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SOME WEIGHTED DYNAMIC INEQUALITIES OF HARDY TYPE WITH KERNELS ON TIME SCALES NABLA CALCULUS

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Abstract. In this paper, we present some properties of the time scale nabla calculus and how to apply it for proving the dynamic inequalities. Also, we prove some weighted dynamic inequalities of Hardy type with kernels on time scales nabla calculus and also, we study the characterizations of the weights for these inequalities in different spaces and for the exponent $p > 1$. The Holder inequality, Jensen inequality, and Minkowski inequality are used to prove our results.

1. Introduction

In 1920, Hardy [9] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n),\quad p>1,\tag{1}$$

where $a(n)\geqslant 0$ for $n\geqslant 1$, $a(n)\in l^{p}(\mathbb{N})$ (i.e. $\sum_{n=1}^{\infty}a^{p}(n)<\infty$) and the constant $(p/(p-1))^{p}$ is the best possible. In [10, Theorem A] Hardy proved the integral version of (1) and showed that if $f\geqslant 0$ and integrable over any finite interval $(0,\lambda)$, where $\lambda\in(0,\infty)$ and $f\in L^{p}(0,\infty)$ and $p>1$, then

$$\int_{0}^{\infty}\left(\frac{1}{\lambda}\int_{0}^{\lambda}f(\tau)d\tau\right)^{p}d\lambda\leqslant\left(\frac{p}{p-1}\right)^{p}\int_{0}^{\infty}f^{p}(\lambda)d\lambda.\tag{2}$$

The constant $(p/(p-1))^{p}$ in (2) is the best possible. In [11] Hardy and Littlewood showed that the inequality (2) holds with reversed sign when $0<p<1$, provided that the integral $\int_{0}^{\lambda}f(t)dt$ is replaced by $\int_{\lambda}^{\infty}f(t)dt$. In particular, it was proved that if $f(\lambda)\geqslant 0$, $\int_{0}^{\infty}f^{p}(\lambda)d\lambda<\infty$, then

$$\int_{0}^{\infty}\left(\frac{1}{\lambda}\int_{\lambda}^{\infty}f(\tau)d\tau\right)^{p}d\lambda>\left(\frac{p}{1-p}\right)^{p}\int_{0}^{\infty}f^{p}(\lambda)d\lambda,\ 0<p<1,$$

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unless $f \equiv 0$. Also, the constant $(p/(1-p))^p$ is the best possible. In 1928, Knopp [14] proved that

$$\int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right)d\zeta \leqslant e\int_0^\infty f(\zeta)d\zeta, \tag{3}$$

where f is a nonnegative and integrable function. The constant e in (3) is the best constant. The inequality (3) is called a Knopp-type inequality. The inequality (3) can be considered as a limit, for p tending to infinity of the classical Hardy integral inequality (2), so for the function $f^{1/p}$, we have

$$\int_0^\infty \left(\frac{1}{\zeta}\int_0^\zeta f^{\frac{1}{p}}(t)dt\right)^p d\zeta \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f(\zeta)d\zeta.$$

Indeed

$$\lim_{p\to\infty}\left(\frac{1}{\zeta}\int_0^\zeta f^{\frac{1}{p}}(t)dt\right)^p = \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right),$$

while $(p/(p-1))^p \to e$ as $p\to\infty$. If we replace $f(t)$ by $f(t)/t$ in (3), then we have that

$$\begin{aligned}\int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln\frac{f(t)}{t}dt\right)d\zeta &= \int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt - \frac{1}{\zeta}\int_0^\zeta (\ln t)\,dt\right)d\zeta \\ &= \int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt - \ln\zeta + 1\right)d\zeta \\ &= e\int_0^\infty \frac{1}{\zeta}\exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right)d\zeta,\end{aligned}$$

where $\int_0^\zeta (\ln t)\,dt = \zeta\ln\zeta - \zeta$. Therefore we have from (3) by replacing $f(t)$ with $f(t)/t$ that

$$\int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right)\frac{d\zeta}{\zeta} \leqslant \int_0^\infty f(\zeta)\frac{d\zeta}{\zeta}. \tag{4}$$

In 2002, Kaijser et al. [12] generalized (4) with a convex function and proved the general Hardy-Knopp inequality

$$\int_0^\infty \Phi\left(\frac{1}{\zeta}\int_0^\zeta f(t)dt\right)\frac{d\zeta}{\zeta} \leqslant \int_0^\infty \Phi(f(\zeta))\frac{d\zeta}{\zeta}, \tag{5}$$

where Φ is a convex function on $\mathbb{R}^+$ and $f:\mathbb{R}^+\to\mathbb{R}^+$ is a locally integrable positive function. In 2003, Čižmešija et al. [8] proved a generalization of the Hardy-Knopp inequality (5) with two different weighted functions. In particular, it was proved that if $0<b\leqslant\infty$, $u:(0,b)\to\mathbb{R}$ is a nonnegative function such that the function $\zeta\to u(\zeta)/\zeta^2$ is locally integrable on $(0,b)$ and Φ is convex on (a,c), where $-\infty\leqslant a<c\leqslant\infty$, the inequality

$$\int_0^b u(\zeta)\Phi\left(\frac{1}{\zeta}\int_0^\zeta f(t)dt\right)\frac{d\zeta}{\zeta} \leqslant \int_0^b \upsilon(\zeta)\Phi(f(\zeta))\frac{d\zeta}{\zeta},$$

holds for all integrable functions $f:(0,b)\to\mathbb{R}$, such that $f(\zeta)\in(a,c)$ for all $\zeta\in(0,b)$ and the function υ is defined by

$$\upsilon(t):=t\int_t^b \frac{u(\zeta)}{\zeta^2}d\zeta, \quad \text{for} \quad t\in(0,b).$$

