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SOME REFINEMENTS OF YOUNG TYPE INEQUALITIES

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Abstract. In this paper, we give some new improvements and reverse improvements of Young type inequalities. The conclusion proved by Yang and Wang [J. Math. Inequal., 17 (2023), 205–217] involved the monotonicity of $\frac{K(x,2)^{\upsilon}(x^{\nu})-(1-\upsilon+\upsilon x)}{\upsilon}$, where $K(x,2)=\frac{(x+1)^2}{4x}$, $x>0$ and $\frac{1}{2}\leqslant\upsilon\leqslant 1$. This article demonstrates the monotonicity of $\frac{M_{\upsilon}^{\upsilon}(x)(x^{\nu})-(1-\upsilon+\upsilon x)}{\upsilon}$, where $M_{\upsilon}(x)=1+\upsilon(1-\upsilon)\frac{(x-1)^2}{x}$, $x\geqslant 1$ and $\frac{1}{2}\leqslant\upsilon\leqslant\frac{3}{4}$. And this implies a main conclusion as follows:

$$\frac{M_{\upsilon}^{\upsilon}(h)a\sharp_{\upsilon}b-a\nabla_{\upsilon}b}{M_{\tau}^{\tau}(h)a\sharp_{\tau}b-a\nabla_{\tau}b}\leqslant\frac{\upsilon}{\tau},$$

where $\frac{1}{2}\leqslant\upsilon<\tau\leqslant\frac{3}{4}$, $b\geqslant a>0$, $M_{\upsilon}(h)=1+\upsilon(1-\upsilon)\frac{(h-1)^2}{h}$ and $h=\frac{b}{a}$.

Furthermore, we can get some related results about operator, Hilbert-Schmidt norm, trace norm by these scalars results.

1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers and $\mathbb{N}^+$ be the set of all positive integers. And we always denote the Kantorovich constant by $K(t,2)=\frac{(t+1)^2}{4t}$ for any $t>0$.

The classical weighted arithmetic-geometric mean inequality reads:

$$\prod_{i=1}^{n}a_i^{p_i}\leqslant\sum_{i=1}^{n}p_ia_i, \tag{1.1}$$

where a_i, $p_i\geqslant 0$ and $\sum_{i=1}^{n}p_i=1$. Then we can get the famous Young's inequality by (1.1) when $n=2$,

$$a^{1-\nu}b^{\nu}\leqslant(1-\nu)a+\nu b. \tag{1.2}$$

Zuo et al. [10] improved (1.2) and Liao et al. [6] gave a reverse of (1.2) as follows:

$$K(h,2)^{r}a\sharp_{\upsilon}b\leqslant a\nabla_{\upsilon}b\leqslant K(h,2)^{R}a\sharp_{\upsilon}b, \tag{1.3}$$

where $a,b>0$, $0\leqslant\upsilon\leqslant 1$, $r=\min\{\upsilon,1-\upsilon\}$, $R=\max\{\upsilon,1-\upsilon\}$ and $h=\frac{b}{a}$.

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Zuo and Li established the following reverse refinement of Young inequality in [9], if $0 \leqslant \upsilon \leqslant 1$ and $a, b > 0$, then

$$a \nabla_\upsilon b \leqslant M_\upsilon^R(h) a \sharp_\upsilon b, \tag{1.4}$$

where $M_\upsilon(t) = 1 + \upsilon(1-\upsilon)\frac{(t-1)^2}{t}$, $R = \max\{\upsilon, 1-\upsilon\}$ and $h = \frac{b}{a}$. And they also gave an inequality for $t > 0$ and $0 \leqslant \upsilon \leqslant 1$ as follows

$$M_\upsilon(t) \leqslant K(t,2). \tag{1.5}$$

Y.H. Ren in [8] established the following improvements regarding Young's inequality:

$$\begin{cases} \dfrac{a\nabla_\nu b - a\sharp_\nu b}{a\nabla_\tau b - a\sharp_\tau b} \leqslant \dfrac{\nu(1-\nu)}{\tau(1-\tau)}, & b - a > 0 \\ \dfrac{a\nabla_\nu b - a\sharp_\nu b}{a\nabla_\tau b - a\sharp_\tau b} \geqslant \dfrac{\nu(1-\nu)}{\tau(1-\tau)}, & b - a < 0 \end{cases} \tag{1.6}$$

Yang and Wang in [12] studied an improvement of inequality (1.3), they obtained the following inequality,

$$\frac{K(h,2)^\upsilon a\sharp_\upsilon b - a\nabla_\upsilon b}{K(h,2)^\tau a\sharp_\tau b - a\nabla_\tau b} \leqslant \frac{\upsilon}{\tau}, \quad \text{(Note: } 0/0 = 0) \tag{1.7}$$

where $\frac{1}{2} < \upsilon \leqslant \tau < 1$, $h = \frac{b}{a}$ and $a, b > 0$.

On the other hand, Ghazanfari, Malekinejad and Talebi in [3] gave a new inequality, which can be stated that if $a, b \geqslant 0$ and $\upsilon \in (0,1]$, then

$$(1-\upsilon^2+\upsilon^3)a + (1-\upsilon^2)b \leqslant \upsilon^{\upsilon-2} a^\upsilon b^{1-\upsilon} + (\sqrt{a}-\sqrt{b})^2. \tag{1.8}$$

In 2020, Yang and Li [11] studied an improvement of inequality (1.8), they obtained the following inequality,

$$(1-\upsilon^{N_1+1}+\upsilon^{N_1+2})a + (1-\upsilon^{N_2+2})b \leqslant \upsilon^{-(1-\upsilon)N_1-\upsilon N_2-1} a^\upsilon b^{1-\upsilon} + (\sqrt{a}-\sqrt{b})^2, \tag{1.9}$$

where $\upsilon \in (0,1]$, $N_1, N_2 \in \mathbb{N}$ and $a, b \geqslant 0$. It was obvious that (1.8) was a special case of inequality (1.9) for $N_1 = 1$, $N_2 = 0$.

