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FRAME INEQUALITIES IN HILBERT SPACES: TWO–SIDED INEQUALITIES WITH NEW STRUCTURES

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Abstract. This paper is devoted to establishing frame inequalities in Hilbert spaces. By using operator theory methods, several two-sided inequalities for frames are presented, which, comparing to previous inequalities on frames and generalized frames, admit new structures.

1. Introduction

Frames in Hilbert spaces, known also as redundant bases, were proposed in 1952 by Duffin and Schaeffer [6], which offered us new ideas to study nonharmonic Fourier series. More than 30 years later, frames were brought back to researcher's attention due to the pioneering work of Daubechies et al. in [5]. Because of the flexibility, frames now have played an important and indispensable role in numerous areas, both in theory and in practice, see [3, 14, 15, 17].

Let $\mathbb{I}$ be a countable index set and $\mathcal{M}$ be a Hilbert space. We denote by $BL(\mathcal{M})$ the set of all bounded linear operators on $\mathcal{M}$ and the symbol $\mathrm{Id}_{\mathcal{M}}$, as usual, is used to denote the identity operator on $\mathcal{M}$.

Recall that a family $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}\subseteq\mathcal{M}$ is called a *frame* for $\mathcal{M}$, if there are constants $0\leqslant C_{\mathcal{F}}\leqslant D_{\mathcal{F}}<+\infty$ such that the inequality

$$C_{\mathcal{F}}\|x\|^2\leqslant\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2\leqslant D_{\mathcal{F}}\|x\|^2 \tag{1.1}$$

holds for each $x\in\mathcal{M}$.

The frame $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ is said to be *Parseval* if $C_{\mathcal{F}}=D_{\mathcal{F}}=1$. Likewise, we call $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ a *Bessel sequence* if the right-hand inequality of (1.1) is required to be satisfied.

Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$. Then it can naturally lead to an invertible operator $S_{\mathcal{F}}$, called the *frame operator* of $\mathcal{F}$, given below

$$S_{\mathcal{F}}:\mathcal{M}\to\mathcal{M},\quad S_{\mathcal{F}}x=\sum_{i\in\mathbb{I}}\langle x,f_i\rangle f_i. \tag{1.2}$$

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By means of $\mathcal{F}$ and $S_{\mathcal{F}}$, we can construct a new frame $\widetilde{\mathcal{F}} = \{\widetilde{f}_i : \widetilde{f}_i = S_{\mathcal{F}}^{-1} f_i\}_{i\in\mathbb{I}}$ for $\mathcal{M}$, which is called the *canonical dual frame* of $\mathcal{F}$.

Recall also that a frame $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ for $\mathcal{M}$ is said to be an *alternate dual frame* of $\mathcal{F}$, if

$$x = \sum_{i\in\mathbb{I}} \langle x, g_i\rangle f_i = \sum_{i\in\mathbb{I}} \langle x, f_i\rangle g_i, \quad \forall x \in \mathcal{M}. \tag{1.3}$$

Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ be a Bessel sequence for $\mathcal{M}$. For any $\sigma \subset \mathbb{I}$, we let $\sigma^c = \mathbb{I} \setminus \sigma$. Then, associated with $\mathcal{F}$, σ and σ^c there are always two self-adjoint operators $S_{\mathcal{F}}^{\sigma}$ and $S_{\mathcal{F}}^{\sigma^c}$, defined by

$$S_{\mathcal{F}}^{\sigma}, S_{\mathcal{F}}^{\sigma^c} : \mathcal{M} \to \mathcal{M}, \quad S_{\mathcal{F}}^{\sigma} x = \sum_{i\in\sigma} \langle x, f_i\rangle f_i, \quad S_{\mathcal{F}}^{\sigma^c} x = \sum_{i\in\sigma^c} \langle x, f_i\rangle f_i. \tag{1.4}$$

Balan et al. in [2] provided us an interesting inequality for Parseval frames, as a derivative product of the famous Parseval frame identity deriving in the process of exploring efficient algorithms for the reconstruction of signals, which we list as follows.

THEOREM A. (see [2, Proposition 4.1]). *Suppose that $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ is a Parseval frame for $\mathcal{M}$. Then for any $\sigma \subset \mathbb{I}$ and any $x \in \mathcal{M}$, we get*

$$\sum_{i\in\sigma} |\langle x, f_i\rangle|^2 + \Big\| \sum_{i\in\sigma^c} \langle x, f_i\rangle f_i \Big\|^2 \geqslant \frac{3}{4}\|x\|^2. \tag{1.5}$$

With the help of the operator $S_{\mathcal{F}}^{\sigma^c}$ given in (1.4), Găvruţa in [9] generalized the inequality in (1.5) for Parseval frames to the setting of general frames.

THEOREM B. (see [9, Theorem 2.2]). *Suppose that $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ is a frame for $\mathcal{M}$ with canonical dual frame $\widetilde{\mathcal{F}} = \{\widetilde{f}_i\}_{i\in\mathbb{I}}$. Then for any $\sigma \subset \mathbb{I}$ and any $x \in \mathcal{M}$, we have*

$$\sum_{i\in\sigma} |\langle x, f_i\rangle|^2 + \sum_{i\in\mathbb{I}} |\langle S_{\mathcal{F}}^{\sigma^c} x, \widetilde{f}_i\rangle|^2 \geqslant \frac{3}{4} \sum_{i\in\mathbb{I}} |\langle x, f_i\rangle|^2.$$

By using the corresponding alternate dual frames, an inequality for general frames was also obtained in [9].

