of generalized Riemann-Liouville fractional integrals defined in [27]:

DEFINITION 1.3. For any $t \in [0,1]$ and $\alpha > 0$,

$$\mathscr{J}^{\alpha}_{m_t(\mathscr{L},\mathscr{R})^+}\mathscr{F}(s) = \frac{1}{\Gamma(\alpha)} \int_{m_t(\mathscr{L},\mathscr{R})}^{s} (s-u)^{\alpha-1}\mathscr{F}(u)\mathrm{d}u, \qquad (s > m_t(\mathscr{L},\mathscr{R}))$$

and

$$\mathscr{J}^{\alpha}_{M_t(\mathscr{L},\mathscr{R})^-}\mathscr{F}(s) = \frac{1}{\Gamma(\alpha)} \int_{s}^{M_t(\mathscr{L},\mathscr{R})} (u-s)^{\alpha-1}\mathscr{F}(u)\mathrm{d}u, \qquad (M_t(\mathscr{L},\mathscr{R}) < s),$$

where $\mathscr{L}(t) : [0,1] \to [a,b]$ and $\mathscr{R}(t) : [0,1] \to [a,b]$ are considered as

$$\mathscr{L}(t) = tb + (1-t)a, \qquad \mathscr{R}(t) = ta + (1-t)b$$

and

$$m_t(\mathscr{L},\mathscr{R}) = \min\{\mathscr{L}(t),\mathscr{R}(t)\}, \qquad M_t(\mathscr{L},\mathscr{R}) = \max\{\mathscr{L}(t),\mathscr{R}(t)\}.$$

Also Euler's Gamma function is defined as ([3, 5])

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}\mathrm{d}t \qquad (\mathrm{Re}(s) > 0).$$

Fractional integrals $\mathscr{J}^{\alpha}_{m_t(\mathscr{L},\mathscr{R})^+}\mathscr{F}(s)$ and $\mathscr{J}^{\alpha}_{M_t(\mathscr{L},\mathscr{R})^-}\mathscr{F}(s)$ in special case ($t = 0,1$) reduce to $J^{\alpha}_{a^+}\mathscr{F}(s)$ and $J^{\alpha}_{b^-}\mathscr{F}(s)$, respectively, which are known as the Riemann-Liouville fractional integrals in literature (also see [10, 17, 32]).

We define generalized Chebyshev functional as follows (see [4, 20, 21]):

DEFINITION 1.4. For any $t \in [0,1] \setminus \{\frac{1}{2}\}$ and any pair of integrable functions $\mathscr{F}, \mathscr{G} : [a,b] \to \mathbb{R}$, the generalized Chebyshev functional is defined by

$$\begin{aligned} \mathscr{T}_t(\mathscr{F},\mathscr{G}) = & \frac{1}{\Delta_t(\mathscr{L},\mathscr{R})} \int_{m_t(\mathscr{L},\mathscr{R})}^{M_t(\mathscr{L},\mathscr{R})} \mathscr{F}(s)\mathscr{G}(s)ds \\ & - \frac{1}{\Delta_t^2(\mathscr{L},\mathscr{R})} \int_{m_t(\mathscr{L},\mathscr{R})}^{M_t(\mathscr{L},\mathscr{R})} \mathscr{F}(s)ds \int_{m_t(\mathscr{L},\mathscr{R})}^{M_t(\mathscr{L},\mathscr{R})} \mathscr{G}(s)ds, \end{aligned}$$

where $\Delta_t^{\alpha}(\mathscr{L},\mathscr{R}) = \big(M_t(\mathscr{L},\mathscr{R}) - m_t(\mathscr{L},\mathscr{R})\big)^{\alpha}$, $\alpha > 0$. For the case that $\mathscr{F} = \mathscr{G}$, we use the notation $\mathscr{T}_t(\mathscr{F})$.

Motivated by [11], we state the following result in connection with generalized Chebyshev functional $\mathscr{T}_t(\mathscr{F},\mathscr{G})$ and its bounds:

THEOREM 1.5. *For any $t \in [0,1]$, If there exist real numbers $\varphi_t, \phi_t, \gamma_t, \Gamma_t$ such that $\varphi_t \leqslant \mathscr{F}(s) \leqslant \phi_t$ and $\gamma_t \leqslant \mathscr{G}(s) \leqslant \Gamma_t$ for all $s \in [m_t(\mathscr{L},\mathscr{R}), M_t(\mathscr{L},\mathscr{R})]$ then:*

$$\big|\mathscr{T}_t(\mathscr{F},\mathscr{G})\big| \leqslant \frac{1}{4}(\phi_t - \varphi_t)(\Gamma_t - \gamma_t).$$

Proof. Technically the proof is similar to what was mentioned in [11]. It is just enough to consider parameter "t" and local bounds of $\mathscr{F}$ and $\mathscr{G}$ on $[m_t(\mathscr{L},\mathscr{R}), M_t(\mathscr{L},\mathscr{R})]$. □

2. Main results

In this section, we obtain a sharp mid-point type inequality in connection with Riemann-Liouville fractional integrals by using the concepts defined in previous section . Also as corollaries, some special sharp and non-sharp mid-point type inequalities are discussed. The following lemma is of interest and needed to obtain the main result of this section. In what follows we consider "m_t" and "M_t" as "$m_t(\mathscr{L},\mathscr{R})$" and "$M_t(\mathscr{L},\mathscr{R})$", briefly.