In [Theorem 2.14, 1], Zuo and Li obtained the following inequality, which can be stated that if $a, b > 0$, $\upsilon \in (0,1]$, $N \in \mathbb{N}$, $r = \min\{\upsilon, 1-\upsilon\}$ and $h = \frac{\upsilon a}{b}$, then

$$(1-\upsilon^{N+1}+\upsilon^{N+2})a + (1-\upsilon^{N+1})b \leqslant K^{-r}(h,2)\upsilon^{-N-1+\upsilon} a^\upsilon b^{1-\upsilon} + (\sqrt{a}-\sqrt{b})^2. \tag{1.10}$$

In this short paper, similar to (1.7), we will give some inequalities. In particular, we get

$$\frac{M_\upsilon^\upsilon(h) a\sharp_\upsilon b - a\nabla_\upsilon b}{M_\tau^\tau(h) a\sharp_\tau b - a\nabla_\tau b} \leqslant \frac{\upsilon}{\tau},$$

where $\frac{1}{2} \leqslant \upsilon \leqslant \tau \leqslant \frac{3}{4}$, $b \geqslant a > 0$, $M_\upsilon(h) = 1 + \upsilon(1-\upsilon)\frac{(h-1)^2}{h}$ and $h = \frac{b}{a}$. Moreover, we will provide some generalizations for inequality (1.9). As applications, we obtain some inequalities for operator, Hilbert-Schmidt norm and trace norm.

2. Main results

By using $K(h,2) \geqslant 1$ and (1.3), we have $K(h,2)^{-r}a\sharp_\upsilon b \leqslant K(h,2)^r a\sharp_\upsilon b \leqslant a\nabla_\upsilon b$ for $a,b>0$, $0 \leqslant \upsilon \leqslant 1$, $r=\min\{\upsilon,1-\upsilon\}$ and $h=\frac{b}{a}$. The inequality (1.7) was obtained based on $K(h,2)^r a\sharp_\upsilon b \leqslant a\nabla_\upsilon b$. Similarly, we obtain the following theorem based on $K(h,2)^{-r}a\sharp_\upsilon b \leqslant a\nabla_\upsilon b$. Note that when $a \neq b$, we have $K(h,2)^{-r}a\sharp_\upsilon b < a\nabla_\upsilon b$.

THEOREM 2.1. *Let $0<\upsilon \leqslant \tau < \frac{1}{2}$, $a,b>0$ and $h=\frac{b}{a} \neq 1$. Then*

$$\frac{a\nabla_\upsilon b - K(h,2)^{-\upsilon}a\sharp_\upsilon b}{a\nabla_\tau b - K(h,2)^{-\tau}a\sharp_\tau b} \geqslant \frac{\upsilon}{\tau}. \tag{2.1}$$

Proof.

$$\text{Let} \quad f(\upsilon) = \frac{1-\upsilon+\upsilon x - K(x,2)^{-\upsilon}x^\upsilon}{\upsilon} = \frac{(1-\upsilon+\upsilon x)-(\frac{2x}{x+1})^{2\upsilon}}{\upsilon}, \quad (x>0).$$

Then

$$f'(\upsilon) = \frac{h(x)}{\upsilon^2}$$

for

$$\begin{aligned} h(x) &= \upsilon\Big[-1+x-2\Big(\frac{2x}{x+1}\Big)^{2\upsilon}\ln\frac{2x}{x+1}\Big] - \Big[(1-\upsilon+\upsilon x)-\Big(\frac{2x}{x+1}\Big)^{2\upsilon}\Big] \\ &= -2\upsilon\Big(\frac{2x}{x+1}\Big)^{2\upsilon}\ln\frac{2x}{x+1} - 1 + \Big(\frac{2x}{x+1}\Big)^{2\upsilon}. \end{aligned}$$

So we have

$$h'(x) = -4\upsilon^2\Big(\frac{2x}{x+1}\Big)^{2\upsilon-1}\frac{2}{(x+1)^2}\ln\frac{2x}{x+1}.$$

We have $h'(x)>0$ for $x\in(0,1)$ and $h'(x)<0$ for $x>1$, which implies $h(x)\leqslant h(1)=0$. It means $f'(\upsilon)\leqslant 0$, so $f(\upsilon)\geqslant f(\tau)$ when $0<\upsilon\leqslant\tau<\frac{1}{2}$. We complete the proof by putting $x=\frac{b}{a}$. $\square$

THEOREM 2.2. *Let $0<\upsilon\leqslant\tau<\frac{1}{2}$ and $m\in\mathbb{N}^+$. If $a>b>0$ and $h=\frac{b}{a}$, then*

$$\frac{(a\nabla_\upsilon b)^m - K(h,2)^{-m\upsilon}(a\sharp_\upsilon b)^m}{(a\nabla_\tau b)^m - K(h,2)^{-m\tau}(a\sharp_\tau b)^m} \geqslant \frac{\upsilon}{\tau}. \tag{2.2}$$

Proof. For any fixed $x\geqslant 1$, letting $f(\upsilon)=(1-\upsilon+\upsilon x)^m-((\frac{2x}{x+1})^{2\upsilon})^m=((1-\upsilon+\upsilon x)-(\frac{2x}{x+1})^{2\upsilon})h(\upsilon)$, where $h(\upsilon)=(1-\upsilon+\upsilon x)^{m-1}+(1-\upsilon+\upsilon x)^{m-2}(\frac{2x}{x+1})^{2\upsilon}+\ldots+(1-\upsilon+\upsilon x)((\frac{2x}{x+1})^{2\upsilon})^{m-2}+((\frac{2x}{x+1})^{2\upsilon})^{m-1}$. So $h'(\upsilon)=(x-1)[(m-1)(1-\upsilon+\upsilon x)^{m-2}+(m-2)(1-\upsilon+\upsilon x)^{m-3}(\frac{2x}{x+1})^{2\upsilon}+\ldots+((\frac{2x}{x+1})^{2\upsilon})^{m-2}]+[2(1-\upsilon+\upsilon x)^{m-2}$

$(\frac{2x}{x+1})^{2\upsilon}+\ldots+2(1-\upsilon+\upsilon x)(m-2)((\frac{2x}{x+1})^{2\upsilon})^{m-2}+2(m-1)((\frac{2x}{x+1})^{2\upsilon})^{m-1}]\ln\frac{2x}{x+1}$. We have $h'(\upsilon)\leqslant 0$ for $x\in(0,1)$, which implies $h(\upsilon)\geqslant h(\tau)$ for $0<\upsilon\leqslant\tau<\frac{1}{2}$. Hence, $\frac{f(\upsilon)}{f(\tau)}=\frac{(1-\upsilon+\upsilon x)^m-((\frac{2x}{x+1})^{2\upsilon})^m}{(1-\tau+\tau x)^m-((\frac{2x}{x+1})^{2\tau})^m}=\frac{((1-\upsilon+\upsilon x)-(\frac{2x}{x+1})^{2\upsilon})h(\upsilon)}{((1-\tau+\tau x)-(\frac{2x}{x+1})^{2\tau})h(\tau)}\geqslant\frac{(1-\upsilon+\upsilon x)-(\frac{2x}{x+1})^{2\upsilon}}{(1-\tau+\tau x)-(\frac{2x}{x+1})^{2\tau}}\geqslant\frac{\upsilon}{\tau}$. Now by taking $x=\frac{b}{a}$, we can get the desired results. $\square$

