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NORMAL APPROXIMATION FOR A RANDOMLY INDEXED BRANCHING PROCESS

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Abstract. Consider a supercritical Galton-Watson process $\{Z_n, n \geqslant 0\}$ and an independent renewal process $\{N(t), t \geqslant 0\}$, one-term Edgeworth expansions and Cramér type moderate deviations for the logarithm of $Z_{N(t)}$ are developed. Examples are also given to illustrate our results.

1. Introduction

A randomly indexed branching process was introduced by Epps [3] as an alternative to geometric Brownian motion for modeling stock prices. Precisely, let $\{Z_n, n \in \mathbb{N}\}$ be a supercritical Galton-Watson process with offspring distribution $\{p_i, i \in \mathbb{N}\}$ and $\{N(t), t \geqslant 0\}$ be a renewal process with the inter-arrival times $\{T_n, n \in \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$, if $\{Z_n\}$ is independent of $\{N(t)\}$, $\{Y_t = Z_{N(t)}, t \geqslant 0\}$ is said to be a randomly indexed branching process (BPRI).

Assuming that the indexing process is a simple Poisson process and four particular offspring distributions, Epps [3] studied the asymptotics of the moments, estimates of certain parameters of the BPRI, and model calibration based on real data from the New York Stock Exchange. When the indexed process is a general renewal process, [8] and [9] investigated the probability of non-extinction, the asymptotic behavior of the moments, and also limiting distributions under normalization. Results on subcritical case were done in [10]. The large deviation results were given in [5]. More recently, in [7], they extended the BPRI to controlled branching processes.

As the population grows exponentially fast on the survival set [6] and so, it is convenient to consider the sequence $\{\log Y_t, t \geqslant 0\}$ rather than $\{Y_t, t \geqslant 0\}$. It turns out that behavior of $\log Y_t$ and $N(t)$ is comparable and both processes admit similar limit properties. In particular, for the sequence $\log Y_t$, asymptotic normalities of BPRI were given in [6] when the subordinate process is a Poisson process. Specifically, let $\{N(t)\}$ be a Poisson process with intensity $\lambda > 0$, assume that $p_0 = 0$ and the offspring mean $m = \sum_i i p_i \in (1, \infty)$, one has

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{\log Y_t - \lambda t \log m}{\sqrt{\lambda t} \log m} \leqslant x \right) = \Phi(x), \quad x \in \mathbb{R},$$

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where $\Phi(x)$ is the cumulative distribution function of a standard normal random variable. Convergence rates in the form of Berry-Esseen type inequalities for above result were investigated in [4].

In this manuscript, we consider the normal approximations for branching process subordinated by a general renewal process. Our first result is about the one-term Edgeworth expansions for the asymptotic normality. Corresponding results for branching process in random environment were given in [2].

Throughout this paper, we assume that the inter-arrival time T is a non-lattice random variable, where a lattice random variable X means that there exists a positive number d such that the support of X is contained in $\{\pm nd, n \in \mathbb{N}\}$.

THEOREM 1.1. (Edgeworth expansions) *Assume that* $p_0 = 0$, $m \in (1,\infty)$ *and* $\mathbb{E}(T^3) < \infty$, *we have*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log Y_t - (\theta t - U)\log m}{\theta\vartheta\sqrt{\theta t}\log m} \leqslant x \right) - \Phi(x) - \frac{Q(x)\phi(x)}{\sqrt{t}} \right| = o(t^{-1/2}),\ \ t \to \infty,$$

where $a_t = o(b_t)$ *means* $a_t/b_t \to 0$ *as* $t \to \infty$, $\theta^{-1} = \mathbb{E}(T), \vartheta^2 = \mathbb{V}ar(T) \in (0,\infty)$, $\phi(x)$ *is the probability density function of a standard normal random variable,*

$$Q(x) = -\frac{\mathbb{E}(\log W)}{\sqrt{\theta}} + \frac{\nu_3 - 3\theta\vartheta^4}{6\vartheta^3\sqrt{\theta}}(x^2-1) - \frac{\theta^2\vartheta^2 - 1}{2\theta\vartheta\sqrt{\theta}},$$

$\nu_3 = \mathbb{E}(T - \mathbb{E}(T))^3$, W *is the limit variable of martingale* $\{W_n = Z_n/m^n, n \in \mathbb{N}\}$ *and* U *is a uniformly distributed random variable on* $(-1/2, 1/2)$ *and is independent of the processes* $\{Z_n\}$ *and* $\{N(t)\}$.