LEMMA 2.1. *For real numbers a,b,α,t with $a<b$, $t \in [0,1] \setminus \{\frac{1}{2}\}$, $\alpha > 0$ and absolutely continuous function $\mathscr{F} : [a,b] \to \mathbb{R}$, the following characterization holds:*

$$\begin{aligned} & \frac{1}{\Delta_t^{\alpha}(\mathscr{L},\mathscr{R})} \int_{m_t}^{M_t} \mathscr{P}(t,s)\mathscr{F}'(s)ds \\ & = \mathscr{F}\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2\Delta_t^{\alpha}(\mathscr{L},\mathscr{R})}\Big(\mathscr{J}_{M_t^-}^{\alpha}\mathscr{F}(m_t) + \mathscr{J}_{m_t^+}^{\alpha}\mathscr{F}(M_t)\Big), \end{aligned}$$

where the bifunction $\mathscr{P}(t,s) : [0,1] \setminus \{\frac{1}{2}\} \times [m_t, M_t] \to \mathbb{R}$ is defined as

$$\mathscr{P}(t,s) = \begin{cases} \frac{(s-m_t)^{\alpha} - (M_t - s)^{\alpha} + \Delta_t^{\alpha}(\mathscr{L},\mathscr{R})}{2}, & m_t \leqslant s \leqslant \frac{a+b}{2}; \\ \frac{(s-m_t)^{\alpha} - (M_t - s)^{\alpha} - \Delta_t^{\alpha}(\mathscr{L},\mathscr{R})}{2}, & \frac{a+b}{2} < s \leqslant M_t. \end{cases}$$

Proof. By the use of integration by parts rule and fundamental theorem of Lebesgue integral calculus, we deduce the desired result:

$$\int_{m_t}^{M_t} \mathscr{P}(t,s)\mathscr{F}'(s)ds = \mathscr{F}\Big(\frac{a+b}{2}\Big)\mathscr{P}\Big(t,\frac{a+b}{2}\Big) - \mathscr{F}(m_t)\mathscr{P}(t,m_t) + \mathscr{F}(M_t)\mathscr{P}(t,M_t)$$
$$-\mathscr{F}\Big(\frac{a+b}{2}\Big).\lim_{s\to\frac{a+b}{2}^+}\mathscr{P}(t,s) - \frac{\alpha}{2}\int_{m_t}^{M_t}\big[(s-m_t)^{\alpha-1}+(M_t-s)^{\alpha-1}\big]\mathscr{F}(s)ds$$
$$= \mathscr{F}\Big(\frac{a+b}{2}\Big)\Delta_t^{\alpha}(\mathscr{L},\mathscr{R}) - \frac{\Gamma(\alpha+1)}{2}\Big[\mathscr{J}^{\alpha}_{M_t^-}\mathscr{F}(m_t) + \mathscr{J}^{\alpha}_{m_t^+}\mathscr{F}(M_t)\Big]. \quad \square$$

COROLLARY 2.2. *With all conditions of Lemma 2.1, for $t = 0,1$ we get (see Lemma 1 in [14])*

$$\int_a^b \mathscr{P}(s)\mathscr{F}'(s)ds = \mathscr{F}\Big(\frac{a+b}{2}\Big)(b-a)^{\alpha} - \frac{\Gamma(\alpha+1)}{2}\Big(\mathscr{J}^{\alpha}_{b^-}\mathscr{F}(a) + \mathscr{J}^{\alpha}_{a^+}\mathscr{F}(b)\Big),$$

where the function $\mathscr{P}(s):[a,b]\to\mathbb{R}$ is defined as

$$\mathscr{P}(s) = \begin{cases} \frac{(s-a)^{\alpha}-(b-s)^{\alpha}+(b-a)^{\alpha}}{2}, & a\leqslant s\leqslant \frac{a+b}{2};\\ \frac{(s-a)^{\alpha}-(b-s)^{\alpha}-(b-a)^{\alpha}}{2}, & \frac{a+b}{2}< s\leqslant b.\end{cases}$$

In more special case we have

$$\int_a^b \mathscr{P}(s)\mathscr{F}'(s)ds = \mathscr{F}\Big(\frac{a+b}{2}\Big)(b-a) - \int_a^b \mathscr{F}(s)ds,$$

where the function $\mathscr{P}(s):[a,b]\to\mathbb{R}$ is defined as

$$\mathscr{P}(s) = \begin{cases} s-a, & a\leqslant s\leqslant \frac{a+b}{2};\\ s-b, & \frac{a+b}{2}< s\leqslant b.\end{cases}$$

which is equivalent to Lemma 2.1 in [15].