In 2005, Kaijser et al. [13] applied Jensen's inequality for convex functions and established an interesting generalization of Hardy's type inequalitiy (2). In particular, they proved that if $0<b\leqslant\infty$, $u:(0,b)\to\mathbb{R}$ and $k:(0,b)\times(0,b)\to\mathbb{R}$ are non-negative functions, such that $0<K(t):=\int_0^t k(t,\vartheta)d\vartheta<\infty$, $t\in(0,b)$ and

$$\upsilon(\zeta):=\zeta\int_\zeta^b u(t)\frac{k(t,\zeta)}{K(t)}\frac{dt}{t}<\infty, \quad \zeta\in(0,b),$$

then

$$\int_0^b u(\zeta)\Phi(A_kf(\zeta))\frac{d\zeta}{\zeta}\leqslant\int_0^b \upsilon(\zeta)\Phi(f(\zeta))\frac{d\zeta}{\zeta}, \tag{6}$$

where Φ is a convex function on an interval $I\subseteq\mathbb{R}$, $f:(0,b)\to\mathbb{R}$ is a function with values in I, and

$$A_kf(\zeta):=\frac{1}{K(\zeta)}\int_0^\zeta k(\zeta,\vartheta)f(\vartheta)d\vartheta, \ \ K(\zeta)=\int_0^\zeta k(\zeta,\vartheta)d\vartheta, \ \ \zeta\in(0,b).$$

Also, in [13] it is proved that if $1<p\leqslant q<\infty$, $s\in(1,p)$ and $0<b<\infty$. Furthermore assume that Φ is a convex and strictly monotone function on (a,c), $-\infty<a<c<\infty$ and assumed that the general Hardy operator A_k defined as following

$$A_kf(\zeta)=\frac{1}{K(\zeta)}\int_0^\zeta k(\zeta,\vartheta)f(\vartheta)d\vartheta, \quad K(\zeta)=\int_0^\zeta k(\zeta,\vartheta)d\vartheta,$$

where $k:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a nonnegative kernel and assume that $u(\zeta)$ and $v(\zeta)$ are nonnegative weighted functions. Then the inequality

$$\left(\int_0^b [\Phi(A_kf(\zeta))]^q u(\zeta)\frac{d\zeta}{\zeta}\right)^{\frac{1}{q}}\leqslant C\left[\int_0^b \Phi^p(f(\zeta))v(\zeta)\frac{d\zeta}{\zeta}\right]^{\frac{1}{p}}, \tag{7}$$

holds for all the nonnegative functions $f(\zeta)$, $a<f(\zeta)<c$, $\zeta\in[0,b]$ and $C>0$, if

$$A(s):=\sup_{0<\vartheta<b}[V(\vartheta)]^{\frac{s-1}{p}}\left(\int_\vartheta^b\left(\frac{k(\zeta,\vartheta)}{K(\zeta)}\right)^q[V(\zeta)]^{\frac{q(p-s)}{p}}u(\zeta)\frac{d\zeta}{\zeta}\right)^{\frac{1}{q}}<\infty,$$

where

$$V(\vartheta)=\int_0^\vartheta [v(t)]^{\frac{-1}{p-1}}t^{\frac{1}{p-1}}dt.$$

In the last decades, a new theory has been discovered to unify the continuous calculus and discrete calculus. It is called a time scale theory. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Many authors esatablished dynamic inequalities and generalized them on time scales. For example, see ([1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

Our aim in this paper is to generalize (6) and (7) by establishing some new weighted dynamic inequalities of Hardy type with kernels on time scales nabla calculus and we present the chain rule on time scale nabla calculus.

The paper is organized as follows. In Section 2, we present some preliminaries concerning the theory of time scales nabla calculus and some basic lemmas needed in Section 3 where we prove the main results. Our main results when $\mathbb{T}\to\mathbb{R}$, we obtain (6) and (7) proved by Kaijser et al. [13]. Also, we will prove some dynamic inequalities on time scales nabla calculus.

2. Preliminaries and basic lemmas

For a time scale $\mathbb{T}$, we define the backward jump operator as following $\rho(\tau):=\sup\{s\in\mathbb{T}: s<\tau\}$. Let $f:\mathbb{T}\to\mathbb{R}$ be a function, we say that f is ld-continuous if it is continuous at each left dense point in $\mathbb{T}$ and the right limit exists as a finite number for all right dense points $t\in\mathbb{T}$. The set of all such ld–continuous functions is ushered by $C_{ld}(\mathbb{T},\mathbb{R})$ and for any function $f:\mathbb{T}\to\mathbb{R}$, the notation $f^{\rho}(\tau)$ denotes $f(\rho(\tau))$. Also, we define a mapping $\nu:\mathbb{T}\to\mathbb{R}^{+}$ by $\nu(t)=t-\rho(t)$ such that if f is nabla differentiable at t, then $\nu(t)f^{\nabla}(t)=f(t)-f^{\rho}(t)$. For more details about the time scale calculus, see ([6], [7]).

The nabla derivative of the product uv and the quotient u/v (where $v(\tau)v^{\rho}(\tau)\neq 0$) are given by

$$\begin{aligned}(uv)^{\nabla}(\tau) &= u^{\nabla}(\tau)v(\tau)+u^{\rho}(\tau)v^{\nabla}(\tau)\\ &= u(\tau)v^{\nabla}(\tau)+u^{\nabla}(\tau)v^{\rho}(\tau),\end{aligned}$$

and

$$\left(\frac{u}{v}\right)^{\nabla}(\tau)=\frac{u^{\nabla}(\tau)v(\tau)-u(\tau)v^{\nabla}(\tau)}{v(\tau)v^{\rho}(\tau)}.$$

LEMMA 1. (Chain rule) *Let $g\in C_{ld}(\mathbb{T})$ and be nabla differentiable and assume that $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Then $f\circ g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable and satisfies*

$$(f\circ g)^{\nabla}(t)=\left[\int_0^1 f'\left(g(t)-h\nu(t)g^{\nabla}(t)\right)dh\right]g^{\nabla}(t).$$

Proof. Applying the same method of proof in [6, Theorem 1.90] and applying the ordinary substitution rule from calculus to obtain

$$f(g(\rho(t))) - f(g(s)) = \int_{g(s)}^{g(\rho(t))} f'(\tau)\,d\tau$$
$$= [g(\rho(t)) - g(s)] \int_0^1 f'[hg(\rho(t)) + (1-h)g(s)]\,dh.$$

