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SIX CLASSES OF NONLINEAR TWO–DIMENSIONAL SYSTEMS OF DIFFERENCE EQUATIONS WHICH ARE SOLVABLE

STEVO STEVIĆ

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Abstract. We present six classes of nonlinear two-dimensional systems of difference equations which are solvable. Some methods for finding their general solutions are described in detail.

1. Introduction and preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ be the set of real numbers. The standard convention

$$\prod_{i=m}^{m-1} c_i = 1,$$

where $m \in \mathbb{N}_0$ and $(c_n)_{n\in I}$ $(I \subset \mathbb{N}_0)$ is a sequence of numbers is used throughout the paper.

The first nontrivial closed-form formulas for solutions to difference equations and systems were obtained during the eighteenth century (see, e.g., the original sources [7, 10, 18, 19] and the books [16, 17]). Since that time a majority of books have one or few chapters on solvability of the equations and systems (see, e.g., [8, 12, 20, 21, 22, 24, 25, 49]). For some recent results on solvability see, for instance, [2, 11, 14, 31, 32, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] and the related references cited therein. Solvable difference equations and systems of difference equations, and the methods employed in dealing with them can be also useful in some comparison results, see, for instance, the difference equations and methods in [5, 6, 36, 37, 38], where naturally occur some linear difference inequalities with constant coefficients

Although there are many solvable difference equations and systems of difference equations, they form a narrow subset of all the equations and systems. Therefore, many other methods are employed in dealing with their solutions (see, for instance [3, 4, 5, 6, 8, 12, 13, 14, 20, 21, 22, 24, 25, 26, 27, 29, 30, 31, 36, 37, 38, 49] and the references cited therein). The main idea in the solvability theory is to find some applicable closed-form formulas for the solutions to the difference equations and systems. However, some of the solvable equations and systems have quite complex closed-form formulas for

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their solutions, so that it might be better to use the qualitative analysis of the equations and systems, as it is the case for many equations and systems for which it is not possible to find closed-form formulas for their solutions. Sometimes it is possible to find their invariants [27, 28, 29, 33, 34], but the class of such equations and systems is also not so big. No matter the facts, it is always nice to have formulas for the general solutions to some new classes of difference equations and systems.

The following lemma is one of the basic results in solvability theory of difference equations. It belongs to D. Bernoulli and De Moivre (see [7] and [10]).

LEMMA 1. *Consider the difference equation*

$$a_2x_{n+2}+a_1x_{n+1}+a_0x_n=0, \quad n\in\mathbb{N}_0, \tag{1}$$

where $a_1,x_0,x_1\in\mathbb{R}$ *and* $a_0,a_2\in\mathbb{R}\setminus\{0\}$*. Then, the following statements hold.*

(a) If $a_1^2\neq 4a_0a_2$*, then*

$$x_n=\frac{(x_1-\lambda_2x_0)\lambda_1^n-(x_1-\lambda_1x_0)\lambda_2^n}{\lambda_1-\lambda_2}, \quad n\in\mathbb{N}_0, \tag{2}$$

where

$$\lambda_1=\frac{-a_1+\sqrt{a_1^2-4a_0a_2}}{2a_2} \quad \textit{and} \quad \lambda_2=\frac{-a_1-\sqrt{a_1^2-4a_0a_2}}{2a_2}.$$

(b) If $a_1^2=4a_0a_2$*, then*

$$x_n=((x_1-\lambda x_0)n+\lambda x_0)\lambda^{n-1}, \quad n\in\mathbb{N}_0, \tag{3}$$

where

$$\lambda=-\frac{a_1}{2a_2}.$$

Equation (1) is the homogeneous linear difference equation of second order with constant coefficients and Lemma 1 shows its solvability by giving its general solution in all the possible cases. Formulas (2) and (3) are interesting since they are not only general, but also concrete ones, by which can be found any solution to equation (1) for specified initial values x_0 and x_1.

Many difference equations and systems are transformed to known solvable ones from which their solvability follows. Let us mention some of the systems of difference equations where such a situation occurs.

In [40] was considered the following two-dimensional system of difference equations

$$x_n=\frac{c_ny_{n-3}}{a_n+b_ny_{n-1}x_{n-2}y_{n-3}}, \quad y_n=\frac{\gamma_nx_{n-3}}{\alpha_n+\beta_nx_{n-1}y_{n-2}x_{n-3}}, \quad n\in\mathbb{N}_0,$$

which was transformed to a solvable linear system of difference equations, in [42] was studied the tree-dimensional system

$$x_{n+1}=\frac{a_n^{(1)}x_{n-2}}{b_n^{(1)}y_nz_{n-1}x_{n-2}+c_n^{(1)}}, \qquad y_{n+1}=\frac{a_n^{(2)}y_{n-2}}{b_n^{(2)}z_nx_{n-1}y_{n-2}+c_n^{(2)}},$$

$$z_{n+1} = \frac{a_n^{(3)} z_{n-2}}{b_n^{(3)} x_n y_{n-1} z_{n-2} + c_n^{(3)}}, \quad n \in \mathbb{N}_0,$$

which was also transformed to a solvable linear system of difference equations, and in [43] was investigated the system

$$x_n = \frac{x_{n-k} y_{n-l}}{b_n x_{n-k} + a_n y_{n-l-k}}, \quad y_n = \frac{y_{n-k} x_{n-l}}{d_n y_{n-k} + c_n x_{n-l-k}}, \quad n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, and as in the case of the previous two systems it was also transformed to a solvable linear system of difference equations.