THEOREM 2.3. *Let $\frac{1}{6}<\upsilon\leqslant\tau<\frac{1}{2}$. If $b>a>0$ and $h=\frac{b}{a}$, then*

$$\frac{a\nabla_\upsilon b-K(h,2)^{-\upsilon}a\sharp_\upsilon b}{a\nabla_\tau b-K(h,2)^{-\tau}a\sharp_\tau b}\leqslant\frac{\upsilon(1-\upsilon)}{\tau(1-\tau)}. \tag{2.3}$$

Proof. Let

$$f(\upsilon)=\frac{1-\upsilon+\upsilon x-K(x,2)^{-\upsilon}x^\upsilon}{\upsilon(1-\upsilon)}=\frac{(1-\upsilon+\upsilon x)-(\frac{2x}{x+1})^{2\upsilon}}{\upsilon(1-\upsilon)},\quad (x\geqslant 1).$$

Then

$$f'(\upsilon)=\frac{h(x)}{\upsilon^2(1-\upsilon)^2}$$

for

$$h(x)=\upsilon(1-\upsilon)\left[-1+x-2\left(\frac{2x}{x+1}\right)^{2\upsilon}\ln\frac{2x}{x+1}\right]+(2\upsilon-1)\left[(1-\upsilon+\upsilon x)-\left(\frac{2x}{x+1}\right)^{2\upsilon}\right].$$

So we have

$$\begin{aligned}h'(x)=\upsilon(1-\upsilon)&\left[1-2\upsilon x^{-2}\left(\frac{2x}{x+1}\right)^{2\upsilon+1}\ln\frac{2x}{x+1}-x^{-2}\left(\frac{2x}{x+1}\right)^{2\upsilon+1}\right]\\&+(2\upsilon-1)\left[\upsilon-\upsilon x^{-2}\left(\frac{2x}{x+1}\right)^{2\upsilon+1}\right]\end{aligned}$$

and

$$\begin{aligned}h''(x)&=\frac{v^2}{2}x^{-4}\left(\frac{2x}{x+1}\right)^{2\upsilon+2}g(x),\\g(x)&=4(1-v)(x+1)\ln\frac{2x}{x+1}-2(1-v)(1+2v)\ln\frac{2x}{x+1}+2x-1,\\g'(x)&=4(1-v)\ln\frac{2x}{x+1}+4(1-v)\frac{1}{x}-2(1-v)(1+2v)\frac{1}{x(x+1)}+2,\\g''(x)&=\frac{2(1-v)}{x^2(x+1)^2}p(x),\\p(x)&=2v+4vx-1.\end{aligned}$$

Because $p'(x)=4v>0$, we have $p(x)>p(1)=6v-1>0$ for $\frac{1}{6}<\upsilon\leqslant\tau<\frac{1}{2}$ and $x>1$. It means $g''(x)>0$ for $\frac{1}{6}<\upsilon\leqslant\tau<\frac{1}{2}$ and $x>1$, which implies $g'(x)>g'(1)=2(v-\frac{5}{4})^2+\frac{15}{8}>0$ and $g(x)>g(1)=1>0$. Hence, we have $h''(x)>0$, $h'(x)>h'(1)=0$ and $h(x)>h(1)=0$ for $\frac{1}{6}<\upsilon\leqslant\tau<\frac{1}{2}$ and $x>1$, it means $f'(v)>0$, which implies $f(v)\leqslant f(\tau)$ for $\frac{1}{6}<\upsilon\leqslant\tau<\frac{1}{2}$ and $x>1$. We complete the proof by putting $x=\frac{b}{a}$. $\square$

COROLLARY 2.4. *Let* $\frac{1}{6} < \upsilon \leqslant \tau < \frac{1}{2}$, $K(h,2) = \frac{(h+1)^2}{4h}$, $h = \frac{b}{a}$. *If* $b > a > 0$, *then*

$$\frac{\upsilon}{\tau} \leqslant \frac{a\nabla_\upsilon b - K(h,2)^{-\upsilon} a\sharp_\upsilon b}{a\nabla_\tau b - K(h,2)^{-\tau} a\sharp_\tau b} \leqslant \frac{\upsilon(1-\upsilon)}{\tau(1-\tau)}. \tag{2.4}$$

LEMMA 2.5. ([1, 9, 10]) *Let* $M_\upsilon(t) = 1 + \upsilon(1-\upsilon)\frac{(t-1)^2}{t}$, *for* $t > 0$ *and* $0 \leqslant \upsilon \leqslant 1$, *it has the following properties:*

- $M_\upsilon(t) = M_\upsilon(\frac{1}{t})$, $M_{\frac{1}{2}}(t) = K(t,2)$;
- $M_\upsilon(t)$ *is decreasing on* $t \in (0,1]$ *and increasing on* $t \in [1,\infty)$. *And* $M_\upsilon(1) = 1$;
- $1 \leqslant M_\upsilon(t) \leqslant K(t,2)$.
- $M_\upsilon(t) = M_{1-\upsilon}(t)$.

By Lemma 2.5, it is easy to see that $M_\upsilon(t)$ is similar to $K(t,2)$. Similar to (1.7), we can conclude the following Theorem 2.7. First of all, we provide a lemma as follows.