Hence, we have

$$(f \circ g)^{\nabla}(t) = \left[\int_0^1 f'\left(g(t) - h\nu(t)g^{\nabla}(t)\right) dh\right] g^{\nabla}(t).$$

The proof is complete. □

LEMMA 2. *If λ is a nabla derivative and increasing function and $\gamma > 1$, then λ^{γ} is also nabla derivative and satisfies that*

$$[\lambda^{\gamma}(t)]^{\nabla} \geqslant \gamma[\lambda^{\rho}(t)]^{\gamma-1}\lambda^{\nabla}(t). \tag{8}$$

Proof. Applying Lemma 1 with $f(\lambda(t)) = \lambda^{\gamma}(t)$, we observe that λ^{γ} is nabla derivative and satisfies that

$$[\lambda^{\gamma}(t)]^{\nabla} = \gamma\left[\int_0^1 \left(\lambda(t) - h\nu(t)\lambda^{\nabla}(t)\right)^{\gamma-1} dh\right]\lambda^{\nabla}(t)$$
$$= \gamma\left[\int_0^1 [\lambda(t) - h(\lambda(t) - \lambda^{\rho}(t))]^{\gamma-1} dh\right]\lambda^{\nabla}(t)$$
$$= \gamma\left[\int_0^1 [(1-h)\lambda(t) + h\lambda^{\rho}(t)]^{\gamma-1} dh\right]\lambda^{\nabla}(t). \tag{9}$$

Since λ is an increasing function, $t \geqslant \rho(t)$ and $\gamma > 1$, we get $\lambda(t) \geqslant \lambda^{\rho}(t)$ and then we have from (9) that

$$[\lambda^{\gamma}(t)]^{\nabla} \geqslant \gamma\left[\int_0^1 [(1-h)\lambda^{\rho}(t) + h\lambda^{\rho}(t)]^{\gamma-1} dh\right]\lambda^{\nabla}(t)$$
$$= \gamma\left[\int_0^1 [\lambda^{\rho}(t)]^{\gamma-1} dh\right]\lambda^{\nabla}(t) = \gamma[\lambda^{\rho}(t)]^{\gamma-1}\lambda^{\nabla}(t),$$

which is (8). The proof is complete. □

DEFINITION 1. [6] A function $F:\mathbb{T}\to\mathbb{R}$ is called a nabla antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\nabla}(t)=f(t)$ holds for all $t\in\mathbb{T}$. We then define the integral of f by

$$\int_a^t f(\tau)\nabla\tau = F(t)-F(a) \quad \text{for all} \quad t\in\mathbb{T}.$$

THEOREM 1. [6] *If $a,\ b\in\mathbb{T}$, $\alpha\in\mathbb{R}$ and $f,\ \lambda$ are ld-continuous functions; then*
(1) $\int_a^b [f(\tau)+\lambda(\tau)]\nabla\tau=\int_a^b f(\tau)\nabla\tau+\int_a^b \lambda(\tau)\nabla\tau$;
(2) $\int_a^b \alpha f(\tau)\nabla\tau=\alpha\int_a^b f(\tau)\nabla\tau$;
(3) $\int_a^a f(\tau)\nabla\tau=0$.

The integration by parts formula on time scales nabla calculus is given by

$$\int_a^b u(\tau)v^{\nabla}(\tau)\nabla\tau=[u(\tau)v(\tau)]_a^b-\int_a^b u^{\nabla}(\tau)v^{\rho}(\tau)\nabla\tau. \tag{10}$$

The Hölder inequality on time scales is given by

$$\int_a^b |f(\tau)\lambda(\tau)|\nabla\tau \leqslant \left[\int_a^b |f(\tau)|^{\gamma}\nabla\tau\right]^{\frac{1}{\gamma}}\left[\int_a^b |\lambda(\tau)|^{\nu}\nabla\tau\right]^{\frac{1}{\nu}}, \tag{11}$$

where $a,\ b\in\mathbb{T},\ f,\ \lambda\in C_{ld}(\mathbb{I},\mathbb{R}),\ \gamma>1$ and $1/\gamma+1/\nu=1$.

THEOREM 2. (Jensen's inequality) *Assume that $\mathbb{T}$ is a time scale with $a,\ b\in\mathbb{T}$ and $c,\ d\in\mathbb{R}$. If $h\in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$, $\lambda:[a,b]_{\mathbb{T}}\to(c,d)$ is rd-continuous and $\Phi:(c,d)\to\mathbb{R}$ is continuous and convex, then*

$$\Phi\left(\frac{1}{\int_a^b h(s)\nabla s}\int_a^b h(t)\lambda(t)\nabla t\right)\leqslant\frac{1}{\int_a^b h(s)\nabla s}\int_a^b h(t)\Phi(\lambda(t))\nabla t. \tag{12}$$

The direction of the inequality (12) will be reversed if Φ is a concave function.

Let $(\Omega,\mathscr{M},\mu_{\Delta})$ and $(\Lambda,\mathscr{L},\lambda_{\Delta})$ be finite dimensional time scale measure spaces. We define the product measure space $(\Omega\times\Lambda,\mathscr{M}\times\mathscr{L},\mu_{\Delta}\times\lambda_{\Delta})$, where $\mathscr{M}\times\mathscr{L}$ is the product $\sigma-$algebra generated by $\{E\times F: E\in\mathscr{M},F\in\mathscr{L}\}$ and $(\mu_{\Delta}\times\lambda_{\Delta})(E\times F)=\mu_{\Delta}(E)\lambda_{\Delta}(F)$.