THEOREM 2.3. *For real numbers a,b,α,t with $a<b$, $t\in[0,1]\setminus\{\frac{1}{2}\}$, $\alpha>0$ and absolutely continuous function $\mathscr{F}:[a,b]\to\mathbb{R}$ with $\mathscr{F}'\in L^2([a,b])$, the following mid-point type inequality holds:*

$$\left|\mathscr{F}\Big(\frac{a+b}{2}\Big) - \frac{\Gamma(\alpha+1)}{2\Delta_t^{\alpha}(\mathscr{L},\mathscr{R})}\Big(\mathscr{J}^{\alpha}_{M_t^-}\mathscr{F}(m_t)+\mathscr{J}^{\alpha}_{m_t^+}\mathscr{F}(M_t)\Big)\right| \leqslant \mathscr{C}_{\alpha}\Delta_t^{1}(\mathscr{L},\mathscr{R})\big(\mathscr{T}_t(\mathscr{F}')\big)^{\frac{1}{2}}, \tag{3}$$

where $\mathscr{C}_{\alpha} = \Big(\frac{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)} + \frac{(\alpha-3)2^{\alpha-2}+1}{2^{\alpha}(\alpha+1)}\Big)^{\frac{1}{2}}$. Also for any $\alpha>0$, the value of $\mathscr{C}_{\alpha}$ is the best possible in the sense that can not be replaced by a smaller one.

Proof. Because of the fact

$$\int_{m_t}^{\frac{a+b}{2}}\left[(s-m_t)^\alpha-(M_t-s)^\alpha+(M_t-m_t)^\alpha\right]ds$$
$$=-\int_{\frac{a+b}{2}}^{M_t}\left[(s-m_t)^\alpha-(M_t-s)^\alpha-(M_t-m_t)^\alpha\right]ds,$$

it is not hard to see that

$$\int_{m_t}^{M_t}\mathscr{P}(t,s)ds=0. \tag{4}$$

So from Lemma 2.1, (4) and Hölder's integral inequality we get

$$\begin{aligned}
I&=\left|\mathscr{F}\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2\Delta_t^\alpha(\mathscr{L},\mathscr{R})}\left(\mathscr{J}_{M_t-}^{\alpha}\mathscr{F}(m_t)+\mathscr{J}_{m_t+}^{\alpha}\mathscr{F}(M_t)\right)\right|\\
&=\frac{1}{\Delta_t^\alpha(\mathscr{L},\mathscr{R})}\left|\int_{m_t}^{M_t}\mathscr{P}(t,s)\left[\mathscr{F}'(s)-\frac{\mathscr{F}(M_t)-\mathscr{F}(m_t)}{\Delta_t^1(\mathscr{L},\mathscr{R})}\right]ds\right|\\
&\leqslant\frac{1}{\Delta_t^\alpha(\mathscr{L},\mathscr{R})}\left\{\int_{m_t}^{M_t}\mathscr{P}^2(t,s)ds\right\}^{\frac{1}{2}}\left\{\int_{m_t}^{M_t}\left[\mathscr{F}'(s)-\frac{\mathscr{F}(M_t)-\mathscr{F}(m_t)}{\Delta_t^1(\mathscr{L},\mathscr{R})}\right]^2ds\right\}^{\frac{1}{2}}.
\end{aligned}$$

Now we calculate the integral of $\mathscr{P}^2(t,s)$ on $[m_t,M_t]$ with respect to the variable "s" by considering the following equivalence:

$$\begin{aligned}
\int_{m_t}^{M_t}\mathscr{P}^2(t,s)ds&=\int_{m_t}^{\frac{a+b}{2}}\left[\frac{(s-m_t)^\alpha-(M_t-s)^\alpha+\Delta_t^\alpha(\mathscr{L},\mathscr{R})}{2}\right]^2ds\\
&\quad+\int_{\frac{a+b}{2}}^{M_t}\left[\frac{(s-m_t)^\alpha-(M_t-s)^\alpha-\Delta_t^\alpha(\mathscr{L},\mathscr{R})}{2}\right]^2ds\\
&=\frac{1}{2}\int_{\frac{a+b}{2}}^{M_t}\left[(s-m_t)^\alpha-(M_t-s)^\alpha-\Delta_t^\alpha(\mathscr{L},\mathscr{R})\right]^2ds,
\end{aligned}$$

which holds by the fact

$$\int_{m_t}^{\frac{a+b}{2}}\left[(s-m_t)^\alpha-(M_t-s)^\alpha+(M_t-m_t)^\alpha\right]^2ds$$
$$=\int_{\frac{a+b}{2}}^{M_t}\left[(s-m_t)^\alpha-(M_t-s)^\alpha-(M_t-m_t)^\alpha\right]^2ds.$$