LEMMA 2.6. *Let* $\frac{1}{2} \leqslant \upsilon \leqslant \frac{3}{4}$ *and* $x \geqslant 1$, *then*

$$\begin{aligned} &\left[x + v(1-v)(x-1)^2\right]^{1-v}[1 + 2v(1-v)(x-1)] \\ &\leqslant x + v^2(x-1)^2 + 2v(2-3v)x(x-1) + 2v^2(1-v)^2(x-1)^3 \end{aligned} \tag{2.5}$$

Proof. According to inequality (1.2), by calculating, we obtain

$$\begin{aligned} &\left[x + v(1-v)(x-1)^2\right]^{1-v}[1 + 2v(1-v)(x-1)] \\ &\leqslant \left[v + (1-v)x + v(1-v)^2(x-1)^2\right][1 + 2v(1-v)(x-1)] \\ &= v + 2v^2(1-v)(x-1) + (1-v)x + 2v(1-v)^2x(x-1) + v(1-v)^2(x-1)^2 \\ &\quad + 2v^2(1-v)^3(x-1)^3. \end{aligned}$$

We only need to prove the following inequality

$$\begin{aligned} &v + 2v^2(1-v)(x-1) + (1-v)x + 2v(1-v)^2x(x-1) \\ &\quad + v(1-v)^2(x-1)^2 + 2v^2(1-v)^3(x-1)^3 \\ &\leqslant x + v^2(x-1)^2 + 2v(2-3v)x(x-1) + 2v^2(1-v)^2(x-1)^3. \end{aligned}$$

This means that we only need to prove that

$$(-v^2 + 3v^3 - v)x - 2v^3(1-v)^2(x-1)^2 - 2v + 5v^2 - 3v^3 \leqslant 0.$$

Let $m(x) = (-v^2 + 3v^3 - v)x - 2v^3(1-v)^2(x-1)^2 - 2v + 5v^2 - 3v^3$, then $m'(x) = -v^2 + 3v^3 - v - 4v^3(1-v)^2(x-1)$.

Because $m'(x) \leqslant 0$ for $x \geqslant 1$ and $m(1) = 4v^2 - 3v \leqslant 0$ under $\frac{1}{2} \leqslant \upsilon \leqslant \frac{3}{4}$, it means $m(x) \leqslant m(1) \leqslant 0$ for $x \geqslant 1$, we complete the proof. $\square$

THEOREM 2.7. *Let* $M_\upsilon(h)=1+\upsilon(1-\upsilon)\frac{(h-1)^2}{h}$, $\frac{1}{2}\leqslant\upsilon\leqslant\tau\leqslant\frac{3}{4}$. *If* $b\geqslant a>0$, $h=\frac{b}{a}$, *then*

$$\frac{M_\upsilon^\upsilon(h)a\sharp_\upsilon b-a\nabla_\upsilon b}{M_\tau^\tau(h)a\sharp_\tau b-a\nabla_\tau b}\leqslant\frac{\upsilon}{\tau}.\ \ (\textit{Note: } 0/0=0) \tag{2.6}$$

Proof.

$$\textit{Let}\quad f(\upsilon)=\frac{[x+\upsilon(1-\upsilon)(x-1)^2]^\upsilon-1+\upsilon-\upsilon x}{\upsilon},\ \ (x\geqslant 1).$$

Then

$$\begin{aligned} f'(v)&=\frac{[x+v(1-v)(x-1)^2]^v\cdot\left\{v\ln[x+v(1-v)(x-1)^2]+v^2\cdot\frac{(x-1)^2(1-2v)}{x+v(1-v)(x-1)^2}-1\right\}+1}{v^2}\\ &=\frac{[x+v(1-v)(x-1)^2]^v}{v^2}\cdot H(x), \end{aligned}$$

where

$$\begin{aligned} H(x)\,&=v\ln\left[x+v(1-v)(x-1)^2\right]+\left[x+v(1-v)(x-1)^2\right]^{-v}\\ &\quad+\frac{v^2(1-2v)(x-1)^2}{x+v(1-v)(x-1)^2}-1. \end{aligned}$$

Then

$$\begin{aligned} H'(x)&=\frac{v[1+v(1-v)(2x-2)]}{x+v(1-v)(x-1)^2}-v\cdot\frac{1+v(1-v)(2x-2)}{[x+v(1-v)(x-1)^2]^{v+1}}\\ &\quad+\frac{\left[v^2(1-2v)(2x-2)\right]}{\left[x+v(1-v)(x-1)^2\right]}-\frac{v^2(1-2v)(x-1)^2[1+v(1-v)(2x-2)]}{\left[x+v(1-v)(x-1)^2\right]^2}\\ &=v\cdot\frac{x+v^2(x-1)^2+2v(2-3v)x(x-1)+2v^2(1-v)^2(x-1)^3}{\left[x+v(1-v)(x-1)^2\right]^2}\\ &\quad-v\cdot\frac{\left[x+v(1-v)(x-1)^2\right]^{1-v}[1+2v(1-v)(x-1)]}{\left[x+v(1-v)(x-1)^2\right]^2}. \end{aligned}$$

By Lemma 2.6, we have $H'(x)\geqslant 0$ for $\frac{1}{2}\leqslant\upsilon\leqslant\frac{3}{4}$ and $x\geqslant 1$. It means $H(x)\geqslant H(1)=0$. Hence $f'(v)\geqslant 0$, it implies that $f(\upsilon)\leqslant f(\tau)$ for $\frac{1}{2}\leqslant\upsilon\leqslant\tau\leqslant\frac{3}{4}$ and $x\geqslant 1$. We complete the proof by putting $x=\frac{b}{a}$. □

REMARK 2.8. In some cases, we note that (2.6) is better than (1.7). For example, if taking $a=1$, $b=2$, $v=\frac{5}{8}$ and $\tau=\frac{3}{4}$, then (2.6) becomes the following

$$\frac{M_\upsilon^\upsilon(h)a\sharp_\upsilon b-a\nabla_\upsilon b}{M_\tau^\tau(h)a\sharp_\tau b-a\nabla_\tau b}=\frac{(2+\frac{15}{64})^{\frac{5}{8}}-\frac{13}{8}}{(2+\frac{3}{16})^{\frac{3}{4}}-\frac{7}{4}}=\frac{0.0278\ldots}{0.0487\ldots}=0.57\ldots<\frac{5}{6};$$

and (1.7) becomes the following

$$\frac{K(h,2)^{\upsilon}a\sharp_{\upsilon}b-a\nabla_{\upsilon}b}{K(h,2)^{\tau}a\sharp_{\tau}b-a\nabla_{\tau}b}=\frac{(1.5)^{\frac{5}{4}}-\frac{13}{8}}{(1.5)^{\frac{3}{2}}-\frac{7}{4}}=\frac{0.035\ldots}{0.087\ldots}=0.402\ldots<\frac{5}{6}.$$

REMARK 2.9. We point out that the condition $\frac{1}{2}\leqslant\upsilon\leqslant\tau\leqslant\frac{3}{4}$ in Theorem 2.7 can not be placed by $\frac{3}{4}\leqslant v\leqslant\tau<1$ or $0<v\leqslant\tau\leqslant\frac{1}{2}$.