THEOREM 3. (Minkowski's inequality [5]) *Let $u,\ v$ and f be nonnegative functions on $\Omega,\ \Lambda$ and $\Omega\times\Lambda$, respectively. If $\alpha\geqslant 1$, then*

$$\left(\int_{\Omega}u(\zeta)\left(\int_{\Lambda}f(\zeta,\vartheta)v(\vartheta)\nabla\vartheta\right)^{\alpha}\nabla\zeta\right)^{\frac{1}{\alpha}}\leqslant\int_{\Lambda}v(\vartheta)\left(\int_{\Omega}f^{\alpha}(\zeta,\vartheta)u(\zeta)\nabla\zeta\right)^{\frac{1}{\alpha}}\nabla\vartheta. \tag{13}$$

3. Main results

Throughout the paper, we will assume that the functions (without mentioning) are nonnegative ld-continuous functions on $[a,b]_{\mathbb{T}}$ and the integrals considered are assumed to exist (finite i.e. convergent). We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. Also, we define the general Hardy operator A_k as following

$$A_k f(\zeta,s) := \frac{1}{K(\zeta,s)} \int_a^{\zeta} k(s,\vartheta) f(\vartheta) \nabla\vartheta, \quad K(\zeta,s) := \int_a^{\zeta} k(s,\vartheta) \nabla\vartheta,$$

where $\zeta,\ s > a$ and $f \in \mathrm{C}_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $k(s,\vartheta) \in \mathrm{C}_{ld}([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}, \mathbb{R})$ are delta integrable and nonnegative functions.

Now, we are ready to state and prove our main results.

THEOREM 4. *Assume that $\mathbb{T}$ is a time scale with $a,\ b \in \mathbb{T}$, $t \geqslant 1$ and $u,\ v$ are nonnegative weighted functions such that*

$$v(\vartheta) = (\rho(\vartheta) - a) \left(\int_{\vartheta}^{b} \left(\frac{k(\zeta,\vartheta)}{K(\rho(\zeta),\zeta)} \right)^t \frac{u(\zeta)}{\rho(\zeta) - a} \nabla\zeta \right)^{\frac{1}{t}}. \tag{14}$$

Furthermore assume that ϕ, ψ are nonnegative functions on (c,d), $-\infty < c < d < \infty$ and ψ is a convex function such that

$$A\psi(\zeta) \leqslant \phi(\zeta) \leqslant B\psi(\zeta), \quad c < \zeta < d, \tag{15}$$

where $A,\ B$ are positive constants, then

$$\int_a^b \phi^t \left(A_k f(\rho(\zeta),\zeta) \right) \frac{u(\zeta)}{\rho(\zeta) - a} \nabla\zeta \leqslant \left(\frac{B}{A} \right)^t \left(\int_a^b \phi(f(\vartheta)) \frac{v(\vartheta)}{\rho(\vartheta) - a} \nabla\vartheta \right)^t, \tag{16}$$

holds for the nonnegative function f.

Proof. Using (15) and Applying Jensen's inequality (where ψ is convex), we obtain

$$\begin{aligned}
&\int_a^b \phi^t \left(A_k f(\rho(\zeta),\zeta) \right) u(\zeta) \frac{\nabla\zeta}{\rho(\zeta) - a} \\
&= \int_a^b \phi^t \left(\frac{1}{K(\rho(\zeta),\zeta)} \int_a^{\rho(\zeta)} k(\zeta,\vartheta) f(\vartheta) \nabla\vartheta \right) \frac{u(\zeta)}{\rho(\zeta) - a} \nabla\zeta \\
&\leqslant B^t \int_a^b \psi^t \left(\frac{1}{K(\rho(\zeta),\zeta)} \int_a^{\rho(\zeta)} k(\zeta,\vartheta) f(\vartheta) \nabla\vartheta \right) \frac{u(\zeta)}{\rho(\zeta) - a} \nabla\zeta \\
&\leqslant B^t \int_a^b \frac{1}{K^t(\rho(\zeta),\zeta)} \left(\int_a^{\rho(\zeta)} k(\zeta,\vartheta) \psi(f(\vartheta)) \nabla\vartheta \right)^t \frac{u(\zeta)}{\rho(\zeta) - a} \nabla\zeta.
\end{aligned} \tag{17}$$

Applying Minkowski's inequality on the term

$$\int_a^b \frac{1}{K^t(\rho(\zeta),\zeta)} \left(\int_a^{\rho(\zeta)} k(\zeta,\vartheta) \psi(f(\vartheta)) \nabla\vartheta \right)^t \frac{u(\zeta)}{\rho(\zeta) - a} \nabla\zeta,$$

with $t \geqslant 1$, we see that

$$\left(\int_a^b \frac{1}{K^t(\rho(\zeta),\zeta)}\left(\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\psi(f(\vartheta))\nabla\vartheta\right)^t \frac{u(\zeta)}{\rho(\zeta)-a}\nabla\zeta\right)^{\frac{1}{t}}$$
$$\leqslant \int_a^b \psi(f(\vartheta))\left(\int_\vartheta^b \left(\frac{k(\zeta,\vartheta)}{K(\rho(\zeta),\zeta)}\right)^t \frac{u(\zeta)}{\rho(\zeta)-a}\nabla\zeta\right)^{\frac{1}{t}}\nabla\vartheta,$$

then

$$\int_a^b \frac{1}{K^t(\rho(\zeta),\zeta)}\left(\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\psi(f(\vartheta))\nabla\vartheta\right)^t \frac{u(\zeta)}{\rho(\zeta)-a}\nabla\zeta$$
$$\leqslant \left[\int_a^b \psi(f(\vartheta))\left(\int_\vartheta^b \left(\frac{k(\zeta,\vartheta)}{K(\rho(\zeta),\zeta)}\right)^t \frac{u(\zeta)}{\rho(\zeta)-a}\nabla\zeta\right)^{\frac{1}{t}}\nabla\vartheta\right]^t$$
$$= \left[\int_a^b \psi(f(\vartheta))\frac{1}{\rho(\vartheta)-a}(\rho(\vartheta)-a)\left(\int_\vartheta^b \left(\frac{k(\zeta,\vartheta)}{K(\rho(\zeta),\zeta)}\right)^t \frac{u(\zeta)}{\rho(\zeta)-a}\nabla\zeta\right)^{\frac{1}{t}}\nabla\vartheta\right]^t. \tag{18}$$