It follows that

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{M_t}\Big[(s-m_t)^{\alpha}-(M_t-s)^{\alpha}-\Delta_t^{\alpha}(\mathscr{L},\mathscr{R})\Big]^2 ds\\
&=\int_{\frac{a+b}{2}}^{M_t}\Big[(s-m_t)^{\alpha}-(M_t-s)^{\alpha}\Big]^2 ds\\
&\quad-2\Delta_t^{\alpha}(\mathscr{L},\mathscr{R})\int_{\frac{a+b}{2}}^{M_t}\Big[(s-m_t)^{\alpha}-(M_t-s)^{\alpha}\Big]ds+\int_{\frac{a+b}{2}}^{M_t}\Delta_t^{2\alpha}(\mathscr{L},\mathscr{R})ds\\
&=\Big[\frac{1}{2\alpha+1}-\frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)}+\frac{1-2^{\alpha}}{2^{\alpha-1}(\alpha+1)}+\frac{1}{2}\Big]\Delta_t^{2\alpha+1}(\mathscr{L},\mathscr{R})\\
&=\Big[\frac{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)}+\frac{(\alpha-3)2^{\alpha-2}+1}{2^{\alpha-1}(\alpha+1)}\Big]\Delta_t^{2\alpha+1}(\mathscr{L},\mathscr{R}),
\end{aligned}$$

which implies that

$$\left\{\int_{m_t}^{M_t}\mathscr{P}^2(t,s)ds\right\}^{\frac{1}{2}}=\mathscr{C}_{\alpha}\Delta_t^{\alpha+\frac{1}{2}}(\mathscr{L},\mathscr{R}).$$

Also we have

$$\begin{aligned}
&\left\{\int_{m_t}^{M_t}\Big[\mathscr{F}'(s)-\frac{\mathscr{F}(M_t)-\mathscr{F}(m_t)}{\Delta_t^1(\mathscr{L},\mathscr{R})}\Big]^2 ds\right\}^{\frac{1}{2}}\\
&=\left\{\int_{m_t}^{M_t}(\mathscr{F}'(s))^2 ds-\frac{1}{\Delta_t^1(\mathscr{L},\mathscr{R})}\Big(\int_{m_t}^{M_t}\mathscr{F}'(s)ds\Big)^2\right\}^{\frac{1}{2}}\\
&=\Big(\Delta_t^1(\mathscr{L},\mathscr{R})\mathscr{T}_t(\mathscr{F}')\Big)^{\frac{1}{2}}.
\end{aligned}$$

By all above results we conclude

$$I\leqslant\mathscr{C}_{\alpha}\Delta_t^1(\mathscr{L},\mathscr{R})\big(\mathscr{T}_t(\mathscr{F}')\big)^{\frac{1}{2}},$$

which gives (3) as well. For the Sharpness of (3), whiteout loss of generality, set $\alpha>0$, $t=0,1$, $a=0$, $b=1$ and define

$$\mathscr{F}(s)=\begin{cases}\frac{s^{\alpha+1}+(1-s)^{\alpha+1}}{2(\alpha+1)}+\frac{s}{2}, & 0\leqslant s\leqslant\frac{1}{2};\\ \frac{s^{\alpha+1}+(1-s)^{\alpha+1}}{2(\alpha+1)}+\frac{(1-s)}{2}, & \frac{1}{2}<s\leqslant 1.\end{cases}$$

Now suppose that there exists a real number $\mathscr{D}>0$ such that

$$\left|\mathscr{F}\Big(\frac{1}{2}\Big)-\frac{\Gamma(\alpha+1)}{2}\Big(\mathscr{J}_{1^-}^{\alpha}\mathscr{F}(0)+\mathscr{J}_{0^+}^{\alpha}\mathscr{F}(1)\Big)\right|\leqslant\mathscr{D}.\big(\mathscr{T}_0(\mathscr{F}')\big)^{\frac{1}{2}},$$

holds. It is not hard to see that

$$\int_0^1\mathscr{F}'(s)ds=0,$$

and

$$\int_0^1 (\mathscr{F}'(s))^2 ds = \left(\frac{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)} + \frac{(\alpha-3)2^{\alpha-2}+1}{2^{\alpha-1}(\alpha+1)}\right),$$

which imply that

$$\left(\mathscr{T}_0(\mathscr{F}')\right)^{\frac{1}{2}} = \mathscr{C}_\alpha. \tag{5}$$