(a) If letting $a=1$, $b=2$, $v=\frac{3}{4}$ and $\tau=\frac{7}{8}$, then

$$\frac{M_{\upsilon}^{\upsilon}(h)a\sharp_{\upsilon}b-a\nabla_{\upsilon}b}{M_{\tau}^{\tau}(h)a\sharp_{\tau}b-a\nabla_{\tau}b}=\frac{(2+\frac{3}{16})^{\frac{3}{4}}-\frac{7}{4}}{(2+\frac{7}{64})^{\frac{7}{8}}-\frac{15}{8}}=\frac{0.048\ldots}{0.046\ldots}>\frac{6}{7}=\frac{v}{\tau}.$$

(b) If letting $a=1$, $b=2$, $v=\frac{1}{4}$ and $\tau=\frac{3}{8}$, then

$$\frac{M_{\upsilon}^{\upsilon}(h)a\sharp_{\upsilon}b-a\nabla_{\upsilon}b}{M_{\tau}^{\tau}(h)a\sharp_{\tau}b-a\nabla_{\tau}b}=\frac{(2+\frac{3}{16})^{\frac{1}{4}}-\frac{5}{4}}{(2+\frac{15}{64})^{\frac{3}{8}}-\frac{11}{8}}=\frac{-0.0338\ldots}{-0.023\ldots}>\frac{2}{3}=\frac{v}{\tau}.$$

REMARK 2.10. We point out that the condition $b\geqslant a>0$ in Theorem 2.7 is also necessary. For example, if taking $a=1$, $b=\frac{1}{2}$, $v=0.74$ and $\tau=0.75$, then

$$\frac{M_{\upsilon}^{\upsilon}(h)a\sharp_{\upsilon}b-a\nabla_{\upsilon}b}{M_{\tau}^{\tau}(h)a\sharp_{\tau}b-a\nabla_{\tau}b}=\frac{(0.5+0.0481)^{0.74}-0.63}{(0.5+\frac{3}{64})^{\frac{3}{4}}-\frac{5}{8}}=\frac{0.0108\ldots}{0.0109\ldots}>\frac{0.74}{0.75}=\frac{v}{\tau}.$$

Now by $a\sharp_{\tau}b=b\sharp_{1-\tau}a$, $a\nabla_{\tau}b=b\nabla_{1-\tau}a$, and $M_{\tau}(h)=M_{1-\tau}(h)=M_{1-\tau}(h^{-1})$, we have the following results by applying Theorem 2.7.

COROLLARY 2.11. *Let* $M_{\upsilon}(h)=1+\upsilon(1-\upsilon)\frac{(h-1)^2}{h}$, $\frac{1}{4}\leqslant\tau\leqslant\upsilon\leqslant\frac{1}{2}$. *If* $a\geqslant b>0$, $h=\frac{b}{a}$, *then*

$$\frac{M_{\upsilon}^{1-\upsilon}(h)a\sharp_{\upsilon}b-a\nabla_{\upsilon}b}{M_{\tau}^{1-\tau}(h)a\sharp_{\tau}b-a\nabla_{\tau}b}\leqslant\frac{1-\upsilon}{1-\tau}.\quad(Note:0/0=0)$$

We will further generalize inequality (1.8) based on inequality (1.9) and (1.10).

THEOREM 2.12. *Suppose that* $a,b\geqslant 0$, $N_1,N_2,N_3,N_4\in\mathbb{N}$, $N_3\geqslant 2$ *and* N_4 *is a positive even number,* $\upsilon\in(0,1)$, *then*

$$(1-\upsilon^{N_1+1}+\upsilon^{N_1+2}-\upsilon^{N_1+3}+\upsilon^{N_1+4}-\upsilon^{N_1+5}+\cdots+\upsilon^{N_1+N_4})a$$
$$+(1-\upsilon^{N_2+2}-\upsilon^{N_2+3}-\cdots-\upsilon^{N_2+N_3})b$$
$$\leqslant K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1-\upsilon}{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}\right]^{\upsilon}a^{\upsilon}b^{1-\upsilon}+(\sqrt{a}-\sqrt{b})^2,\tag{2.7}$$

where $r=\min\{\upsilon,1-\upsilon\}$ *and* $h=\frac{\upsilon^{N_1-N_2}(1-\upsilon^{N_4})a}{(1+\upsilon)(1-\upsilon^{N_3-1})b}$.

Proof. By simple calculation and using the first inequality of (1.3), we have

$$\begin{aligned}
&K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1-\upsilon}{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}\right]^{\upsilon}a^{\upsilon}b^{1-\upsilon}+(\sqrt{a}-\sqrt{b})^2\\
&-(1-\upsilon^{N_1+1}+\upsilon^{N_1+2}-\upsilon^{N_1+3}+\upsilon^{N_1+4}-\upsilon^{N_1+5}+\cdots+\upsilon^{N_1+N_4})a\\
&-(1-\upsilon^{N_2+2}-\upsilon^{N_2+3}-\cdots-\upsilon^{N_2+N_3})b\\
=&K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1-\upsilon}{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}\right]^{\upsilon}a^{\upsilon}b^{1-\upsilon}-2\sqrt{ab}\\
&+(1-\upsilon)\left[\frac{\upsilon^{N_1+1}(1-\upsilon^{N_4})}{1-\upsilon^2}a\right]+\upsilon\left[\frac{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}{1-\upsilon}b\right]\\
\geqslant&K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1-\upsilon}{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}\right]^{\upsilon}a^{\upsilon}b^{1-\upsilon}-2\sqrt{ab}\\
&+K^{r}(h,2)\left[\frac{\upsilon^{N_1+1}(1-\upsilon^{N_4})}{1-\upsilon^2}\right]^{1-\upsilon}\left[\frac{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}{1-\upsilon}\right]^{\upsilon}a^{1-\upsilon}b^{\upsilon}\\
=&\left\{K^{\frac{-r}{2}}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{\frac{1-\upsilon}{2}}\left[\frac{1-\upsilon}{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}\right]^{\frac{\upsilon}{2}}a^{\frac{\upsilon}{2}}b^{\frac{1-\upsilon}{2}}\right.\\
&\left.-K^{\frac{r}{2}}(h,2)\left[\frac{\upsilon^{N_1+1}(1-\upsilon^{N_4})}{1-\upsilon^2}\right]^{\frac{1-\upsilon}{2}}\left[\frac{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}{1-\upsilon}\right]^{\frac{\upsilon}{2}}a^{\frac{1-\upsilon}{2}}b^{\frac{\upsilon}{2}}\right\}^2\\
\geqslant&0. \quad \square
\end{aligned}$$