Substituting (18) into (17), we have from (14) that

$$\int_a^b \phi^t\left(A_k f(\rho(\zeta),\zeta)\right)u(\zeta)\frac{\nabla\zeta}{\rho(\zeta)-a}$$
$$\leqslant B^t\left[\int_a^b \psi(f(\vartheta))\frac{1}{\rho(\vartheta)-a}(\rho(\vartheta)-a)\left(\int_\vartheta^b \left(\frac{k(\zeta,\vartheta)}{K(\rho(\zeta),\zeta)}\right)^t \frac{u(\zeta)}{\rho(\zeta)-a}\nabla\zeta\right)^{\frac{1}{t}}\nabla\vartheta\right]^t$$
$$= B^t\left[\int_a^b \psi(f(\vartheta))\frac{1}{\rho(\vartheta)-a}v(\vartheta)\nabla\vartheta\right]^t,$$

and then we have from (15) that

$$\int_a^b \phi^t\left(A_k f(\rho(\zeta),\zeta)\right)u(\zeta)\frac{\nabla\zeta}{\rho(\zeta)-a} \leqslant \left(\frac{B}{A}\right)^t\left[\int_a^b \phi(f(\vartheta))\frac{1}{\rho(\vartheta)-a}v(\vartheta)\nabla\vartheta\right]^t,$$

which is the desired inequality (16). The proof is complete. □

REMARK 1. When $\mathbb{T}=\mathbb{R}$, $a=0$, $\rho(\zeta)=\zeta$, $t=1$ and $A=B$, we get the inequality (6) proved by Kaijser et al. [13].

REMARK 2. When $\mathbb{T}=\mathbb{N}$, $a=0$, $\rho(n)=n-1$, the inequality (16) reduces to the discrete inequality

$$\sum_{n=1}^{N}\phi^t\left(\frac{1}{\sum_{m=1}^{n}k(n,m)}\sum_{m=1}^{n}k(n,m)f(m)\right)\frac{u(n)}{n-1}$$
$$\leqslant \left(\frac{B}{A}\right)^t\left[\sum_{n=1}^{N}\phi(f(n))\frac{v(n)}{n-1}\right]^t, \quad \text{for } t\geqslant 1 \text{ and } N\in\mathbb{N}.$$

REMARK 3. If $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$, a, $b \in \mathbb{T}$, $t \geqslant 1$ and u, v are nonnegative sequences such that

$$v(\vartheta) = (\vartheta/q - a)\left(\sum_{\zeta=q\vartheta}^{b}(q-1)\zeta\left(\frac{k(\zeta,\vartheta)}{K(\zeta/q,\zeta)}\right)^{t}\frac{u(\zeta)}{\zeta/q-a}\right)^{\frac{1}{t}}.$$

Furthermore assume that ϕ, ψ are nonnegative sequences on (c,d), $-\infty < c < d < \infty$ and ψ is a convex such that

$$A\psi(\zeta) \leqslant \phi(\zeta) \leqslant B\psi(\zeta), \quad c < \zeta < d,$$

where A, B are positive constants, then

$$\sum_{\zeta=qa}^{b}(q-1)\zeta\phi^{t}\left(A_k f(\zeta/q,\zeta)\right)\frac{u(\zeta)}{\zeta/q-a} \leqslant \left(\frac{B}{A}\right)^{t}\left(\sum_{\vartheta=qa}^{b}(q-1)\phi(f(\vartheta))\frac{\vartheta v(\vartheta)}{\vartheta/q-a}\right)^{t},$$

holds for the nonnegative sequence f.

In the following theorem, we characterize the weighted functions for dynamic inequalities with kernels in different spaces.

THEOREM 5. *Assume that* $\mathbb{T}$ *is a time scale with* a, $b \in \mathbb{T}$, $0 < p < 1 < s < \infty$ *and* $1 < q < \infty$. *Also, we assume that* ϕ *is a nonnegative and convex function on* (c,d), $-\infty < c < d < \infty$, *and* u, v *are nonnegative weighted functions. Then the inequality*

$$\left(\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)}\right)^{\frac{p}{q}} \leqslant C\left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\nabla\vartheta\right]^{p}, \tag{19}$$

holds for the nonnegative function f *and* $C > 0$, *if*

$$A(s) = \sup_{\vartheta\in[a,b]_{\mathbb{T}}}\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\left(\int_{\vartheta}^{b} k^{\frac{q}{p}}(\zeta,\vartheta)\left(\frac{\left[V^{\rho}(\zeta)\right]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta\right)^{\frac{1}{q}} < \infty,$$

where

$$V(\vartheta) = \int_a^{\vartheta}\left[v(t)\right]^{\frac{-1}{1-p}}(\rho(t)-a)^{\frac{1}{1-p}}\nabla t.$$

Proof. By applying Jensen's inequality, we get that

$$
\begin{aligned}
&\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)} \\
&= \int_a^b \left[\phi\left(\frac{1}{K(\rho(\zeta),\zeta)}\int_a^{\rho(\zeta)} k(\zeta,\vartheta) f(\vartheta)\nabla\vartheta\right)\right]^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta \\
&\leqslant \int_a^b \left(\frac{1}{K(\rho(\zeta),\zeta)}\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi(f(\vartheta))\nabla\vartheta\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta \\
&= \int_a^b \frac{1}{K^{\frac{q}{p}}(\rho(\zeta),\zeta)}\left(\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi(f(\vartheta))\nabla\vartheta\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta. \qquad (20)
\end{aligned}
$$

Define a function g as following

$$
\phi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} = \phi(g(\vartheta)), \qquad (21)
$$

and then we have that

$$
\begin{aligned}
&\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi(f(\vartheta))\nabla\vartheta \\
&= \int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi^p(g(\vartheta))\left[V^\rho(\vartheta)\right]^{1-s}\left[V^\rho(\vartheta)\right]^{s-1}\left[v(\vartheta)\right]^{-1}\left(\rho(\vartheta)-a\right)\nabla\vartheta. \qquad (22)
\end{aligned}
$$