On the other hand

$$\mathscr{F}\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^\alpha}{\alpha+1} + \frac{1}{4},$$

and

$$\mathscr{J}_{1^-}^{\alpha}\mathscr{F}(0) = \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}\mathscr{F}(s)ds = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mathscr{F}(s)ds = \mathscr{J}_{0^+}^{\alpha}\mathscr{F}(1). \tag{6}$$

If we calculate (6) directly, we obtain that

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2}\left(\mathscr{J}_{1^-}^{\alpha}\mathscr{F}(0)+\mathscr{J}_{0^+}^{\alpha}\mathscr{F}(1)\right)\\ &= \frac{\alpha}{2\alpha+2}\left(\frac{1}{2\alpha+1}+\frac{\Gamma(\alpha)\Gamma(\alpha+2)}{\Gamma(2\alpha+2)}\right) + \frac{\alpha(2^{-\alpha-2})}{\alpha+1} + \frac{2^{-\alpha-2}(2^{\alpha+1}-\alpha-2)}{\alpha+1}.\end{aligned}$$

So, further calculations show that

$$\begin{aligned}&\left|\mathscr{F}\left(\frac{1}{2}\right) - \frac{\Gamma(\alpha+1)}{2}\left(\mathscr{J}_{1^-}^{\alpha}\mathscr{F}(0)+\mathscr{J}_{0^+}^{\alpha}\mathscr{F}(1)\right)\right|\\ &= \left|\frac{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)} + \frac{\frac{1}{2}+(\alpha+1)2^{\alpha-2}-2^{\alpha-1}-\frac{\alpha}{4}-\frac{1}{4}(2^{\alpha+1}-\alpha-2)}{2^\alpha(\alpha+1)}\right|\\ &= \left|\frac{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}{2\Gamma(2\alpha+2)} + \frac{(\alpha-3)2^{\alpha-2}+1}{2^\alpha(\alpha+1)}\right| = \mathscr{C}_\alpha^2.\end{aligned}$$

Then by (5), we get

$$\mathscr{C}_\alpha^2 \leqslant \mathscr{D}\left(\mathscr{T}_0(\mathscr{F})\right)^{\frac{1}{2}} = \mathscr{D}\mathscr{C}_\alpha.$$

This means that

$$\mathscr{D} \geqslant \mathscr{C}_\alpha. \quad \square$$

REMARK 2.4. With all conditions of of Theorem 2.3, for special case of $\alpha = 1$ we have:

$$\left|\mathscr{F}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \mathscr{F}(s)ds\right| \leqslant \frac{\Delta_t^1(\mathscr{L},\mathscr{R})}{\sqrt{12}}\left(\mathscr{T}_t(\mathscr{F}')\right)^{\frac{1}{2}}, \tag{7}$$

and for $t=0$ and $t=1$ we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\mathscr{J}_{b^-}^{\alpha}\mathscr{F}(a)+\mathscr{J}_{a^+}^{\alpha}\mathscr{F}(b)\right)\right|\leqslant\mathscr{C}_{\alpha}(b-a)\left(\mathscr{T}_0(\mathscr{F}')\right)^{\frac{1}{2}}. \tag{8}$$

In more special case, by merging (7) and (8) we deduce

$$\left|\mathscr{F}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\mathscr{F}(s)ds\right|\leqslant\frac{b-a}{\sqrt{12}}\left(\mathscr{T}_0(\mathscr{F}')\right)^{\frac{1}{2}}. \tag{9}$$

Also a non-sharp kind of (9) can be

$$\left|\mathscr{F}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\mathscr{F}(s)ds\right|\leqslant\sqrt{\frac{b-a}{12}}\|\mathscr{F}'\|_2.$$

3. Examples and applications

EXAMPLE 3.1. Consider the function $\mathscr{F}(t)=\ln(\Gamma(t))$, $t>0$. This function is convex on $(0,\infty)$, hence it is absolutely continuous on $[x,x+1]\subset(0,\infty)$ (see [27]). From (9), we obtain

$$\begin{aligned}0&\leqslant\ln\Gamma\big(x+\frac{1}{2}\big)-\int_x^{x+1}\ln(\Gamma(t))\mathrm{d}t \qquad (10)\\&\leqslant\frac{1}{\sqrt{12}}\left\{\int_x^{x+1}\left[(\ln(\Gamma(t))'\right]^2\mathrm{d}t-\left[\int_x^{x+1}(\ln(\Gamma(t))'\mathrm{d}t\right]^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Consider the notion of *digamma function* which is defined as the logarithmic derivative of the gamma function [1, 2, 18]:

$$\psi(t)=\frac{\Gamma'(t)}{\Gamma(t)}\qquad\mathscr{R}(t)>0,\qquad(\mathscr{R}(t)\text{ is the real part of }t)$$