Consequently, we have the following Berry-Esseen type inequality, which extends the result established in [4] where the subordinated process is a Poisson Process.

COROLLARY 1.2. (Berry-Esseen inequality) *Assume that* $p_0 = 0$, $m \in (1,\infty)$ *and* $\mathbb{E}(T^3) < \infty$, *we have*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log Y_t - \theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m} \leqslant x \right) - \Phi(x) \right| \leqslant C \cdot t^{-1/2}$$

for some positive absolute constant C.

Edgeworth expansions investigate the absolute deviations between the target distribution and normal distribution, while a Cramér-type moderate deviation theorem quantifies the relative error of the tail probability approximation. It provides theoretical justification when the limiting tail probability can be used to estimate the tail probability under study.

THEOREM 1.3. (Cramér type moderate deviations) *Assume that* $p_0=0$, $m\in(1,\infty)$ *and* $\mathbb{E}(\exp(s_0 T)) < \infty$ *for some positive* s_0, *we have for* $0 \leqslant x = o(t^{1/6})$, *as* $t \to \infty$,

$$\frac{1 - F_t(x)}{1 - \Phi(x)} = 1 + O(1)\frac{(1 + x^3)\mathbb{E}(T^3)}{\sqrt{t}}, \tag{1.1}$$

where $O(1)$ *is a sequence of real numbers bounded by a universal constant for* $t \geqslant 0$ *and*

$$F_t(x) = \mathbb{P}\left(\frac{\log Y_t - \theta t \log m}{\theta \vartheta \sqrt{\theta t} \log m} \leqslant x\right).$$

As a consequence of (1.1),

$$\frac{1 - F_t(x)}{1 - \Phi(x)} \to 1 \tag{1.2}$$

as $t \to \infty$, uniformly in $x \in [0, o(t^{1/6}))$. It is known in general that $o(t^{1/6})$ is the largest possible value for the range of x such that (1.2) holds.

The rest of the paper is organized as follows. In Section 2, we prove a one-term Edgeworth expansion for the asymptotic normality. Two examples are also given to illustrated our result. Section 3 is devoted to the Cramér type moderate deviations.

2. Edgeworth expansions

A crucial result about the one-term Edgeworth expansions of a renewal process are needed to prove Theorem 1.1, which is due to Babu et al. [1].

LEMMA 2.1. *For a renewal process* $\{N(t), t > 0\}$, *if* $\vartheta > 0$, $\mathbb{E}(T^3) < \infty$, T *is a non-lattice random variable and* U *is a uniformly distributed random variable on* $(-1/2, 1/2)$ *and is independent of the processes* $\{N(t)\}$, *then as* $t \to \infty$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{N(t) - \theta t + U}{\theta \vartheta \sqrt{\theta t}} \leqslant x\right) - \Phi(x) - \frac{S(x)\phi(x)}{\sqrt{t}} \right| = o(t^{-1/2}),$$

where

$$S(x) = \frac{\nu_3 - 3\theta \vartheta^4}{6\vartheta^3 \sqrt{\theta}}(x^2 - 1) - \frac{\theta^2 \vartheta^2 - 1}{2\theta \vartheta \sqrt{\theta}}.$$