THEOREM C. (see [9, Theorem 3.2]). *Suppose that $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ is a frame for $\mathcal{M}$ with an alternate dual frame $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$. Then for any $\sigma \subset \mathbb{I}$ and any $x \in \mathcal{M}$, we have*

$$\operatorname{Re} \sum_{i\in\sigma} \langle x, g_i\rangle \langle f_i, x\rangle + \Big\| \sum_{i\in\sigma^c} \langle x, g_i\rangle f_i \Big\|^2 \geqslant \frac{3}{4}\|x\|^2.$$

In recent years, much attention has been paid to the generalization of frame inequalities and many interesting results are obtained (see [10, 16, 19, 20] for example), which enrich the inequality theory of frames. Particularly, the author in [18] showed us the following two-sided inequalities for g-frames in Hilbert C^*-modules, an extension of Hilbert spaces.

THEOREM D. (see [18, Theorem 2.4]). *Let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a* g*-frame for* $\mathcal{N}$*, a Hilbert* C^**-module, with canonical dual* g*-frame* $\{\widetilde{\Lambda}_i\}_{i\in\mathbb{I}}$*. Then for any* $\sigma\subset\mathbb{I}$ *and any* $x\in\mathcal{N}$*, we have*

$$0\leqslant\sum_{i\in\sigma}|\Lambda_i x|^2-\sum_{i\in\mathbb{I}}|\widetilde{\Lambda}_i S_\sigma x|^2\leqslant\frac{1}{4}\sum_{i\in\mathbb{I}}|\Lambda_i x|^2.$$

$$\frac{1}{2}\sum_{i\in\mathbb{I}}|\Lambda_i x|^2\leqslant\sum_{i\in\mathbb{I}}|\widetilde{\Lambda}_i S_\sigma x|^2+\sum_{i\in\mathbb{I}}|\widetilde{\Lambda}_i S_{\sigma^c} x|^2\leqslant\sum_{i\in\mathbb{I}}|\Lambda_i x|^2.$$

Later on, Li and Leng in [12] obtained several new types of two-sided inequalities for fusion frames, where a parameter λ is involved.

THEOREM E. (see [12, Theorem 3]). *Suppose that* $\{(W_i,\omega_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{M}$ *with the fusion frame operator* S *and that* $\{(S^{-1}W_i,\omega_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\{(W_i,\omega_i)\}_{i\in\mathbb{I}}$*. Then for any* $\lambda\in[0,2]$*, for all* $\sigma\subset\mathbb{I}$ *and all* $x\in\mathcal{M}$*, we have*

$$\begin{aligned}\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(x)\|^2&\geqslant\sum_{i\in\sigma}\omega_i^2\|\pi_{W_i}(x)\|^2+\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(S^{-1}S'_{\sigma^c}x)\|^2\\&\geqslant(\lambda-\frac{\lambda^2}{4})\sum_{i\in\sigma}\omega_i^2\|\pi_{W_i}(x)\|^2+(1-\frac{\lambda^2}{4})\sum_{i\in\sigma^c}\omega_i^2\|\pi_{W_i}(x)\|^2.\end{aligned}$$

THEOREM F. (see [12, Theorem 5]). *Suppose that* $\{(W_i,\omega_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{M}$ *with the fusion frame operator* S *and that* $\{(S^{-1}W_i,\omega_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\{(W_i,\omega_i)\}_{i\in\mathbb{I}}$*. Then for any* $\lambda\in[1,2]$*, for all* $\sigma\subset\mathbb{I}$ *and all* $x\in\mathcal{M}$*, we have*

$$\begin{aligned}0&\leqslant\sum_{i\in\sigma}\omega_i^2\|\pi_{W_i}(x)\|^2-\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(S^{-1}S'_{\sigma}x)\|^2\\&\leqslant(\lambda-1)\sum_{i\in\sigma^c}\omega_i^2\|\pi_{W_i}(x)\|^2+\left(1-\frac{\lambda}{2}\right)^2\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(x)\|^2.\end{aligned}$$

THEOREM G. (see [12, Theorem 6]). *Suppose that* $\{(W_i,\omega_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{M}$ *with the fusion frame operator* S *and that* $\{(S^{-1}W_i,\omega_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\{(W_i,\omega_i)\}_{i\in\mathbb{I}}$*. Then for any* $\lambda\in[1,2]$*, for all* $\sigma\subset\mathbb{I}$ *and all* $x\in\mathcal{M}$*, we have*

$$\begin{aligned}&\left(2\lambda-\frac{\lambda^2}{2}-1\right)\sum_{i\in\sigma}\omega_i^2\|\pi_{W_i}(x)\|^2+\left(1-\frac{\lambda^2}{2}\right)\sum_{i\in\sigma^c}\omega_i^2\|\pi_{W_i}(x)\|^2\\&\leqslant\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(S^{-1}S'_{\sigma}x)\|^2+\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(S^{-1}S'_{\sigma^c}x)\|^2\\&\leqslant\lambda\sum_{i\in\mathbb{I}}\omega_i^2\|\pi_{W_i}(x)\|^2.\end{aligned}$$

Some two-sided inequalities for generalized frames with the same structures as those in [12] are also given, see [13] for continuous fusion frames, and [8] for continuous g-frames. We refer to [1, 4, 7, 11] for more information on the generalized versions of frames mentioned above.