THEOREM 2.13. *Suppose that* $a,b\geqslant 0$, $N_1,N_2,N_3,N_4\in\mathbb{N}$, $N_3\geqslant 2$ *and* N_4 *is a positive even number,* $\upsilon\in(0,1)$, *then*

$$\begin{aligned}
&(1-\upsilon^{N_1+1}+\upsilon^{N_1+2}-\upsilon^{N_1+3}+\upsilon^{N_1+4}-\upsilon^{N_1+5}+\cdots+\upsilon^{N_1+N_4})a\\
&+(1-\upsilon^{N_2+2}+\upsilon^{N_2+3}-\upsilon^{N_2+4}+\cdots+(-1)^{N_3-1}\upsilon^{N_2+N_3})b\\
\leqslant&K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1+\upsilon}{\upsilon^{N_2+1}\left[1-(-\upsilon)^{N_3-1}\right]}\right]^{\upsilon}a^{\upsilon}b^{1-\upsilon}+(\sqrt{a}-\sqrt{b})^2,
\end{aligned} \tag{2.8}$$

where $r=\min\{\upsilon,1-\upsilon\}$ *and* $h=\frac{\upsilon^{N_1-N_2}(1-\upsilon^{N_4})(1+\upsilon)a}{(1-\upsilon^2)[1-(-\upsilon)^{N_3-1}]b}$.

Proof. By simple calculation and using the first inequality of (1.3), we have

$$\begin{aligned}
&K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1+\upsilon}{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}\right]^{\upsilon}a^{\upsilon}b^{1-\upsilon}+(\sqrt{a}-\sqrt{b})^2\\
&-(1-\upsilon^{N_1+1}+\upsilon^{N_1+2}-\upsilon^{N_1+3}+\upsilon^{N_1+4}-\upsilon^{N_1+5}+\cdots+\upsilon^{N_1+N_4})a\\
&-(1-\upsilon^{N_2+2}+\upsilon^{N_2+3}-\upsilon^{N_2+4}+\cdots+(-1)^{N_3-1}\upsilon^{N_2+N_3})b
\end{aligned}$$

$$
\begin{aligned}
&= K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1+\upsilon}{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}\right]^{\upsilon} a^{\upsilon}b^{1-\upsilon}-2\sqrt{ab}\\
&\quad +(1-\upsilon)\left[\frac{\upsilon^{N_1+1}(1-\upsilon^{N_4})}{1-\upsilon^2}a\right]+\upsilon\left[\frac{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}{1+\upsilon}b\right]\\
&\geqslant K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1+\upsilon}{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}\right]^{\upsilon} a^{\upsilon}b^{1-\upsilon}-2\sqrt{ab}\\
&\quad +K^{r}(h,2)\left[\frac{\upsilon^{N_1+1}(1-\upsilon^{N_4})}{1-\upsilon^2}\right]^{1-\upsilon}\left[\frac{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}{1+\upsilon}\right]^{\upsilon} a^{1-\upsilon}b^{\upsilon}\\
&= \left\{K^{\frac{-r}{2}}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{\frac{1-\upsilon}{2}}\left[\frac{1+\upsilon}{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}\right]^{\frac{\upsilon}{2}} a^{\frac{\upsilon}{2}}b^{\frac{1-\upsilon}{2}}\right.\\
&\quad \left. -K^{\frac{r}{2}}(h,2)\left[\frac{\upsilon^{N_1+1}(1-\upsilon^{N_4})}{1-\upsilon^2}\right]^{\frac{1-\upsilon}{2}}\left[\frac{\upsilon^{N_2+1}[1-(-\upsilon)^{N_3-1}]}{1+\upsilon}\right]^{\frac{\upsilon}{2}} a^{\frac{1-\upsilon}{2}}b^{\frac{\upsilon}{2}}\right\}^2\\
&\geqslant 0. \quad \square
\end{aligned}
$$

3. Operators inequalities

In this section, we mainly give an operator inequality for the improved Young inequality. Before giving the main result of this part, we need to recall certain useful knowledge.

Let $\mathbb{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators acting on a complex (separable) Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$ and I be the identity. An operator $A\in\mathbb{B}(\mathcal{H})$ is said to be positive semi-definite (denote by $A\geqslant 0$) if $\langle Ax,x\rangle\geqslant 0$ for all vectors $x\in\mathcal{H}$. If $\langle Ax,x\rangle>0$ for all nonzero vectors $x\in\mathcal{H}$, A is said to be positive (denotes $A>0$). For self-adjoint operators $A,B\in\mathbb{B}(\mathcal{H})$, $A\leqslant B$ means $B-A$ is positive semi-definite operator.

For positive invertible operators $A,B\in\mathbb{B}(\mathcal{H})$, the weighted operator arithmetic mean and geometric mean of A and B defined, respectively, by

$$A\nabla_{\nu}B=(1-\nu)A+\nu B,\quad A\sharp_{\nu}B=A^{\frac12}(A^{-\frac12}BA^{-\frac12})^{\nu}A^{\frac12},$$

where $\nu\in[0,1]$.

LEMMA 3.1. ([7]) *Let $X\in\mathbb{B}(\mathcal{H})$ be self-adjoint. If f and g are both continuous functions with $f(t)\geqslant g(t)$ for $t\in Sp(X)$ (where the sign $Sp(X)$ denotes the spectrum of operator X), then $f(X)\geqslant g(X)$.*

THEOREM 3.2. *Let $A,B\in\mathbb{B}(\mathcal{H})$, $0<\upsilon\leqslant\tau<\frac12$, if $0<mI\leqslant B\leqslant m'I\leqslant M'I\leqslant A\leqslant MI$, $h'=\frac{m'}{M'}$, $h=\frac{m}{M}$, then we have*

$$A\nabla_{\upsilon}B\geqslant\frac{\upsilon}{\tau}\left[A\nabla_{\tau}B-K(h',2)^{-\tau}(A\sharp_{\tau}B)\right]+K(h,2)^{-\upsilon}(A\sharp_{\upsilon}B). \tag{3.1}$$

Proof. Letting $a = 1$ in inequality (2.1), then we obtain

$$1\nabla_\upsilon b - K(b,2)^{-\upsilon}(1\sharp_\upsilon b) \geqslant \frac{\upsilon}{\tau}\left[1\nabla_\tau b - K(b,2)^{-\tau}(1\sharp_\tau b)\right].$$