The proof of Theorem 1.1. For any real x,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\log Y_t - (\theta t + U)\log m}{\theta \vartheta \sqrt{\theta t} \log m} \leqslant x\right) \\
&= \mathbb{P}\left(\frac{N(t) - \theta t + U}{\theta \vartheta \sqrt{\theta t}} \leqslant x + \frac{N(t) - \theta t - U}{\theta \vartheta \sqrt{\theta t}} - \frac{\log Y_t - (\theta t + U)\log m}{\theta \vartheta \sqrt{\theta t} \log m}\right) \\
&= \mathbb{P}\left(\frac{N(t) - \theta t + U}{\theta \vartheta \sqrt{\theta t}} \leqslant x - \frac{\log Y_t - N(t)\log m}{\theta \vartheta \sqrt{\theta t} \log m}\right) \\
&= \mathbb{P}\left(\frac{N(t) - \theta t + U}{\theta \vartheta \sqrt{\theta t}} \leqslant x - \frac{\log W(t)}{\theta \vartheta \sqrt{\theta t} \log m}\right),
\end{aligned}$$

where $W(t) = W_{N(t)}$, $W_n = Z_n/m^n$ and m is the offspring mean. Define

$$x_t = x - \frac{\log W(t)}{\theta\vartheta\sqrt{\theta t}\log m},$$

according to Lemma 2.1, conditioning on $W(t)$, we have

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\log Y_t - (\theta t + U)\log m}{\theta\vartheta\sqrt{\theta t}\log m} \leqslant x | W(t)\right) - \Phi(x_t) - \frac{S(x_t)\phi(x_t)}{\sqrt{t}}\right| = o(t^{-1/2}),$$

as $t \to \infty$. By Taylor series expansions

$$\begin{aligned}\Phi(x_t) &= \Phi(x) + (x_t - x)\phi(x) - \frac{1}{2}(x_t - x)^2\phi(\theta_1(t))\theta_1(t)\\ &= \Phi(x) - \frac{\log W(t)}{\sqrt{\lambda t}}\phi(x) - \frac{(\log W(t))^2}{2\lambda t}\phi(\theta_1(t))\theta_1(t),\end{aligned}$$

$$\begin{aligned}x_t^2\phi(x_t) &= x^2\phi(x) + (x_t - x)\{2\theta_2(t) - \theta_2^3(t)\}\phi(\theta_2(t))\\ &= x^2\phi(x) + \frac{\log W(t)}{\sqrt{\lambda t}}\{2\theta_2(t) - \theta_2^3(t)\}\phi(\theta_2(t)),\end{aligned}$$

$$\phi(x_t) = \phi(x) - (x_t - x)\theta_3(t)\phi(\theta_3(t)) = \phi(x) - \frac{\log W(t)}{\sqrt{\lambda t}}\theta_3(t)\phi(\theta_3(t))$$

for some $\theta_1(t), \theta_2(t), \theta_3(t)$ between x_t and x. By Lemma 2 of [4], for $k = 1,2$, $\sup_n \mathbb{E}|\log W_n|^k < \infty$. Since $y^k\phi(y)$ is uniformly bounded for any $k \geqslant 0$, we have

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\log Y_t - (\theta t + U)\log m}{\theta\vartheta\sqrt{\theta t}\log m} \leqslant x\right) - \Phi(x) - \frac{\tilde{Q}(x,t)\phi(x)}{\sqrt{t}}\right| = o(t^{-1/2}),$$

as $t \to \infty$, where

$$\tilde{Q}(x,t) = -\frac{\mathbb{E}(\log W(t))}{\sqrt{\theta}} + \frac{\nu_3 - 3\theta\vartheta^4}{6\vartheta^3\sqrt{\theta}}(x^2 - 1) - \frac{\theta^2\vartheta^2 - 1}{2\theta\vartheta\sqrt{\theta}}.$$

It is sufficient to show that $|\mathbb{E}(\log W(t)) - \mathbb{E}(\log W)| = o(1)$, as $t \to \infty$. In fact, from Lemma 2 of [4], there exists constants $C > 0$ and $r \in (0,1)$ such that

$$|\mathbb{E}(\log W_n) - \mathbb{E}(\log W)| \leqslant \mathbb{E}|\log W_n - \log W| \leqslant Cr^n.$$

Accordingly,

$$|\mathbb{E}(\log W(t)) - \mathbb{E}(\log W)| \leqslant \sum_{n=0}^{\infty}\mathbb{E}|\log W_n - \log W|\mathbb{P}(N(t) = n) \leqslant C\mathbb{E}(r^{N(t)}).$$