Motivated by above works, in this paper we establish some two-sided inequalities for frames from the point of view of operator theory, which differ in structures from previous inequalities for frames and generalized frames.

2. Main results and their proofs

To prove our theorems, we require the following lemma.

LEMMA 2.1. *Suppose that $W,T \in BL(\mathcal{M})$ satisfy $W+T=\mathrm{Id}_{\mathcal{M}}$. The following statements hold.*

(1) For each $\lambda \in \mathbb{R}$ and each $x \in \mathcal{M}$, we have

$$\|Wx\|^2+\lambda(\langle Tx,x\rangle+\langle x,Tx\rangle)=\|Tx\|^2+(1-\lambda)(\langle Wx,x\rangle+\langle x,Wx\rangle)+(2\lambda-1)\|x\|^2 \geqslant (2\lambda-\lambda^2)\|x\|^2.$$

(2) For each $\lambda \in [0,\frac{1}{2}]$ and each $x \in \mathcal{M}$ we get

$$\|Wx\|^2+\lambda(\langle Tx,x\rangle+\langle x,Tx\rangle) \leqslant \frac{3\lambda+2(1-2\lambda)\|W\|^2+\lambda\|W-T\|^2}{2}\|x\|^2.$$

Proof. (1) It is similar to [16, Proposition 3.6], we omit the details.
(2) We obtain, for each $\lambda \in [0,\frac{1}{2}]$ and each $x \in \mathcal{M}$, that

$$\begin{aligned}
&\|Wx\|^2+\lambda(\langle Tx,x\rangle+\langle x,Tx\rangle)\\
&\quad=\|Wx\|^2+\lambda(\langle x,x\rangle-\langle Wx,x\rangle+\langle x,x\rangle-\langle x,Wx\rangle)\\
&\quad=2\lambda\langle x,x\rangle+(1-2\lambda)\langle Wx,Wx\rangle-\lambda(\langle Wx,x\rangle-\langle Wx,Wx\rangle)\\
&\qquad-\lambda(\langle x,Wx\rangle-\langle Wx,Wx\rangle)\\
&\quad=2\lambda\langle x,x\rangle+(1-2\lambda)\langle Wx,Wx\rangle-\lambda\langle Wx,Tx\rangle-\lambda\langle Tx,Wx\rangle\\
&\quad=\frac{3\lambda}{2}\langle x,x\rangle+(1-2\lambda)\langle Wx,Wx\rangle+\frac{\lambda}{2}\langle (W+T)x,(W+T)x\rangle\\
&\qquad-\lambda\langle Wx,Tx\rangle-\lambda\langle Tx,Wx\rangle\\
&\quad=\frac{3\lambda}{2}\langle x,x\rangle+(1-2\lambda)\langle Wx,Wx\rangle+\frac{\lambda}{2}\langle (W-T)x,(W-T)x\rangle\\
&\quad\leqslant\frac{3\lambda}{2}\|x\|^2+(1-2\lambda)\|W\|^2\|x\|^2+\frac{\lambda}{2}\|W-T\|^2\|x\|^2\\
&\quad=\frac{3\lambda+2(1-2\lambda)\|W\|^2+\lambda\|W-T\|^2}{2}\|x\|^2,
\end{aligned}$$

and we arrive at the conclusion. □

THEOREM 2.2. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$. Then for every $\lambda \in [1,+\infty)$, for any $\sigma \subset \mathbb{I}$ and any $x \in \mathcal{M}$, we have*

$$\begin{aligned}
\sum_{i\in\sigma}|\langle x,f_i\rangle|^2-\lambda\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2 &\leqslant \sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma}x,f_i\rangle|^2-\lambda\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2\\
&\leqslant(\lambda^3-\lambda^2+1)\sum_{i\in\sigma}|\langle x,f_i\rangle|^2\\
&\quad+(\lambda^3-3\lambda^2+2\lambda-1)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2.
\end{aligned} \tag{2.1}$$

Proof. For any $\sigma \subset \mathbb{I}$, from (1.4) we see that $S_{\mathcal{F}}^{\sigma} + S_{\mathcal{F}}^{\sigma^c} = S_{\mathcal{F}}$. Hence

$$S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} S_{\mathcal{F}}^{-\frac{1}{2}} + S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-\frac{1}{2}} = S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}} S_{\mathcal{F}}^{-\frac{1}{2}} = \mathrm{Id}_{\mathcal{M}}.$$