Under our conditions, we have $I \geqslant h'I = \frac{m'}{M'}I \geqslant X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geqslant hI = \frac{m}{M}I$, and then $Sp(X) \subseteq [h,h'] \subseteq (0,1)$. The operator X has a positive spectrum, then by Lemma 3.1, we have

$$I\nabla_\upsilon X \geqslant \frac{\upsilon}{\tau}\left[I\nabla_\tau X - \max_{h\leqslant x\leqslant h'} K(x,2)^{-\tau}(I\sharp_\tau X)\right] + \min_{h\leqslant x\leqslant h'} K(x,2)^{-\upsilon}(I\sharp_\upsilon X).$$

By the monotonicity of the function $K(h,2)$, we have

$$I\nabla_\upsilon X \geqslant \frac{\upsilon}{\tau}\left[I\nabla_\tau X - K(h',2)^{-\tau}(I\sharp_\tau X)\right] + K(h,2)^{-\upsilon}(I\sharp_\upsilon X).$$

We complete the proof of multiplying $A^{\frac{1}{2}}$ on both left and right sides. $\square$

THEOREM 3.3. *Let $A,B \in \mathbb{B}(\mathcal{H})$, $\frac{1}{2} < \upsilon \leqslant \tau < \frac{3}{4}$, if $0 < mI \leqslant A \leqslant m'I \leqslant M'I \leqslant B \leqslant MI$, $h' = \frac{m'}{M'}, h = \frac{m}{M}$, $M_\upsilon(h) = 1 + \upsilon(1-\upsilon)\frac{(h-1)^2}{h}$, then we have*

$$A\nabla_\upsilon B \geqslant \frac{\upsilon}{\tau}\left[A\nabla_\tau B - M_\tau^\tau(h)(A\sharp_\tau B)\right] + M_v^v(h')(A\sharp_\upsilon B). \tag{3.2}$$

Proof. Combination inequality (2.6) with the proof process of Theorem 3.2, we can get the proof easily, so we omit it. $\square$

Similarly, using Theorem2.12 and Theorem2.13, we can obtain the following two results.

THEOREM 3.4. *Let $M' \geqslant m' > 0$ and $A,B \in \mathbb{B}(\mathcal{H})$, satisfy $0 < mI \leqslant A \leqslant m'I < \rho M'I \leqslant B \leqslant \rho MI$ or $0 < \rho mI \leqslant B \leqslant \rho m'I < M'I \leqslant A \leqslant MI$, then we have*

$$\begin{aligned}
&(1-\upsilon^{N_1+1}+\upsilon^{N_1+2}-\upsilon^{N_1+3}+\upsilon^{N_1+4}-\upsilon^{N_1+5}+\cdots+\upsilon^{N_1+N_4})A\\
&+(1-\upsilon^{N_2+2}-\upsilon^{N_2+3}-\cdots-\upsilon^{N_2+N_3})B\\
\leqslant{}& K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1-\upsilon}{\upsilon^{N_2+1}(1-\upsilon^{N_3-1})}\right]^{\upsilon}A\sharp_\upsilon B + 2(A\nabla B - A\sharp B),
\end{aligned}$$

where $v \in [0,1]$, $r = \min\{\upsilon, 1-\upsilon\}$, $h = \frac{m'}{M'}$, $\rho = \frac{\upsilon^{N_1-N_2}(1-\upsilon^{N_4})}{(1+\upsilon)(1-\upsilon^{N_3-1})}$, $N_1,N_2,N_3,N_4 \in \mathbb{N}$, $N_3 \geqslant 2$ and N_4 is a positive even number.

THEOREM 3.5. *Let $M' \geqslant m' > 0$ and $A,B \in \mathbb{B}(\mathcal{H})$, satisfy $0 < mI \leqslant A \leqslant m'I < \rho M'I \leqslant B \leqslant \rho MI$ or $0 < \rho mI \leqslant B \leqslant \rho m'I < M'I \leqslant A \leqslant MI$, then we have*

$$\begin{aligned}
&(1-\upsilon^{N_1+1}+\upsilon^{N_1+2}-\upsilon^{N_1+3}+\upsilon^{N_1+4}-\upsilon^{N_1+5}+\cdots+\upsilon^{N_1+N_4})A\\
&+(1-\upsilon^{N_2+2}+\upsilon^{N_2+3}-\upsilon^{N_2+4}+\cdots+(-1)^{N_3-1}\upsilon^{N_2+N_3})B\\
\leqslant{}& K^{-r}(h,2)\left[\frac{1-\upsilon^2}{\upsilon^{N_1+1}(1-\upsilon^{N_4})}\right]^{1-\upsilon}\left[\frac{1+\upsilon}{\upsilon^{N_2+1}\left[1-(-\upsilon)^{N_3-1}\right]}\right]^{\upsilon}A\sharp_\upsilon B + 2(A\nabla B - A\sharp B),
\end{aligned}$$

where $v \in [0,1]$, $r = \min\{\upsilon, 1-\upsilon\}$, $h = \frac{m'}{M'}$, $\rho = \frac{\upsilon^{N_1-N_2}(1-\upsilon^{N_4})(1+\upsilon)}{(1-\upsilon^2)[1-(-\upsilon)^{N_3-1}]}$ and $N_1,N_2,N_3,N_4 \in \mathbb{N}$, $N_3 \geqslant 2$ and N_4 is a positive even number.

4. Hilbert-Schmidt type inequalities

Let $\mathcal{M}_n(\mathbb{C})$ denote the set of all $n \times n$ complex matrices and let $\mathcal{M}_n^+(\mathbb{C})$ denote the set of all $n \times n$ positive semi-definite matrices in $\mathcal{M}_n(\mathbb{C})$. A matrix norm $|||.|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathcal{M}_n(\mathbb{C})$. For $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\sum_{i=1}^{n} s_i^2(A)} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. The Hilbert-Schmidt norm is unitarily invariant. For the equivalence of the point-wise order (scalar inequality) and Hilbert-Schmidt norm inequality, we recommend the reader to see [11, 12].