In [39] we found solutions to the max-type system of difference equations

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{y_n}{x_n}\right\}, \quad y_{n+1} = \max\left\{\frac{A}{y_n}, \frac{x_n}{y_n}\right\}, \quad n \in \mathbb{N}_0,$$

for the case $y_0, x_0 \geqslant A > 0$, $y_0/x_0 \geqslant \max\{A, 1/A\}$, but this time reducing it to a solvable product type system of difference equations. For solvability of some product type systems of difference equations see, for instance, [45, 46] and the related references therein.

For some other difference equations and systems of difference equations which can be transformed to known solvable equations and systems see, for instance, [11, 31, 44].

One of the well known solvable difference equations is the bilinear difference equation

$$z_{n+1} = \frac{a z_n + b}{c z_n + d}, \quad n \in \mathbb{N}_0. \tag{4}$$

Many historical facts on the difference equation can be found in the recent paper [41]. For some results and applications of the equation, see, for instance, [1, 8, 9, 14, 15, 16, 17, 20, 23, 24, 25, 41, 47]. It should be noticed that equation (4) is not linear, although its solvability follows from the solvability of a linear equation, more precisely, from the solvability of equation (1).

In the second half of the nineties Papaschinopoulos and Schinas started studying long-term behaviour of concrete two-dimensional systems of difference equations (see, for instance, [27, 28, 29, 30, 33, 34] and the references therein). One of the first studies of symmetric two-dimensional systems of difference equations was conducted in [26]. These papers, among other things, motivated us to study some close-to-symmetric and cyclic systems of difference equations (see, e.g., [35, 39, 40, 41, 42, 43, 45, 46, 47] and the related references therein).

Motivated by all above mentioned, here we continue our investigation of solvability of some concrete systems of difference equations. The systems are two-dimensional and concrete, but they have several parameters, so they are, in fact, some classes of difference equations. We show that they are connected to the bilinear difference equation in a quite natural way, from which their solvability follows. For each of the systems of difference equations we find some formulas for the general solution in some interesting ways.

2. Main results

This section presents our main results in this paper. We present six classes of nonlinear two-dimensional systems of difference equations which are solvable. Some interesting methods for finding their general solutions are described in detail.

2.1. First system

First, we consider the following two-dimensional system of difference equations

$$x_{n+1}=\frac{ax_ny_n+b}{y_n},\quad y_{n+1}=\frac{y_n}{cx_ny_n+d},\quad n\in\mathbb{N}_0, \tag{5}$$

where $a,b,c,d\in\mathbb{R}$, $c^2+d^2\neq 0$ and $x_0,y_0\in\mathbb{R}$.

Note that if $y_{n_0}=0$ for some $n_0\in\mathbb{N}_0$, then x_{n_0+1} is not defined. Hence, from now on we only consider the solutions to system (5) such that

$$y_n\neq 0,\quad n\in\mathbb{N}_0. \tag{6}$$

Besides, we also assume that

$$cx_ny_n+d\neq 0,\quad n\in\mathbb{N}_0,$$

so that the sequence y_n is defined for each $n\in\mathbb{N}_0$.

Multiplying the corresponding sides of the equations in (5) we obtain

$$x_{n+1}y_{n+1}=\frac{ax_ny_n+b}{cx_ny_n+d},\quad n\in\mathbb{N}_0. \tag{7}$$

Using the change of variables

$$z_n=x_ny_n,\quad n\in\mathbb{N}_0, \tag{8}$$

equation (7) is transformed to equation (4).

The cases $c\neq 0$ and $c=0$ are treated in different ways, because we consider them separately.

Case $c\neq 0$. If in equation (4) we employ the change of variables

$$z_n=\frac{u_{n+1}}{u_n}+f,\quad n\in\mathbb{N}_0, \tag{9}$$

(see, e.g., [8, 17, 24]), where we assume that $u_n\neq 0$, $n\in\mathbb{N}_0$, we have

$$\left(\frac{u_{n+2}}{u_{n+1}}+f\right)\left(c\frac{u_{n+1}}{u_n}+cf+d\right)-\left(a\frac{u_{n+1}}{u_n}+af+b\right)=0,$$

for $n\in\mathbb{N}_0$, from which for $f=-\frac{d}{c}$, we obtain

$$\left(\frac{u_{n+2}}{u_{n+1}}-\frac{d}{c}\right)\left(c\frac{u_{n+1}}{u_n}\right)-\left(a\frac{u_{n+1}}{u_n}-\frac{ad}{c}+b\right)=0,$$

for $n \in \mathbb{N}_0$, that is

$$c^2 u_{n+2} - c(a+d)u_{n+1} + (ad-bc)u_n = 0, \tag{10}$$

for $n \in \mathbb{N}_0$.