which is strictly increasing, strictly concave on $(0,\infty)$ and satisfies the following recurrence relation:

$$\psi(t+1)=\psi(t)+\frac{1}{t}.$$

Now, if we use Raabe's formula [25]

$$\int_x^{x+1}\ln(\Gamma(t))\mathrm{d}t=\ln\sqrt{2\pi}+x\ln(x)-x,$$

and Theorem 1.5 in (10), we get

$$\begin{aligned}0&\leqslant\ln\Gamma\Big(x+\frac{1}{2}\Big)-\ln\left(\sqrt{2\pi}\left(\frac{x}{e}\right)^x\right) \qquad (11)\\&\leqslant\frac{1}{4\sqrt{3}}(\psi(x+1)-\psi(x))=\frac{1}{4\sqrt{3}x},\qquad x>0.\end{aligned}$$

From (11) we obtain

$$\lim_{x\to\infty}\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}}=1,$$

which is the basis of well-known generalized Stirling's formula [8]:

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^{s}\qquad\text{as}\qquad x\to\infty.$$

EXAMPLE 3.2. Consider hyperbolic function $\mathscr{F}(x)=\tanh(x)$, $x\in\mathbb{R}$. This function is continuous and differentiable on $[a,b]\subset\mathbb{R}$. Also it satisfies a Lipschitz condition for $\mathscr{M}=1$ since

$$|\tanh(b)-\tanh(a)|\leqslant\left|\frac{1}{\cosh^{2}_{x\in(a,b)}(x)}\right||b-a|\leqslant|b-a|,$$

for all $a,b\in\mathbb{R}$ (Mean-Value Theorem [24]). So $\mathscr{F}$ is absolutely continuous on $\mathbb{R}$ whereas $|\mathscr{F}'|$ is not convex neccessarily and we can not use Theorem 1.1. In order to obtain an appropriate result we consider (3), where $t\in[0,1]\setminus\frac{1}{2}$. According to the fact that (Theorem 1.5)

$$\mathscr{T}_{t}\left(\frac{1}{\cosh^{2}(x)}\right)\leqslant\frac{1}{4}(\phi_{t}-\varphi_{t})^{2}\leqslant\frac{1}{4},\qquad x\in[a,b]$$

we have

$$\left|\tanh\left(\frac{a+b}{2}\right)-\frac{1}{\Delta_{t}(\mathscr{L},\mathscr{R})}\ln\frac{\cosh(M_{t})}{\cosh(m_{t})}\right|\leqslant\frac{\Delta_{t}(\mathscr{L},\mathscr{R})}{4\sqrt{3}},\tag{12}$$

which is interesting in its own way. For $t=0,1$ and so closed values of a an b (let $b\to a$), (12) implies that

$$\tanh\left(\frac{a+b}{2}\right)(b-a)\approx\ln\frac{\cosh(b)}{\cosh(a)}.$$

3.1. Mid-point formula

Here, we present a mid-point formula to approximate the Riemann intgral of an absolutely continuous function and give a bound for approximation error.

Choose a partition $\mathscr{D}=\{a=x_0,x_1,\ldots,x_{n-1},x_n=b\}$ of interval $[a,b]$ and set $h_i=x_{i+1}-x_i$, $i\in\{0,1,\cdots,n\}$. Consider the following formula

$$\int_a^b\mathscr{F}(s)\mathrm{d}s=T(\mathscr{F},\mathscr{D})+E(\mathscr{F},\mathscr{D}),$$

where we define

$$T(\mathscr{F},\mathscr{D})=\sum_{i=0}^{n-1}\mathscr{F}\left(\frac{x_i+x_{i+1}}{2}\right)h_i,$$

and $E(\mathscr{F},\mathscr{P})$ as the approximation error. Suppose that $\mathscr{F}$ satisfies all conditions of Theorem 2.3 on $[a,b]$. For each $i\in\{0,1,\cdots,n\}$, define

$$\mathscr{T}_0^i(\mathscr{F}')=\frac{1}{h_i}\int_{x_i}^{x_{i+1}}(\mathscr{F}'(s))^2\mathrm{d}s-\frac{1}{h_i^2}\Big(\int_{x_i}^{x_{i+1}}\mathscr{F}'(s)\mathrm{d}s\Big)^2.$$

So we have

$$\begin{aligned}\Big|\int_a^b\mathscr{F}(s)\mathrm{d}s-\sum_{i=0}^{n-1}\mathscr{F}\Big(\frac{x_i+x_{i+1}}{2}\Big)h_i\Big| &\leqslant \sum_{i=0}^{n-1}\Big|\int_{x_i}^{x_{i+1}}\mathscr{F}(s)\mathrm{d}s-\mathscr{F}\Big(\frac{x_i+x_{i+1}}{2}\Big)h_i\Big|\\ &\leqslant \frac{1}{\sqrt{12}}\sum_{i=0}^{n-1}h_i^2\sqrt{\mathscr{T}_0^i(\mathscr{F}')}.\end{aligned}$$