Using Lemma 3.1 of [5], $\mathbb{E}(r^{N(t)})$ has an exponential decay rate, we complete the proof of Theorem 1.1. □

EXAMPLE 2.2. Assume that offspring distribution $\{p_k, k \geqslant 0\}$ of the Galton-Watson process $\{Z_n\}$ satisfies $p_0 = 0$ and $p_k = 1/2^k$ for any $k \geqslant 1$, $\{N(t)\}$ is a Poisson process with intensity $\lambda = 1$, then as $t \to \infty$,

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\log Y_t - (t-U)\log 2}{\sqrt{t}\log 2} \leqslant x\right) - \Phi(x) - \left(\gamma - \frac{(x^2-1)}{6}\right)\frac{\phi(x)}{\sqrt{t}}\right| = o(t^{-1/2}),$$

where γ is the famous Euler-Mascheroni constant $0.5772\cdots$.

Proof. Denote the probability generating function of the offspring distribution by f, that is, $f(s) = \sum_i p_i s^i$ for $s \in [0,1]$. In this example, we have

$$f(s) = \sum_{i=1}^{\infty} \frac{1}{2^i} s^i = \frac{s/2}{1-s/2} = \frac{s}{2-s}.$$

By iteration, the probability generating function of Z_n is

$$f_n(s) = \frac{s}{2^n - (2^n-1)s}.$$

Denote the Laplace transformation of W by $\ell(v)$, one has

$$\ell(v) = \lim_{n\to\infty} f_n\left(\exp\left(-\frac{v}{m^n}\right)\right) = \lim_{n\to\infty}\frac{\exp\left(-\frac{v}{2^n}\right)}{2^n - (2^n-1)\exp\left(-\frac{v}{2^n}\right)} = \frac{1}{1+v}.$$

Consequently, W has an exponential distribution with parameter 1. Thus,

$$\mathbb{E}(\log W) = \int_0^{\infty} e^{-x}\log x dx = -\gamma.$$

Note that $\theta = \mathbb{E}(T) = 1$, $\vartheta^2 = \mathbb{V}\mathrm{ar}(T) = 1$ and

$$v_3 = \mathbb{E}(T - \mathbb{E}(T))^3 = \int_0^{\infty}(x-1)^3 e^{-x} dx = 2,$$

we complete the proof of Example 2.2. $\square$

Finite-sample (sample size is 10000) performance are showed in Figure 1–Figure 4. The results indicate that Edgeworth expansion is more efficient than the central limit theorem.

EXAMPLE 2.3. Assume that the probability generating function of the offspring distribution $\{p_k, k \geqslant 0\}$ satisfies

$$f(s) = \frac{s}{(\sqrt{2} - (\sqrt{2}-1)\sqrt{s})^2}, \quad s \in [0,1],$$

$\{N(t)\}$ is a renewal process where the inter-arrival time T has probability density function