Taking $U = S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} S_{\mathcal{F}}^{-\frac{1}{2}}$ and $V = S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-\frac{1}{2}}$. Then it is easy to check that $\langle Ux, x\rangle \geqslant 0$ and $\langle Vx, x\rangle \geqslant 0$ for each $x \in \mathcal{M}$, and that $UV = VU$. Thus

$$0 \leqslant UV = (\mathrm{Id}_{\mathcal{M}} - V)V = V - V^2 = S_{\mathcal{F}}^{-\frac{1}{2}} (S_{\mathcal{F}}^{\sigma^c} - S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c}) S_{\mathcal{F}}^{-\frac{1}{2}},$$

from which we conclude that $S_{\mathcal{F}}^{\sigma^c} - S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} \geqslant 0$. Now for any $x \in \mathcal{M}$ and any $\lambda \in [1, +\infty)$, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} |\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma} x, f_i\rangle|^2 - \lambda \sum_{i\in\mathbb{I}} |\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, f_i\rangle|^2 \\
&\quad = \langle S_{\mathcal{F}} S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma} x, S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma} x\rangle - \lambda \langle S_{\mathcal{F}} S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad = \langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma} x, S_{\mathcal{F}}^{\sigma} x\rangle - \lambda \langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad = \langle S_{\mathcal{F}}^{-1} (S_{\mathcal{F}} - S_{\mathcal{F}}^{\sigma^c}) x, (S_{\mathcal{F}} - S_{\mathcal{F}}^{\sigma^c}) x\rangle - \lambda \langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad = \langle S_{\mathcal{F}} x, x\rangle - 2\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle + \langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle - \lambda \langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad = (\lambda+1)\langle S_{\mathcal{F}} x, x\rangle - \lambda \langle S_{\mathcal{F}} x, x\rangle - 2\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle + (1-\lambda)\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad = (\lambda-1)\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle + (\lambda+1)\langle S_{\mathcal{F}}^{\sigma} x, x\rangle - \lambda \langle S_{\mathcal{F}} x, x\rangle + (1-\lambda)\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad = (\lambda-1)(\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle - \langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle) + \langle S_{\mathcal{F}}^{\sigma} x, x\rangle + \lambda (\langle S_{\mathcal{F}}^{\sigma} x, x\rangle - \langle S_{\mathcal{F}} x, x\rangle) \\
&\quad \geqslant \langle S_{\mathcal{F}}^{\sigma} x, x\rangle - \lambda \langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle = \sum_{i\in\sigma} |\langle x, f_i\rangle|^2 - \lambda \sum_{i\in\sigma^c} |\langle x, f_i\rangle|^2.
\end{aligned} \tag{2.2}$$

Letting $W = S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-\frac{1}{2}}$ and $T = S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} S_{\mathcal{F}}^{-\frac{1}{2}}$, and replacing x by $S_{\mathcal{F}}^{\frac{1}{2}} x$ in Lemma 2.1(1) leads to

$$\begin{aligned}
\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle &= \langle S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\frac{1}{2}} x, S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma^c} S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\frac{1}{2}} x\rangle \\
&\geqslant (2\lambda - \lambda^2)\langle S_{\mathcal{F}}^{\frac{1}{2}} x, S_{\mathcal{F}}^{\frac{1}{2}} x\rangle - \lambda (\langle S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\frac{1}{2}} x, S_{\mathcal{F}}^{\frac{1}{2}} x\rangle \\
&\qquad + \langle S_{\mathcal{F}}^{\frac{1}{2}} x, S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} S_{\mathcal{F}}^{-\frac{1}{2}} S_{\mathcal{F}}^{\frac{1}{2}} x\rangle) \\
&= (2\lambda - \lambda^2)\langle S_{\mathcal{F}} x, x\rangle - 2\lambda \langle S_{\mathcal{F}}^{\sigma} x, x\rangle \\
&= (2\lambda - \lambda^2)\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle - \lambda^2 \langle S_{\mathcal{F}}^{\sigma} x, x\rangle.
\end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} |\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma} x, f_i\rangle|^2 - \lambda \sum_{i\in\mathbb{I}} |\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, f_i\rangle|^2 \\
&\quad = \langle S_{\mathcal{F}}^{\sigma} x, x\rangle - \langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle - (\lambda-1)\langle S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^{\sigma^c} x, S_{\mathcal{F}}^{\sigma^c} x\rangle \\
&\quad \leqslant \langle S_{\mathcal{F}}^{\sigma} x, x\rangle - \langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle + \lambda^2(\lambda-1)\langle S_{\mathcal{F}}^{\sigma} x, x\rangle - (2\lambda - \lambda^2)(\lambda-1)\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle \qquad (2.4) \\
&\quad = (\lambda^3 - \lambda^2 + 1)\langle S_{\mathcal{F}}^{\sigma} x, x\rangle + (\lambda^3 - 3\lambda^2 + 2\lambda - 1)\langle S_{\mathcal{F}}^{\sigma^c} x, x\rangle \\
&\quad = (\lambda^3 - \lambda^2 + 1)\sum_{i\in\sigma} |\langle x, f_i\rangle|^2 + (\lambda^3 - 3\lambda^2 + 2\lambda - 1)\sum_{i\in\sigma^c} |\langle x, f_i\rangle|^2.
\end{aligned}$$

This together with (2.2) gives (2.1), and we are done. □

Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a Parseval frame for $\mathcal{M}$. Then $S_{\mathcal{F}}=\mathrm{Id}_{\mathcal{M}}$. For any $\sigma\subset\mathbb{I}$ and any $x\in\mathcal{M}$, we have

$$\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma}x,f_i\rangle|^2=\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{\sigma}x,f_i\rangle|^2=\|S_{\mathcal{F}}^{\sigma}x\|^2=\Big\|\sum_{i\in\sigma}\langle x,f_i\rangle f_i\Big\|^2,$$

and similarly,

$$\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2=\Big\|\sum_{i\in\sigma^c}\langle x,f_i\rangle f_i\Big\|^2.$$

Combination of above facts and Theorem 2.2 can immediately lead to the following result.