Then, we conclude that

$$|E(\mathscr{F},\mathscr{P})|\leqslant\frac{1}{\sqrt{12}}\sum_{i=0}^{n-1}h_i^2\sqrt{\mathscr{T}_0^i(\mathscr{F}')}.$$

If for all $i\in\{0,1,\cdots,n\}$ we set $h_i=\frac{b-a}{n}$, then by the use of Cauchy-Schwarz inequality [35] we get

$$\sum_{i=0}^{n-1}\sqrt{\mathscr{T}_0^i(\mathscr{F}')}\leqslant\sqrt{n\sum_{i=0}^{n-1}\mathscr{T}_0^i(\mathscr{F}')}\leqslant n\sqrt{\mathscr{T}_0(\mathscr{F}')},$$

which finally implies that

$$|E(\mathscr{F},\mathscr{P})|\leqslant\frac{(b-a)^2}{\sqrt{12}n}\sqrt{\mathscr{T}_0(\mathscr{F}')}.$$

3.2. Special means

For $a,b>0$, consider the following well-known numerical special means:

$$A(a,b)=\frac{a+b}{2},\qquad \text{arithmetic mean;}$$

$$G(a,b)=\sqrt{ab},\qquad \text{geometric mean;}$$

furthermore for $a\neq b$

$$L(a,b)=\frac{b-a}{\ln(b)-\ln(a)},\qquad \text{logarithmic mean;}$$

$$L_p(a,b)=\Big[\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\Big]^{\frac{1}{p}},\quad p\in\mathbb{R}\setminus\{-1,0\}\qquad \text{generalized logarithmic mean.}$$

Note that $L_p^p(a,b)=(L_p(a,b))^p$. Two famous inequalities are happened between above special means ([7]):

$$G(a,b)\leqslant L(a,b)\leqslant A(a,b),\tag{13}$$

and

$$A^p(a,b) \leqslant L_p^p(a,b) \leqslant A(a^p,b^p). \tag{14}$$

To estimate differences between the left and middle parts of (13) and (14), consider $\mathscr{F}(x) = b^x a^{1-x}, x \in [0,1]$ and $a \neq b$. It is not hard to see that $\mathscr{F}$ is absolutely continuous because of

$$\mathscr{F}''(x) = \mathscr{F}(x)(\ln(a) - \ln(b))^2 > 0$$

on $[0,1]$ and also

$$\int_0^1 \mathscr{F}(x)\mathrm{d}x = L(a,b) \qquad \text{and} \qquad \int_0^1 [\mathscr{F}'(x)]^2\mathrm{d}x = \frac{1}{2}(a^2-b^2)(\ln(a)-\ln(b)).$$

So from Corollary 2.4, we have

$$\Big|G(a,b) - L(a,b)\Big| \leqslant \frac{\ln(b)-\ln(a)}{\sqrt{12}} G\big(A(a,b),L(a,b)\big).$$

Also if we consider the function $\mathscr{F}(x) = x^p$ on $[a,b]$, the following inequality holds:

$$\Big|A^p(a,b) - L_p^p(a,b)\Big| \leqslant \frac{p(b-a)(b^{p-1}-a^{p-1})}{2\sqrt{12}}.$$

Acknowledgement. The authors would like to thank the referee(s) for valuable comments and suggestions.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover Publications, New York, (1964).

[2] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comput. **66** (217) (1997), 373–389.

[3] R. BEALS AND R. WONG, *Special Functions: A Graduate Text*, Cambridge University Press, Cambridge, (2010).

[4] P. L. CHEBYSHEV, *Sur less expressions approximatives des intégrales définies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov, **2** (1882), 93–98 (in Russian).

[5] P. J. DAVIS, *Leonhard Euler's integral: a historical profile of the gamma function*, Am. Math. Mon. **66** (10) (1959), 849–869.

[6] S. S. DRAGOMIR, Y. J. CHO AND S. S. KIM, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl. **245** (2000), 489–501.

[7] S. S. DRAGOMIR AND C. E. M. PEARCE, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, (2000), online: `http://ajmaa.org/RGMIA/monographs.php/`.

[8] J. DUTKA, *The early history of the factorial function*, Arch. Hist. Exact Sci. **43** (3) (1991), 225–249.

[9] M. ESHAGHI GORDJI, M. ROSTAMIAN DELAVAR AND M. DE LA SEN, *On ϕ-convex function*, J. Math. Inequal. **10** (1) (2016), 173–183.