$$p(x) = \begin{cases} 4xe^{-2x}, & x > 0; \\ 0, & x \leqslant 0, \end{cases}$$

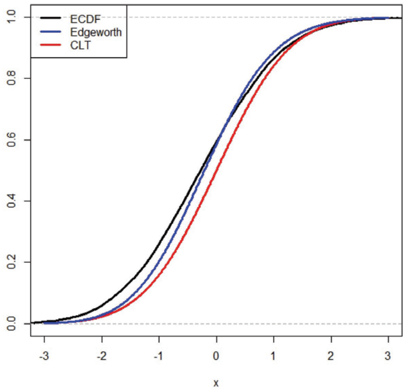

Figure 1: $t = 10$

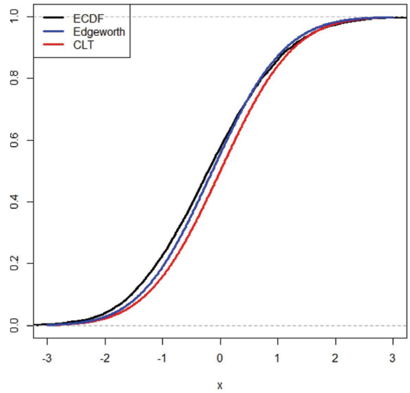

Figure 2: $t = 20$

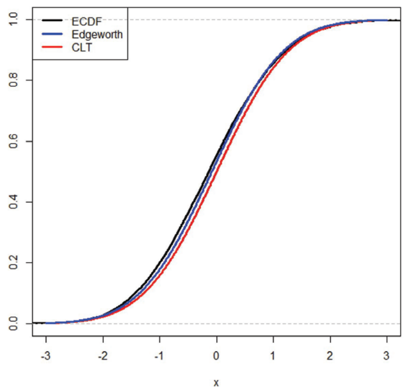

Figure 3: $t = 50$

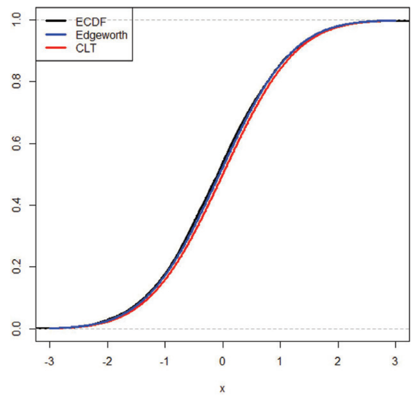

Figure 4: $t = 100$

then as $t \longrightarrow \infty$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\log Y_t - (t-U)\log\sqrt{2}}{(\sqrt{2t}\log\sqrt{2})/2} \leqslant x \right) - \Phi(x) - \frac{Q(x)\phi(x)}{\sqrt{t}} \right| = o(t^{-1/2}),$$

where

$$Q(x) = \gamma - 1 + \log 2 - \frac{1}{6\sqrt{2}}(x^2 - 4).$$

Proof. By iteration, the probability generating function of Z_n is

$$f_n(s) = \frac{s}{(2^{n/2} - (2^{n/2} - 1)\sqrt{s})^2}.$$

The Laplace transformation of W satisfies

$$\ell(v)=\lim_{n\to\infty} f_n\left(\exp\left(-\frac{v}{m^n}\right)\right)=\lim_{n\to\infty}\frac{\exp\left(-\frac{v}{\sqrt{2}^n}\right)}{\left(2^{n/2}-(2^{n/2}-1)\sqrt{\exp\left(-\frac{v}{\sqrt{2}^n}\right)}\right)^2}$$
$$=\frac{4}{(2+v)^2}.$$

Consequently, W has a gamma distribution with parameters $(2,2)$. Thus,

$$\mathbb{E}(\log W)=\int_0^\infty 4xe^{-2x}\log x dx \overset{u=2x}{=} \int_0^\infty ue^{-u}\log u du-\log 2=-(\gamma-1)-\log 2.$$

Note that $\theta=\mathbb{E}(T)=1, \vartheta^2=\mathbb{V}\mathrm{ar}(T)=1/2$ and

$$\nu_3=\mathbb{E}(T-\mathbb{E}(T))^3=\int_0^\infty 4(x-1)^3xe^{-2x}dx=1/2,$$

we complete the proof of Example 2.3. □

The proof of Corollary 1.2. Note that

$$\mathbb{P}\left(\frac{\log Y_t-\theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m}\leqslant x\right)=\mathbb{P}\left(\frac{\log Y_t-(\theta t-U)\log m}{\theta\vartheta\sqrt{\theta t}\log m}\leqslant x+\frac{U}{\theta\vartheta\sqrt{\theta t}}\right),$$

according to Theorem 1.1,

$$\mathbb{P}\left(\frac{\log Y_t-\theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m}\leqslant x\right)\leqslant \mathbb{E}(\Phi(x_t))+\frac{\mathbb{E}(Q(x_t)\phi(x_t))}{\sqrt{t}}+C_0(t^{-1/2}) \tag{2.1}$$

and

$$\mathbb{P}\left(\frac{\log Y_t-\theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m}\leqslant x\right)\geqslant \mathbb{E}(\Phi(x_t))+\frac{\mathbb{E}(Q(x_t)\phi(x_t))}{\sqrt{t}}-C_0(t^{-1/2}) \tag{2.2}$$