COROLLARY 2.3. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a Parseval frame for $\mathcal{M}$. Then for each $\lambda\in[1,+\infty)$, for any $\sigma\subset\mathbb{I}$ and any $x\in\mathcal{M}$, we have*

$$\begin{aligned}\sum_{i\in\sigma}|\langle x,f_i\rangle|^2-\lambda\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2&\leqslant\Big\|\sum_{i\in\sigma}\langle x,f_i\rangle f_i\Big\|^2-\lambda\Big\|\sum_{i\in\sigma^c}\langle x,f_i\rangle f_i\Big\|^2\\&\leqslant(\lambda^3-\lambda^2+1)\sum_{i\in\sigma}|\langle x,f_i\rangle|^2\\&\quad+(\lambda^3-3\lambda^2+2\lambda-1)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2.\end{aligned}$$

REMARK 2.4. We can obtain [9, Theorem 1.3] when taking $\lambda=1$ in Theorem 2.2.

THEOREM 2.5. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$. Then for each $\lambda\in[\frac{1}{2},+\infty)$, for any $\sigma\subset\mathbb{I}$ and any $x\in\mathcal{M}$, we have*

$$\begin{aligned}(4\lambda-1)\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2+(1-\lambda^2)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2&\\\leqslant\sum_{i\in\sigma}|\langle x,f_i\rangle|^2+(1+2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2&\\\leqslant\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2+(1+\lambda^2)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2.&\end{aligned}\tag{2.5}$$

Proof. By (2.3), we obtain, for each $x\in\mathcal{M}$, that

$$\begin{aligned}&\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2-\sum_{i\in\sigma}|\langle x,f_i\rangle|^2\\&\quad=\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,S_{\mathcal{F}}^{\sigma^c}x\rangle-\langle S_{\mathcal{F}}^{\sigma}x,x\rangle\\&\quad\geqslant(2\lambda-\lambda^2)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-\lambda^2\langle S_{\mathcal{F}}^{\sigma}x,x\rangle-\langle S_{\mathcal{F}}^{\sigma}x,x\rangle\\&\quad=(2\lambda-\lambda^2)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(1+\lambda^2)\langle S_{\mathcal{F}}x,x\rangle+(1+\lambda^2)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle\\&\quad=(1+2\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(1+\lambda^2)\langle S_{\mathcal{F}}x,x\rangle\\&\quad=(1+2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2-(1+\lambda^2)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2,\end{aligned}\tag{2.6}$$

which tells us that

$$\sum_{i\in\sigma}|\langle x,f_i\rangle|^2+(1+2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2 \leqslant \sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2+(1+\lambda^2)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2. \tag{2.7}$$

Now by Lemma 2.1(1) (taking $S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma^c}S_{\mathcal{F}}^{-\frac{1}{2}}$, $S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}S_{\mathcal{F}}^{-\frac{1}{2}}$ and $S_{\mathcal{F}}^{\frac{1}{2}}x$, respectively, instead of W, T and x) we arrive at

$$\begin{aligned}\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma}x,S_{\mathcal{F}}^{\sigma}x\rangle &= \langle S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\frac{1}{2}}x,S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\frac{1}{2}}x\rangle\\ &\geqslant ((2\lambda-\lambda^2)-(2\lambda-1))\langle S_{\mathcal{F}}^{\frac{1}{2}}x,S_{\mathcal{F}}^{\frac{1}{2}}x\rangle\\ &\quad -(1-\lambda)(\langle S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma^c}S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\frac{1}{2}}x,S_{\mathcal{F}}^{\frac{1}{2}}x\rangle+\langle S_{\mathcal{F}}^{\frac{1}{2}}x,S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma^c}S_{\mathcal{F}}^{-\frac{1}{2}}S_{\mathcal{F}}^{\frac{1}{2}}x\rangle)\\ &= (1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle-2(1-\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle.\end{aligned}$$

Noting also that $S_{\mathcal{F}}^{\sigma}S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma}\leqslant S_{\mathcal{F}}^{\sigma}$ and $S_{\mathcal{F}}^{\sigma^c}S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}\leqslant S_{\mathcal{F}}^{\sigma^c}$, we have

$$\begin{aligned}&\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,S_{\mathcal{F}}^{\sigma^c}x\rangle-\langle S_{\mathcal{F}}^{\sigma}x,x\rangle\\ &\quad\leqslant \langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma}x,S_{\mathcal{F}}^{\sigma}x\rangle\\ &\quad\leqslant \langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle+2(1-\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle\\ &\quad= (1+2\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(4\lambda-2)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle\\ &\quad\leqslant (1+2\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(4\lambda-2)\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,S_{\mathcal{F}}^{\sigma^c}x\rangle-(1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle\end{aligned}$$

for each $x\in\mathcal{M}$ and each $\lambda\in[\frac{1}{2},+\infty)$, giving that

$$\langle S_{\mathcal{F}}^{\sigma}x,x\rangle+(1+2\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle\geqslant(4\lambda-1)\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,S_{\mathcal{F}}^{\sigma^c}x\rangle+(1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle.$$