[10] R. GORENFLO AND F. MAINARDI, *Fractional Calculus, Integral and Differential Equations of Fractional Order*, Springer Verlag, Wien and New York, (1997), 223–276.

[11] G. GRÜSS, *fiber das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, Math. Z. **39** (1935), 215–226.

[12] J. HADAMARD, *Étude sur les propriètés des fonctions entiéres et en particulier d'une fontion considérée par Riemann*, J. Math. Pures. Appl. **58** (1893) 171–215.

[13] C. HERMITE, *Sur deux limites d'une intégrale définie*, Mathesis, **3** (1883), 82–83.

[14] M. IQBAL, M. IQBAL BHATTI AND K. NAZEER, *Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc. **52** (3) (2015), 707–0716.

[15] U. S. KIRMACI, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp. **147** (1) (2004), 137–146.

[16] U. S. KIRMACI, M. KLARIČIĆ BAKULA, M. E. ÖZDEMIR AND J. PEČARIĆ, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput. **193** (1) (2007), 26–35.

[17] V. KIRYAKOVA, *Generalized Fractional Calculus and Applications*, John Wiley & Sons Inc., New York, (1994).

[18] K. S. KÖLBIG, *The polygamma function* $\psi^k(x)$ *for* $x = 1/4$ *and* $x = 3/4$, J. Comput. Appl. Math. **75** (1) (1996), 43–46.

[19] D. S. MITRINOVIĆ AND I. B. LACKOVIĆ, *Hermite and convexity*, Aequationes Math. **28** (1985) 229–232.

[20] D. S. MITRINOVIĆ, J. PEČARIĆ AND A. M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[21] C. P. NICULESCU AND L. E. PERSSON, *Convex Functions and Their Applications: A Contemporary Approach*, Springer, CMS Books in Mathematics, Berlin, (2006).

[22] C. E. M. PEARCE AND A. M. RUBINOV, *P-functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl. **240** (1999) 92–104.

[23] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc., (1992).

[24] M. H. PROTTER AND C. B. MORREY JR, *Intermediate Calculus* (second ed.), Springer, New York, (1985).

[25] J. L. RAABE, *Angen aherte bestimmung der factorenfolge* $1.2.3.4.5\dots n = \Gamma(1+n) = \int x^n e^{-x} dx$, *wenn n eine sehr grosse zahl ist*, J. Reine Angew. Math. **25** (1843), 146–159.

[26] A. W. ROBERT AND D. E. VARBEG, *Convex Functions*, Academic Press, New York and London (1973).

[27] M. ROSTAMIAN DELAVAR, *On Fejér's inequality: generalizations and applications*, J. Ineq. Appl. (2023), 2023:42.

[28] M. ROSTAMIAN DELAVAR AND S. S. DRAGOMIR, *On* η*-convexity*, Math. Ineq. Appl. **20** (2017), 203–216.

[29] M. ROSTAMIAN DELAVAR AND S. S. DRAGOMIR, *Weighted trapezoidal inequalities related to the area balance of a function with applications*, Appl. Math. Comput. **340** (2019), 5–14.

[30] M. ROSTAMIAN DELAVAR AND S. S. DRAGOMIR, *Hermite-Hadamard's mid-point type inequalities for generalized fractional integrals*, RACSAM. (2020), 114:73.

[31] M. ROSTAMIAN DELAVAR AND M. DE LA SEN, *A mapping associated to h-convex version of the Hermite-Hadamard inequality with applications*, J. Math. Inequal. **14** (2) (2020), 329–335.

[32] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, (1993).

[33] M. Z. SARIKAYA, E. SET, H. YALDIZ AND N. BAŞAK, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. **57** (2013), 2403–2407.

[34] H. H. SOHRAB, *Basic Real Analysis* (2nd ed.), Birkhäuser, (2003).

[35] J. M. STEELE, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, The Mathematical Association of America, (2004).

[36] N. UJEVIĆ, *Sharp inequalities of Simpson type and Ostrowski type*, Comput. Math. Appl. **48** (2004), 145–151.

[37] G.-S. YANG, *Inequalities of Hadamard type for Lipschitzian mappings*, J. Math. Anal. Appl. **260** (2001), 230–238.

(Received November 26, 2023)

Mohsen Rostamian Delavar
Department of Mathematics
Faculty of Basic Sciences, University of Bojnord
P.O. Box 1339, Bojnord 94531, Iran
e-mail: m.rostamian@ub.ac.ir

Mohsen Kian
Department of Mathematics
Faculty of Basic Sciences, University of Bojnord
P.O. Box 1339, Bojnord 94531, Iran
e-mail: kian@ub.ac.ir

Manuel De la Sen
Institute of Research and Development of Processes
University of Basque Country, Campus of Leioa (Bizkaia) – Aptdo
644-Bilbao, Bilbao, 48080, Spain
e-mail: manuel.delasen@ehu.es