Applying Hölder's inequality (11) with $\gamma = 1/p > 1$ and $\nu = 1/(1-p)$, (where $0 < p < 1$) on the term

$$
\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi^p(g(\vartheta))\left[V^\rho(\vartheta)\right]^{1-s}\left[V^\rho(\vartheta)\right]^{s-1}\left[v(\vartheta)\right]^{-1}\left(\rho(\vartheta)-a\right)\nabla\vartheta,
$$

we obtain that

$$
\begin{aligned}
&\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi^p(g(\vartheta))\left[V^\rho(\vartheta)\right]^{1-s}\left[V^\rho(\vartheta)\right]^{s-1}\left[v(\vartheta)\right]^{-1}\left(\rho(\vartheta)-a\right)\nabla\vartheta \\
&\leqslant \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi(g(\vartheta))\left[V^\rho(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^p \\
&\quad\times\left(\int_a^{\rho(\zeta)} \left[V^\rho(\vartheta)\right]^{\frac{s-1}{1-p}}\left[v(\vartheta)\right]^{\frac{-1}{1-p}}\left(\rho(\vartheta)-a\right)^{\frac{1}{1-p}}\nabla\vartheta\right)^{1-p}. \qquad (23)
\end{aligned}
$$

Substituting (23) into (22), we see that

$$
\begin{aligned}
&\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\phi(f(\vartheta))\nabla\vartheta \\
&\leqslant \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi(g(\vartheta))\left[V^\rho(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^p \\
&\quad\times\left(\int_a^{\rho(\zeta)} \left[V^\rho(\vartheta)\right]^{\frac{s-1}{1-p}}\left[v(\vartheta)\right]^{\frac{-1}{1-p}}\left(\rho(\vartheta)-a\right)^{\frac{1}{1-p}}\nabla\vartheta\right)^{1-p},
\end{aligned}
$$

and then we have from (20) that

$$\int_a^b [\phi(A_k f(\rho(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)}$$
$$\leqslant \int_a^b \frac{u(\zeta)}{(\rho(\zeta)-a)K^{\frac{q}{p}}(\rho(\zeta),\zeta)}\left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi(g(\vartheta))\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^q$$
$$\times\left(\int_a^{\rho(\zeta)}\left[V^{\rho}(\vartheta)\right]^{\frac{s-1}{1-p}}[v(\vartheta)]^{\frac{-1}{1-p}}(\rho(\vartheta)-a)^{\frac{1}{1-p}}\nabla\vartheta\right)^{\frac{q(1-p)}{p}}\nabla\zeta. \tag{24}$$

Since

$$V(\vartheta)=\int_a^{\vartheta}[v(t)]^{\frac{-1}{1-p}}(\rho(t)-a)^{\frac{1}{1-p}}\nabla t,$$

then

$$V^{\nabla}(\vartheta)=[v(\vartheta)]^{\frac{-1}{1-p}}(\rho(\vartheta)-a)^{\frac{1}{1-p}}>0. \tag{25}$$

Thus the function V is increasing. Applying Lemma (2) with

$$\lambda(\vartheta)=V(\vartheta)=\int_a^{\vartheta}[v(t)]^{\frac{-1}{1-p}}(\rho(t)-a)^{\frac{1}{1-p}}\nabla t,$$

and (note that $0<p<1<s<\infty$)

$$\gamma=(s-p)/(1-p)=1+(s-1)/(1-p)>1,$$

to get

$$\left[V^{\frac{s-p}{1-p}}(\vartheta)\right]^{\nabla}\geqslant\left(\frac{s-p}{1-p}\right)\left[V^{\rho}(\vartheta)\right]^{\frac{s-1}{1-p}}V^{\nabla}(\vartheta)$$
$$=\left(\frac{s-p}{1-p}\right)\left[V^{\rho}(\vartheta)\right]^{\frac{s-1}{1-p}}[v(\vartheta)]^{\frac{-1}{1-p}}(\rho(\vartheta)-a)^{\frac{1}{1-p}}, \tag{26}$$

and then by integrating the two sides of (26) with respect to ϑ from a to $\rho(\zeta)$, we see (where $p<s$ and $0<p<1$) that

$$\int_a^{\rho(\zeta)}\left[V^{\frac{s-p}{1-p}}(\vartheta)\right]^{\nabla}\nabla\vartheta\geqslant\left(\frac{s-p}{1-p}\right)\int_a^{\rho(\zeta)}\left[V^{\rho}(\vartheta)\right]^{\frac{s-1}{1-p}}[v(\vartheta)]^{\frac{-1}{1-p}}(\rho(\vartheta)-a)^{\frac{1}{1-p}}\nabla\vartheta.$$

Thus (note that $V(a)=0$)

$$\int_a^{\rho(\zeta)}\left[V^{\rho}(\vartheta)\right]^{\frac{s-1}{1-p}}[v(\vartheta)]^{\frac{-1}{1-p}}(\rho(\vartheta)-a)^{\frac{1}{1-p}}\nabla\vartheta$$
$$\leqslant\left(\frac{1-p}{s-p}\right)\int_a^{\rho(\zeta)}\left[V^{\frac{s-p}{1-p}}(\vartheta)\right]^{\nabla}\nabla\vartheta$$
$$=\left(\frac{1-p}{s-p}\right)\left[V^{\rho}(\zeta)\right]^{\frac{s-p}{1-p}}. \tag{27}$$

Substituting (27) into (24), we have

$$\begin{aligned}
&\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)} \\
&\leqslant \left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} \int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi(g(\vartheta))\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^q \\
&\quad\times\left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta. \qquad (28)
\end{aligned}$$

From (28) and the definition of g in (21), we obtain

$$\begin{aligned}
&\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)} \\
&\leqslant \left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} \int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^q \\
&\quad\times\left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta. \qquad (29)
\end{aligned}$$

Applying the Minkowski inequality on the term

$$\begin{aligned}
&\int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^q \\
&\quad\times\left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta,
\end{aligned}$$

with $q>1,$ we observe that

$$\begin{aligned}
&\int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\phi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\nabla\vartheta\right)^q \\
&\quad\times\left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta \\
&\leqslant \Bigg[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} \\
&\quad\times\left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}}\left(\int_\vartheta^b k^{\frac{q}{p}}(\zeta,\vartheta)\left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta\right)^{\frac{1}{q}}\nabla\vartheta\Bigg]^q. \qquad (30)
\end{aligned}$$