That is,

$$\sum_{i\in\sigma}|\langle x,f_i\rangle|^2+(1+2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2 \geqslant (4\lambda-1)\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2+(1-\lambda^2)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2.$$

This along with (2.7) gives (2.5), and we obtain the result. □

It has been shown in the proof of Theorem 2.5 that

$$\begin{aligned}\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,S_{\mathcal{F}}^{\sigma^c}x\rangle-\langle S_{\mathcal{F}}^{\sigma}x,x\rangle &\leqslant \langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle+2(1-\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle\\ &= (3-2\lambda)\langle S_{\mathcal{F}}^{\sigma^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{F}}x,x\rangle.\end{aligned}$$

In other words,

$$\sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2-\sum_{i\in\sigma}|\langle x,f_i\rangle|^2 \leqslant (3-2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2+(\lambda^2-1)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2.$$

This fact together with (2.6) immediately yields

THEOREM 2.6. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$. Then for each $\lambda\in\mathbb{R}$, for any $\sigma\subset\mathbb{I}$ and any $x\in\mathcal{M}$, we have*

$$\begin{aligned}(1+2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2-(1+\lambda^2)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2 &\leqslant \sum_{i\in\mathbb{I}}|\langle S_{\mathcal{F}}^{-1}S_{\mathcal{F}}^{\sigma^c}x,f_i\rangle|^2-\sum_{i\in\sigma}|\langle x,f_i\rangle|^2\\ &\leqslant (3-2\lambda)\sum_{i\in\sigma^c}|\langle x,f_i\rangle|^2+(\lambda^2-1)\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2.\end{aligned}$$

Suppose that $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ is a frame for $\mathcal{M}$ and that $\mathcal{G}=\{g_i\}_{i\in\mathbb{I}}\subseteq\mathcal{M}$ is an alternate dual frame of $\mathcal{F}$. Then for any $\sigma\subset\mathbb{I}$, $\mathcal{F}$ and $\mathcal{G}$ can naturally induce two bounded linear operators $\mathcal{U}^{\sigma},\mathcal{U}^{\sigma^c}:\mathcal{M}\to\mathcal{M}$ given below

$$\mathcal{U}^{\sigma}x=\sum_{i\in\sigma}\langle x,g_i\rangle f_i,\quad \mathcal{U}^{\sigma^c}x=\sum_{i\in\sigma^c}\langle x,g_i\rangle f_i,\quad \forall x\in\mathcal{M}. \tag{2.8}$$

By means of $\mathcal{U}^{\sigma}$ and $\mathcal{U}^{\sigma^c}$ we state a two-sided inequality as follows.

THEOREM 2.7. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$ and $\mathcal{G}=\{g_i\}_{i\in\mathbb{I}}\subseteq\mathcal{M}$ be an alternate dual frame of $\mathcal{F}$. Then for each $\lambda\in[0,1]$, for any $\sigma\subset\mathbb{I}$ and any $x\in\mathcal{M}$, we have*

$$\begin{aligned}(\lambda-\lambda^2)\Big\|\sum_{i\in\mathbb{I}}\langle x,g_i\rangle f_i\Big\|^2-\lambda\mathrm{Re}\sum_{i\in\sigma^c}\langle x,g_i\rangle\langle f_i,x\rangle &\leqslant \Big\|\sum_{i\in\sigma}\langle x,g_i\rangle f_i\Big\|^2-\lambda\mathrm{Re}\sum_{i\in\sigma}\langle x,g_i\rangle\langle f_i,x\rangle\\ &\leqslant \frac{\lambda(\|\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c}\|^2-1)+4(1-\lambda)\|\mathcal{U}^{\sigma}\|^2}{4}\|x\|^2.\end{aligned}$$

Proof. Clearly, $\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c}=\mathrm{Id}_{\mathcal{M}}$. Thus

$$\begin{aligned}&\Big\|\sum_{i\in\sigma}\langle x,g_i\rangle f_i\Big\|^2-\lambda\mathrm{Re}\sum_{i\in\sigma}\langle x,g_i\rangle\langle f_i,x\rangle\\ &=\langle\mathcal{U}^{\sigma}x,\mathcal{U}^{\sigma}x\rangle-\frac{\lambda}{2}(\langle\mathcal{U}^{\sigma}x,x\rangle+\langle x,\mathcal{U}^{\sigma}x\rangle)\\ &=\langle\mathcal{U}^{\sigma}x,\mathcal{U}^{\sigma}x\rangle-\frac{\lambda}{2}(\langle\mathcal{U}^{\sigma}x,(\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c})x\rangle+\langle(\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c})x,\mathcal{U}^{\sigma}x\rangle)\\ &=\langle\mathcal{U}^{\sigma}x,\mathcal{U}^{\sigma}x\rangle-\frac{\lambda}{2}\langle\mathcal{U}^{\sigma}x,\mathcal{U}^{\sigma}x\rangle-\frac{\lambda}{2}\langle\mathcal{U}^{\sigma}x,\mathcal{U}^{\sigma^c}x\rangle\\ &\quad-\frac{\lambda}{2}\langle\mathcal{U}^{\sigma}x,\mathcal{U}^{\sigma}x\rangle-\frac{\lambda}{2}\langle\mathcal{U}^{\sigma^c}x,\mathcal{U}^{\sigma}x\rangle\end{aligned}$$