for some positive absolute constant C_0, where

$$x_t=x+\frac{U}{\theta\vartheta\sqrt{\theta t}}.$$

Note that

$$Q(x)=-\frac{\mathbb{E}(\log W)}{\sqrt{\theta}}+\frac{\nu_3-3\theta\vartheta^4}{6\vartheta^3\sqrt{\theta}}(x^2-1)-\frac{\theta^2\vartheta^2-1}{2\theta\vartheta\sqrt{\theta}}$$

and $y^k\phi(y)$ is uniformly bounded for any $k \geqslant 0$, we have

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\log Y_t-\theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m}\leqslant x\right)-\mathbb{E}(\Phi(x_t))\right|\leqslant C_1\cdot t^{-1/2}$$

for some positive absolute constant C_1. By Taylor series expansion

$$\begin{aligned}\Phi(x_t) &= \Phi(x)+(x_t-x)\phi(x)-\frac{1}{2}(x_t-x)^2\phi(\theta_1(t))\theta_1(t)\\ &= \Phi(x)-\frac{U}{\theta\vartheta\sqrt{\theta t}}\phi(x)-\frac{U^2}{\theta^3\vartheta^2 t}\phi(\theta_1(t))\theta_1(t),\end{aligned}$$

for some $\theta_1(t)$ between x_t and x. Since U is a uniformly distributed random variable on $(-1/2,1/2)$ and is independent of the processes $\{Z_n\}$ and $\{N(t)\}$, we complete the proof of Corollary 1.2 by dominated convergence theorem. □

3. Cramér type moderate deviations

A basic result on partial sums of a sequence of i.i.d. random variables is needed to prove Theorem 1.3, see [11] for instance.

LEMMA 3.1. *Let $X_1,X_2,\cdots$ be i.i.d. random variables with $\mathbb{E}(X_1)=0$, $\mathbb{E}(X_1^2)=1$ and $\mathbb{E}(\exp(s_0|X_1|))<\infty$ for some positive s_0, set $U_n=\sum_{i=1}^n X_i/\sqrt{n}$, one has*

$$\frac{1-\mathbb{P}(U_n\leqslant z)}{1-\Phi(x)}=1+O(1)\frac{(1+z^3)\mathbb{E}|X_1|^3}{\sqrt{n}}$$

for $0\leqslant z\leqslant n^{1/6}/\sqrt[3]{\mathbb{E}|X_1|^3}$.

The proof of Theorem 1.3.

$$\begin{aligned}1-F_t(x) &= \mathbb{P}\left(\frac{\log Y_t-\theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m}>x\right)\\ &= \mathbb{P}\left(\frac{N(t)-\theta t}{\theta\vartheta\sqrt{\theta t}}>x-\frac{\log Y_t-\theta t\log m}{\theta\vartheta\sqrt{\theta t}\log m}+\frac{N(t)-\theta t}{\theta\vartheta\sqrt{\theta t}}\right)\\ &= \mathbb{P}\left(\frac{N(t)-\theta t}{\theta\vartheta\sqrt{\theta t}}>x-\frac{\log W(t)}{\theta\vartheta\sqrt{\theta t}\log m}\right).\end{aligned} \tag{3.1}$$

For any real $0\leqslant x=o(t^{1/6})$, define

$$a_t=a_t(x)=\theta t+x\theta\vartheta\sqrt{\theta t}-\frac{\log W(t)}{\log m},\quad n_t=[a_t]+1,$$

where $[a]$ is the integer part of a. By (3.1), one has

$$1-F_t(x)=\mathbb{P}(N(t)>a_t)=\mathbb{P}(N(t)\geqslant n_t)=\mathbb{P}(S_{n_t}\leqslant t),$$