Substituting (30) into (29), we get

$$\int_a^b [\phi(A_k f(\rho(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)}$$
$$\leqslant \left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} \left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\right.$$
$$\left.\times [V^\rho(\vartheta)]^{\frac{1-s}{p}} \left(\int_\vartheta^b k^{\frac{q}{p}}(\zeta,\vartheta)\left(\frac{[V^\rho(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta\right)^{\frac{1}{q}} \nabla\vartheta\right]^q$$
$$\leqslant \left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} A^q(s) \left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\nabla\vartheta\right]^q,$$

and then

$$\left(\int_a^b [\phi(A_k f(\rho(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)}\right)^{\frac{p}{q}}$$
$$\leqslant \left(\frac{1-p}{s-p}\right)^{1-p} A^p(s) \left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\nabla\vartheta\right]^p,$$

which is the desired inequality (19) with $C = \left(\frac{1-p}{s-p}\right)^{1-p} A^p(s)$. The proof is complete. □

THEOREM 6. *Assume that $\mathbb{T}$ is a time scale with $a,\ b \in \mathbb{T}$, $0 < p < 1 < s < \infty$ and $1 < q < \infty$. Also, we assume that ϕ, ψ are nonnegative functions on (c,d), $-\infty < c < d < \infty$ and ψ is a convex function such that*

$$A\psi \leqslant \phi \leqslant B\psi, \tag{31}$$

where $A,\ B$ are positive constants and $u,\ v$ are nonnegative weighted functions. Then the inequality

$$\left(\int_a^b [\phi(A_k f(\rho(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)}\right)^{\frac{p}{q}}$$
$$\leqslant C \left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\nabla\vartheta\right]^p, \tag{32}$$

holds for the nonnegative function f and $C > 0$, if

$$D(s) = \sup_{\vartheta\in[a,b]_{\mathbb{T}}} [V^\rho(\vartheta)]^{\frac{1-s}{p}} \left(\int_\vartheta^b k^{\frac{q}{p}}(\zeta,\vartheta)\left(\frac{[V^\rho(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta\right)^{\frac{1}{q}} < \infty, \tag{33}$$

where

$$V(\vartheta)=\int_a^{\vartheta}[v(t)]^{\frac{-1}{1-p}}(\rho(t)-a)^{\frac{1}{1-p}}\nabla t.$$

Proof. From (31) and by applying the Jensen inequality, we see that

$$\begin{aligned}
&\int_a^b[\phi(A_kf(\rho(\zeta),\zeta))]^{\frac{q}{p}}u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)}\\
&\leqslant B^{\frac{q}{p}}\int_a^b[\psi(A_kf(\rho(\zeta),\zeta))]^{\frac{q}{p}}u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)}\\
&=B^{\frac{q}{p}}\int_a^b\left[\psi\left(\frac{1}{K(\rho(\zeta),\zeta)}\int_a^{\rho(\zeta)}k(\zeta,\vartheta)f(\vartheta)\nabla\vartheta\right)\right]^{\frac{q}{p}}\frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta\\
&\leqslant B^{\frac{q}{p}}\int_a^b\left(\frac{1}{K(\rho(\zeta),\zeta)}\int_a^{\rho(\zeta)}k(\zeta,\vartheta)\psi(f(\vartheta))\nabla\vartheta\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta\\
&=B^{\frac{q}{p}}\int_a^b\frac{1}{K^{\frac{q}{p}}(\rho(\zeta),\zeta)}J^{\frac{q}{p}}(\zeta)\frac{u(\zeta)}{(\rho(\zeta)-a)}\nabla\zeta, \qquad (34)
\end{aligned}$$

where

$$J(\zeta)=\int_a^{\rho(\zeta)}k(\zeta,\vartheta)\psi(f(\vartheta))\nabla\vartheta. \qquad (35)$$

Define a function g such that

$$\psi^{\frac{1}{p}}(f(\vartheta))\frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}=\psi(g(\vartheta)). \qquad (36)$$

Substituting (36) into (35), we obtain

$$J(\zeta)=\int_a^{\rho(\zeta)}k(\zeta,\vartheta)\psi^p(g(\vartheta))[v(\vartheta)]^{-1}(\rho(\vartheta)-a)\nabla\vartheta.$$

Note that

$$J(\zeta)=\int_a^{\rho(\zeta)}k(\zeta,\vartheta)\psi^p(g(\vartheta))[V^{\rho}(\vartheta)]^{1-s}[V^{\rho}(\vartheta)]^{s-1}[v(\vartheta)]^{-1}(\rho(\vartheta)-a)\nabla\vartheta. \qquad (37)$$

By applying the Hölder inequality (11) with $\gamma=1/p>1$ and $\nu=1/(1-p)$, (where $0<p<1$) on the term

$$\int_a^{\rho(\zeta)}k(\zeta,\vartheta)\psi^p(g(\vartheta))[V^{\rho}(\vartheta)]^{1-s}[V^{\rho}(\vartheta)]^{s-1}[v(\vartheta)]^{-1}(\rho(\vartheta)-a)\nabla\vartheta,$$

we see that

$$\int_a^{\rho(\zeta)} k(\zeta,\vartheta)\psi^p(g(\vartheta))\,[V^\rho(\vartheta)]^{1-s}\,[V^\rho(\vartheta)]^{s-1}\,[v(\vartheta)]^{-1}\,(\rho\,(\vartheta)-a)\nabla\vartheta$$
$$\leqslant \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi(g(\vartheta))\,[V^\rho(\vartheta)]^{\frac{1-s}{p}}\nabla\vartheta\right)^p$$
$$\times\left(\int_a^{\rho(\zeta)} [V^\rho(\vartheta)]^{\frac{s-1}{1-p}}\,[v(\vartheta)]^{\frac{-1}{1-p}}\,(\rho\,(\vartheta)-a)^{\frac{1}{1-p}}\nabla\vartheta\right)^{1-p}. \tag{38}$$

Substituting (38) into (37), we observe that

$$J(\zeta)\leqslant \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi(g(\vartheta))\,[V^\rho(\vartheta)]^{\frac{1-s}{p}}\nabla\vartheta\right)^p$$
$$\times\left(\int_a^{\rho(\zeta)} [V^\rho(\vartheta)]^{\frac{s-1}{1-p}}\,[v(\vartheta)]^{\frac{-1}{1-p}}\,(\rho\,(\vartheta)-a)^{\frac{1}{1-p}}\nabla\vartheta\right)^{1-p}. \tag{39}$$