$$
\begin{aligned}
&= (1-\lambda)\langle \mathcal{U}^{\sigma}x, \mathcal{U}^{\sigma}x\rangle - \frac{\lambda}{2}\langle \mathcal{U}^{\sigma}x, \mathcal{U}^{\sigma^c}x\rangle - \frac{\lambda}{2}\langle \mathcal{U}^{\sigma^c}x, \mathcal{U}^{\sigma}x\rangle \\
&= -\frac{\lambda}{4}\langle x,x\rangle + (1-\lambda)\langle \mathcal{U}^{\sigma}x, \mathcal{U}^{\sigma}x\rangle + \frac{\lambda}{4}\langle (\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c})x, (\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c})x\rangle \\
&\qquad - \frac{\lambda}{2}\langle \mathcal{U}^{\sigma}x, \mathcal{U}^{\sigma^c}x\rangle - \frac{\lambda}{2}\langle \mathcal{U}^{\sigma^c}x, \mathcal{U}^{\sigma}x\rangle \\
&= -\frac{\lambda}{4}\|x\|^2 + (1-\lambda)\langle \mathcal{U}^{\sigma}x, \mathcal{U}^{\sigma}x\rangle + \frac{\lambda}{4}\langle (\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c})x, (\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c})x\rangle \\
&\leqslant -\frac{\lambda}{4}\|x\|^2 + (1-\lambda)\|\mathcal{U}^{\sigma}\|^2\|x\|^2 + \frac{\lambda}{4}\|\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c}\|^2\|x\|^2 \\
&= \frac{\lambda(\|\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c}\|^2-1) + 4(1-\lambda)\|\mathcal{U}^{\sigma}\|^2}{4}\|x\|^2
\end{aligned}
$$

for each $x \in \mathcal{M}$ and each $\lambda \in [0,1]$. For the opposite inequality, we obtain

$$
\|\mathcal{U}^{\sigma}x\|^2 \geqslant (2\lambda-\lambda^2)\|x\|^2 - 2\lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x, x\rangle,
$$

by Lemma 2.1(1). Hence

$$
\begin{aligned}
&\Big\|\sum_{i\in\sigma}\langle x,g_i\rangle f_i\Big\|^2 - \lambda \mathrm{Re}\sum_{i\in\sigma}\langle x,g_i\rangle\langle f_i,x\rangle \\
&\quad = \|\mathcal{U}^{\sigma}x\|^2 - \lambda \mathrm{Re}\langle \mathcal{U}^{\sigma}x,x\rangle \\
&\quad \geqslant (2\lambda-\lambda^2)\|x\|^2 - 2\lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x,x\rangle - \lambda \mathrm{Re}\langle \mathcal{U}^{\sigma}x,x\rangle \\
&\quad = (\lambda-\lambda^2)\|x\|^2 + \frac{\lambda}{2}\langle (\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c})x,x\rangle + \frac{\lambda}{2}\langle x,(\mathcal{U}^{\sigma}+\mathcal{U}^{\sigma^c})x\rangle \\
&\qquad\quad - 2\lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x,x\rangle - \frac{\lambda}{2}\langle \mathcal{U}^{\sigma}x,x\rangle - \frac{\lambda}{2}\langle x,\mathcal{U}^{\sigma}x\rangle \\
&\quad = (\lambda-\lambda^2)\|x\|^2 - 2\lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x,x\rangle + \frac{\lambda}{2}\langle \mathcal{U}^{\sigma^c}x,x\rangle + \frac{\lambda}{2}\langle x,\mathcal{U}^{\sigma^c}x\rangle \\
&\quad = (\lambda-\lambda^2)\|x\|^2 - 2\lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x,x\rangle + \lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x,x\rangle \\
&\quad = (\lambda-\lambda^2)\|x\|^2 - \lambda \mathrm{Re}\langle \mathcal{U}^{\sigma^c}x,x\rangle \\
&\quad = (\lambda-\lambda^2)\Big\|\sum_{i\in\mathbb{I}}\langle x,g_i\rangle f_i\Big\|^2 - \lambda \mathrm{Re}\sum_{i\in\sigma^c}\langle x,g_i\rangle\langle f_i,x\rangle.
\end{aligned}
$$

This completes the proof. □

Alternative inequalities involving the operators $\mathcal{U}^{\sigma}$ and $\mathcal{U}^{\sigma^c}$ defined by (2.8) can be also established.