THEOREM 4.1. *Let $X \in \mathbb{M}_n$ and $A, B \in \mathbb{M}_n$ be positive for $0 < \upsilon \leqslant \tau < \frac{1}{2}$. Then we have*

$$\begin{aligned} &||(1-\upsilon)AX + \upsilon XB)||_2^2 \\ &\geqslant \frac{\upsilon}{\tau}\left[||(1-\tau)AX + \tau XB)||_2^2 - K_{max}^{-2\tau}||A^{1-\tau}XB^{\tau}||_2^2\right] + K_{min}^{-2v}||A^{1-\upsilon}XB^{\upsilon}||_2^2, \end{aligned} \tag{4.1}$$

where $K_{min}^{-2v} = \min\{K^{-2v}(\frac{x_l}{\lambda_i}, 2) : 1 \leqslant i, l \leqslant n\}$ and $K_{max}^{-2v} = \max\{K^{-2v}(\frac{x_l}{\lambda_i}, 2) : 1 \leqslant i, l \leqslant n\}$ and λ_i, x_l are eigenvalues of A, B respectively such that $\lambda_i \geqslant x_l$ for any $i.l \in \{1, 2, \ldots, n\}$.

Proof. Since A, B are positive definite matrices, it follows by spectral theorem that there exist unitary matrices $U, V \in \mathbb{M}_n$ such that $A = U\Lambda_1 U^*$ and $B = V\Lambda_1 V^*$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $\Lambda_2 = diag(x_1, x_2, \cdots, x_n)$ for λ_i, x_i are eigenvalues of A, B respectively, so $\lambda_i, x_i > 0$, $i = 1, 2, \cdots, n$. Let $Y = U^*XV = [y_{il}]$. Then $(1-\upsilon)AX + \upsilon XB = U\left[(1-\upsilon)\Lambda_1 Y + \upsilon Y\Lambda_2\right]V^* = U\left[((1-\upsilon)\lambda_i + \upsilon x_l)\, y_{il}\right]V^*$ and $A^{1-\upsilon}XB^{\upsilon} = U\left[(\lambda_i^{1-\upsilon}x_l^{\upsilon})y_{il}\right]V^*$. By Theorem 2.2 and the unitarily invariant of the Hilbert-Schmidt norm, we have

$$\begin{aligned} &||(1-\upsilon)AX + \upsilon XB)||_2^2 - K_{min}^{-2v}||A^{1-\upsilon}XB^{\upsilon}||_2^2 \\ &\geqslant \sum_{i,l=1}^{n} ((1-\upsilon)\lambda_i + \upsilon x_l)^2\, |y_{il}|^2 - \sum_{i,l=1}^{n} K(\frac{x_l}{\lambda_i}, 2)^{-2\upsilon}(\lambda_i^{1-\upsilon}x_l^{\upsilon})^2\, |y_{il}|^2 \\ &= \sum_{i,l=1}^{n} \left[((1-\upsilon)\lambda_i + \upsilon x_l)^2 - K(\frac{x_l}{\lambda_i}, 2)^{-2\upsilon}(\lambda_i^{1-\upsilon}x_l^{\upsilon})^2\right] |y_{il}|^2 \\ &\geqslant \frac{\upsilon}{\tau}\sum_{i,l=1}^{n} \left[((1-\tau)\lambda_i + \tau x_l)^2 - K(\frac{x_l}{\lambda_i}, 2)^{-2\tau}(\lambda_i^{1-\tau}x_l^{\tau})^2\right] |y_{il}|^2 \end{aligned}$$

$$= \frac{\upsilon}{\tau}\left[\sum_{i,l=1}^{n}((1-\tau)\lambda_i+\tau x_l)^2|y_{il}|^2-\sum_{i,l=1}^{n}K(\frac{x_l}{\lambda_i},2)^{-2\tau}(\lambda_i^{1-\tau}x_l^{\tau})^2|y_{il}|^2\right]$$
$$\geqslant \frac{\upsilon}{\tau}\left[||(1-\tau)AX+\tau XB)||_2^2-K_{max}^{-2\tau}||A^{1-\tau}XB^{\tau}||_2^2\right]. \quad \square$$

5. Trace norm inequalities

We recall that the trace norm is defined by $\|A\|_1 = \mathrm{tr}(|A|)$ for any $A \in \mathcal{M}_n(\mathbb{C})$.

THEOREM 5.1. *Let $A,B \in \mathbb{M}_n$ be positive and $0 < \upsilon \leqslant \tau < \frac{1}{2}$. If $h = \frac{\mathrm{tr}B}{\mathrm{tr}A} \neq 1$, then we have*

$$\frac{||(1-\upsilon)A+\upsilon B||_1 - K(h,2)^{-\upsilon}||A||_1^{1-\upsilon}||B||_1^{\upsilon}}{||(1-\tau)A+\tau B||_1 - K(h,2)^{-\tau}||A||_1^{1-\tau}||B||_1^{\tau}} \geqslant \frac{\upsilon}{\tau}.$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}
&||(1-\upsilon)A+\upsilon B||_1 \\
&= \mathrm{tr}((1-\upsilon)A+\upsilon B) = (1-\upsilon)\mathrm{tr}(A)+\upsilon\mathrm{tr}(B) \\
&\geqslant \frac{\upsilon}{\tau}((1-\tau)\mathrm{tr}(A)+\tau\mathrm{tr}(B)-K(h,2)^{-\tau}\mathrm{tr}(A)^{1-\tau}tr(B)^{\tau})+K(h,2)^{-\upsilon}\mathrm{tr}(A)^{1-\upsilon}\mathrm{tr}(B)^{\upsilon} \\
&= \frac{\upsilon}{\tau}(||(1-\tau)A+\tau B||_1 - K(h,2)^{-\tau}||A||_1^{1-\tau}||B||_1^{\tau})+K(h,2)^{-\upsilon}||A||_1^{1-\upsilon}||B||_1^{\upsilon}. \quad \square
\end{aligned}$$

Similarly, by Theorem 2.7, we also have

THEOREM 5.2. *Let $A,B \in \mathbb{M}_n$ be positive and $\frac{1}{2} \leqslant \upsilon \leqslant \tau \leqslant \frac{3}{4}$. If $B \geqslant A$, $M_\upsilon(h) = 1+\upsilon(1-\upsilon)\frac{(h-1)^2}{h}$ and $h = \frac{trB}{trA}$, then we have*

$$\frac{M_\nu^\nu(h)||A||_1^{1-\upsilon}||B||_1^{\upsilon} - ||(1-\upsilon)A+\upsilon B||_1}{\upsilon} \leqslant \frac{M_\tau^\tau(h)||A||_1^{1-\tau}||B||_1^{\tau} - ||(1-\tau)A+\tau B||_1}{\tau}.$$

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