Substituting (39) into (34), we have that

$$\int_a^b [\phi\,(A_kf\,(\rho(\zeta),\zeta))]^{\frac{q}{p}}\,u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)}$$
$$\leqslant B^{\frac{q}{p}}\int_a^b \frac{1}{K^{\frac{q}{p}}(\rho(\zeta),\zeta)}\frac{u(\zeta)}{(\rho(\zeta)-a)}$$
$$\times\left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi(g(\vartheta))\,[V^\rho(\vartheta)]^{\frac{1-s}{p}}\nabla\vartheta\right)^q$$
$$\times\left(\int_a^{\rho(\zeta)} [V^\rho(\vartheta)]^{\frac{s-1}{1-p}}\,[v(\vartheta)]^{\frac{-1}{1-p}}\,(\rho\,(\vartheta)-a)^{\frac{1}{1-p}}\nabla\vartheta\right)^{\frac{q(1-p)}{p}}\nabla\zeta. \tag{40}$$

Since

$$V(\vartheta)=\int_a^\vartheta [v(t)]^{\frac{-1}{1-p}}\,(\rho\,(t)-a)^{\frac{1}{1-p}}\nabla t,$$

then by substituting (27) into (40), we see that

$$\int_a^b [\phi\,(A_kf\,(\rho(\zeta),\zeta))]^{\frac{q}{p}}\,u(\zeta)\frac{\nabla\zeta}{(\rho(\zeta)-a)}$$
$$\leqslant B^{\frac{q}{p}}\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}}\int_a^b\left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi(g(\vartheta))\,[V^\rho(\vartheta)]^{\frac{1-s}{p}}\nabla\vartheta\right)^q$$
$$\times(V^\rho(\zeta))^{\frac{q(s-p)}{p}}\frac{u(\zeta)}{(\rho(\zeta)-a)}\frac{1}{K^{\frac{q}{p}}(\rho(\zeta),\zeta)}\nabla\zeta. \tag{41}$$

From (41) and the definition of g in (36), we obtain

$$\begin{aligned}
&\int_a^b [\phi\left(A_k f\left(\rho(\zeta),\zeta\right)\right)]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)} \\
&\leqslant B^{\frac{q}{p}} \left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} \\
&\quad \times \int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} [V^{\rho}(\vartheta)]^{\frac{1-s}{p}} \nabla\vartheta\right)^q \\
&\quad \times \frac{u(\zeta)}{(\rho(\zeta)-a)} \left[\frac{(V^{\rho}(\zeta))^{s-p}}{K(\rho(\zeta),\zeta)}\right]^{\frac{q}{p}} \nabla\zeta. \qquad (42)
\end{aligned}$$

By applying Minkowski's inequality on the term

$$\begin{aligned}
&\int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} [V^{\rho}(\vartheta)]^{\frac{1-s}{p}} \nabla\vartheta\right)^q \\
&\quad \times \frac{u(\zeta)}{(\rho(\zeta)-a)} \left[\frac{(V^{\rho}(\zeta))^{s-p}}{K(\rho(\zeta),\zeta)}\right]^{\frac{q}{p}} \nabla\zeta,
\end{aligned}$$

with $q > 1,$ we observe that

$$\begin{aligned}
&\int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} [V^{\rho}(\vartheta)]^{\frac{1-s}{p}} \nabla\vartheta\right)^q \\
&\quad \times \left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)} \nabla\zeta \\
&\leqslant \left[\int_a^b \psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}}\right. \\
&\quad \left. \times [V^{\rho}(\vartheta)]^{\frac{1-s}{p}} \left(\int_{\vartheta}^b k^{\frac{q}{p}}(\zeta,\vartheta) \left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)} \nabla\zeta\right)^{\frac{1}{q}} \nabla\vartheta\right]^q. \qquad (43)
\end{aligned}$$

Substituting (33) into (43), we see that

$$\int_a^b \left(\int_a^{\rho(\zeta)} k^{\frac{1}{p}}(\zeta,\vartheta)\psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} \left[V^{\rho}(\vartheta)\right]^{\frac{1-s}{p}} \nabla\vartheta \right)^q \times \left(\frac{[V^{\rho}(\zeta)]^{s-p}}{K(\rho(\zeta),\zeta)} \right)^{\frac{q}{p}} \frac{u(\zeta)}{(\rho(\zeta)-a)} \nabla\zeta \leqslant D^q(s) \left[\int_a^b \psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} \nabla\vartheta \right]^q. \tag{44}$$

Substituting (44) into (42), we get

$$\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)} \leqslant B^{\frac{q}{p}} \left(\frac{1-p}{s-p} \right)^{\frac{q(1-p)}{p}} D^q(s) \left[\int_a^b \psi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} \nabla\vartheta \right]^q.$$

From (31), we have that

$$\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)} \leqslant \left(\frac{B}{A}\right)^{\frac{q}{p}} \left(\frac{1-p}{s-p} \right)^{\frac{q(1-p)}{p}} D^q(s) \left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} \nabla\vartheta \right]^q,$$

and then

$$\left(\int_a^b \left[\phi\left(A_k f(\rho(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta) \frac{\nabla\zeta}{(\rho(\zeta)-a)} \right)^{\frac{p}{q}} \leqslant \left(\frac{B}{A}\right) \left(\frac{1-p}{s-p} \right)^{1-p} D^p(s) \left[\int_a^b \phi^{\frac{1}{p}}(f(\vartheta)) \frac{v^{\frac{1}{p}}(\vartheta)}{(\rho(\vartheta)-a)^{\frac{1}{p}}} \nabla\vartheta \right]^p,$$

which is the desired inequality (32) with the constant $C = \left(\frac{B}{A}\right)\left(\frac{1-p}{s-p}\right)^{1-p} D^p(s)$. The proof is complete. □

REMARK 4. As a special case of Theorem 6 when $A = B = 1$, we get Theorem 5.

4. Conclusion

In this paper, we establish some new weighted dynamic inequalities of Hardy type with kernels in different spaces and for one parameter $p > 1$ on time scales nabla calculus.

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