THEOREM 2.8. *Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}} \subseteq \mathcal{M}$ be an alternate dual frame of $\mathcal{F}$. Then for each $\lambda \in [0,\frac{1}{2}]$, for any $\sigma \subset \mathbb{I}$ and any $x \in \mathcal{M}$,* *we have*

$$(2\lambda-\lambda^2)\Big\|\sum_{i\in\mathbb{I}}\langle x,g_i\rangle f_i\Big\|^2 \leqslant \Big\|\sum_{i\in\sigma}\langle x,g_i\rangle f_i\Big\|^2 + 2\lambda\mathrm{Re}\sum_{i\in\sigma^c}\langle x,g_i\rangle\langle f_i,x\rangle$$
$$\leqslant \frac{3\lambda+2(1-2\lambda)\|\mathcal{U}^{\sigma}\|^2+\lambda\|\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c}\|^2}{2}\|x\|^2.$$

Proof. It is a direct consequence of Lemma 2.1(2), when replacing W and T respectively by $\mathcal{U}^{\sigma}$ and $\mathcal{U}^{\sigma^c}$. □

REMARK 2.9. Theorem B can be obtained if we take $\lambda=\frac{1}{2}$ in the left-hand inequality of Theorem 2.8.

THEOREM 2.10. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{M}$ and $\mathcal{G}=\{g_i\}_{i\in\mathbb{I}}\subseteq\mathcal{M}$ be an alternate dual frame of $\mathcal{F}$. Then for each $\lambda\geqslant 0$, for any $\sigma\subset\mathbb{I}$ and any $x\in\mathcal{M}$, we have*

$$4\lambda^2\mathrm{Re}\sum_{i\in\sigma}\langle x,g_i\rangle\langle f_i,x\rangle-\lambda^2(1+2\lambda)\Big\|\sum_{i\in\mathbb{I}}\langle x,g_i\rangle f_i\Big\|^2$$
$$\leqslant \Big\|\sum_{i\in\sigma}\langle x,g_i\rangle f_i\Big\|^2-2\lambda\mathrm{Re}\sum_{i\in\sigma^c}\langle x,g_i\rangle\langle f_i,x\rangle+2\lambda\Big\|\sum_{i\in\sigma^c}\langle x,g_i\rangle f_i\Big\|^2$$
$$\leqslant \frac{-\lambda+2\|\mathcal{U}^{\sigma}\|^2+\lambda\|\mathcal{U}^{\sigma}-\mathcal{U}^{\sigma^c}\|^2}{2}\|x\|^2.$$

Proof. The proof of the right-hand inequality is similar to Theorem 2.7. For the left-hand inequality, we have, by Lemma 2.1(1), that

$$\|\mathcal{U}^{\sigma^c}x\|^2-2\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle$$
$$\geqslant(1-\lambda^2)\|x\|^2-(2-2\lambda)\mathrm{Re}\langle\mathcal{U}^{\sigma}x,x\rangle-2\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle$$
$$=(1-\lambda^2)\|x\|^2+2\lambda\mathrm{Re}\langle\mathcal{U}^{\sigma}x,x\rangle-2(\mathrm{Re}\langle\mathcal{U}^{\sigma}x,x\rangle+\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle)$$
$$=(1-\lambda^2)\|x\|^2-2\|x\|^2+2\lambda\mathrm{Re}\langle\mathcal{U}^{\sigma}x,x\rangle$$
$$=2\lambda\mathrm{Re}\langle\mathcal{U}^{\sigma}x,x\rangle-(1+\lambda^2)\|x\|^2$$

for each $x\in\mathcal{M}$ and each $\lambda\geqslant 0$. Again by Lemma 2.1(1),

$$\Big\|\sum_{i\in\sigma}\langle x,g_i\rangle f_i\Big\|^2-2\lambda\mathrm{Re}\sum_{i\in\sigma^c}\langle x,g_i\rangle\langle f_i,x\rangle+2\lambda\Big\|\sum_{i\in\sigma^c}\langle x,g_i\rangle f_i\Big\|^2$$
$$=\|\mathcal{U}^{\sigma}x\|^2-2\lambda\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle+2\lambda\|\mathcal{U}^{\sigma^c}x\|^2$$
$$\geqslant(2\lambda-\lambda^2)\|x\|^2-2\lambda\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle-2\lambda\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle+2\lambda\|\mathcal{U}^{\sigma^c}x\|^2$$
$$\geqslant(2\lambda-\lambda^2)\|x\|^2+2\lambda(\|\mathcal{U}^{\sigma^c}x\|^2-2\mathrm{Re}\langle\mathcal{U}^{\sigma^c}x,x\rangle)$$

$$\begin{aligned}&\geqslant (2\lambda-\lambda^2)\|x\|^2+2\lambda\big(2\lambda\mathrm{Re}\langle\mathcal{U}^\sigma x,x\rangle-(1+\lambda^2)\|x\|^2\big)\\&=4\lambda^2\mathrm{Re}\langle\mathcal{U}^\sigma x,x\rangle-\lambda^2(1+2\lambda)\|x\|^2\\&=4\lambda^2\mathrm{Re}\sum_{i\in\sigma}\langle x,g_i\rangle\langle f_i,x\rangle-\lambda^2(1+2\lambda)\Big\|\sum_{i\in\mathbb{I}}\langle x,g_i\rangle f_i\Big\|^2,\end{aligned}$$

and the proof is finished. □

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