where $S_n = T_1 + \cdots + T_n$. Conditioning on $W(t)$, we have

$$\begin{aligned}\mathbb{P}\left(\frac{\log Y_t - \theta t \log m}{\theta \vartheta \sqrt{\theta t} \log m} > x \middle| W(t)\right) &= \mathbb{P}(S_{n_t} \leqslant t | W(t)) \\ &= \mathbb{P}\left(-\frac{S_{n_t} - \theta^{-1} n_t}{\vartheta \sqrt{n_t}} \geqslant -\frac{t - \theta^{-1} n_t}{\vartheta \sqrt{n_t}} \middle| W(t)\right) \\ &= \mathbb{P}\left(-\frac{S_{n_t} - \theta^{-1} n_t}{\vartheta \sqrt{n_t}} \geqslant x_t \middle| W(t)\right),\end{aligned}$$

where

$$x_t = -\frac{t - \theta^{-1} n_t}{\vartheta \sqrt{n_t}} = \frac{n_t - \theta t}{\theta \vartheta \sqrt{n_t}}.$$

It is well known that $W(t) \xrightarrow{a.s.} W$, which has an absolute continuous distribution on $(0, \infty)$, then with probability one, we have $x_t = o(n_t^{1/6})$ as $t \to \infty$. According to Lemma 3.1,

$$\mathbb{P}\left(\frac{\log Y_t - \theta t \log m}{\theta \vartheta \sqrt{\theta t} \log m} > x \middle| W(t)\right) = (1 - \Phi(x_t))\left(1 + O(1)\frac{(1 + x_t^3)\mathbb{E}T^3}{\sqrt{n_t}}\right).$$

By Taylor series expansions

$$\Phi(x_t) = \Phi(x) + (x_t - x)\phi(\theta_1(t)), \quad x_t^3 = x^3 + 3(x_t - x)\theta_2^2(t)$$

for some $\theta_1(t), \theta_2(t)$ between x_t and x. Note that $\sqrt{n_t}/\sqrt{\theta t} \to 1$ as $t \to \infty$ and

$$x_t - x = \frac{n_t - \theta t}{\theta \vartheta \sqrt{n_t}} - x = O(t^{-1/2}),$$

we complete the proof of Theorem 1.3. □

EXAMPLE 3.2. Consider Example 2.2 again. We choose $x = t^{1/10}, t^{1/9}, t^{1/8}, t^{1/7}, t^{1/6}$ for $t = 20, 50, 100$ and 200 respectively. The finite-sample (sample size is 1000000) performance are showed in Table 1–Table 4, where 'ETP' means empirical tail probabilities:

$$\mathbb{P}\left(\frac{\log Y_t - t \log 2}{\sqrt{t} \log 2} > x\right)$$

The results indicate that our result on Cramér type moderate deviation works well.

Table 1: $t = 20$

	$x = t^{1/10}$	$x = t^{1/9}$	$x = t^{1/8}$	$x = t^{1/7}$	$x = t^{1/6}$
ETP	0.08063800	0.07467100	0.06756800	0.05875100	0.04842800
$1-\Phi(x)$	0.08862307	0.08151534	0.07294329	0.06249916	0.04972263

Table 2: $t = 50$

	$x = t^{1/10}$	$x = t^{1/9}$	$x = t^{1/8}$	$x = t^{1/7}$	$x = t^{1/6}$
ETP	0.06426400	0.05711300	0.04870300	0.03864400	0.02716500
$1-\Phi(x)$	0.06960255	0.06123942	0.05147794	0.04017329	0.02746793

Table 3: $t = 100$

	$x = t^{1/10}$	$x = t^{1/9}$	$x = t^{1/8}$	$x = t^{1/7}$	$x = t^{1/6}$
ETP	0.0529910	0.04525500	0.03636300	0.02635000	0.01604900
$1-\Phi(x)$	0.0564953	0.04764789	0.03767899	0.02676022	0.01560305

Table 4: $t = 200$

	$x = t^{1/10}$	$x = t^{1/9}$	$x = t^{1/8}$	$x = t^{1/7}$	$x = t^{1/6}$
ETP	0.04265000	0.03455900	0.02557200	0.01642200	0.008093000
$1-\Phi(x)$	0.04469291	0.03580038	0.02623682	0.01651728	0.007797225

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