Now we consider two subcases.

Case $c \neq 0$, $(a+d)^2 \neq 4(ad-bc)$. The conditions $c \neq 0$ and $(a+d)^2 \neq 4(ad-bc)$ imply that the characteristic polynomial

$$p_2(\lambda) = c^2\lambda^2 - c(a+d)\lambda + (ad-bc) \tag{11}$$

associated with the linear equation (10) has two different zeros which are given by

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a-d)^2+4bc}}{2c}.$$

Lemma 1 tells us that the general solution to equation (10) in this case is given by the formula

$$u_n = \frac{(u_1-\lambda_2 u_0)\lambda_1^n - (u_1-\lambda_1 u_0)\lambda_2^n}{\lambda_1-\lambda_2}, \tag{12}$$

for $n \in \mathbb{N}_0$.

Using (12) in (9), as well as the choice $f = -\frac{d}{c}$, we get

$$\begin{aligned} z_n &= \frac{(u_1-\lambda_2 u_0)\lambda_1^{n+1} - (u_1-\lambda_1 u_0)\lambda_2^{n+1}}{(u_1-\lambda_2 u_0)\lambda_1^n - (u_1-\lambda_1 u_0)\lambda_2^n} - \frac{d}{c} \\ &= \frac{(z_0-\lambda_2+\frac{d}{c})\lambda_1^{n+1} - (z_0-\lambda_1+\frac{d}{c})\lambda_2^{n+1}}{(z_0-\lambda_2+\frac{d}{c})\lambda_1^n - (z_0-\lambda_1+\frac{d}{c})\lambda_2^n} - \frac{d}{c}, \end{aligned} \tag{13}$$

for $n \in \mathbb{N}_0$.

From this and (8) we have

$$z_n = x_n y_n = \frac{(x_0y_0-\lambda_2+\frac{d}{c})\lambda_1^{n+1} - (x_0y_0-\lambda_1+\frac{d}{c})\lambda_2^{n+1}}{(x_0y_0-\lambda_2+\frac{d}{c})\lambda_1^n - (x_0y_0-\lambda_1+\frac{d}{c})\lambda_2^n} - \frac{d}{c}, \tag{14}$$

for $n \in \mathbb{N}_0$.

Now note that system (5) can be written in the form

$$x_{n+1} = \frac{az_n+b}{y_n}, \quad y_{n+1} = \frac{y_n}{cz_n+d}, \quad n \in \mathbb{N}_0, \tag{15}$$

where the sequence $(z_n)_{n\in\mathbb{N}_0}$ is given by formula (14).

From the second equation in (15) we easily get

$$y_n = y_0 \prod_{j=0}^{n-1} \frac{1}{cz_j+d}, \quad n \in \mathbb{N}_0. \tag{16}$$

Using (16) in the first equation in system (15) we get

$$x_n = \frac{az_{n-1}+b}{y_0}\prod_{j=0}^{n-2}(cz_j+d), \tag{17}$$

for $n \in \mathbb{N}$.

Case $c \neq 0,\ (a+d)^2 = 4(ad-bc)$. The conditions $c \neq 0$ and $(a+d)^2 = 4(ad-bc)$ imply that the characteristic polynomial (11) associated with the linear equation (10) has two equal zeros which are given by

$$\lambda_{1,2} = \frac{a+d}{2c}.$$

In view of Lemma 1 we have that the general solution to equation (10) in this case is given by the formula

$$u_n = ((u_1-\lambda_1 u_0)n + \lambda_1 u_0)\lambda_1^{n-1}, \tag{18}$$

for $n \in \mathbb{N}_0$.

Using (18) in (9), as well as the choice $f = -\frac{d}{c}$, we obtain

$$\begin{aligned} z_n &= \frac{((u_1-\lambda_1 u_0)(n+1)+\lambda_1 u_0)\lambda_1^n}{((u_1-\lambda_1 u_0)n+\lambda_1 u_0)\lambda_1^{n-1}} - \frac{d}{c}, \\ &= \frac{((z_0-\lambda_1+\frac{d}{c})(n+1)+\lambda_1)\lambda_1}{(z_0-\lambda_1+\frac{d}{c})n+\lambda_1} - \frac{d}{c}, \end{aligned} \tag{19}$$

for $n \in \mathbb{N}_0$.

From this and (8) we have

$$z_n = x_n y_n = \frac{((x_0y_0-\lambda_1+\frac{d}{c})(n+1)+\lambda_1)\lambda_1}{(x_0y_0-\lambda_1+\frac{d}{c})n+\lambda_1} - \frac{d}{c}, \tag{20}$$

for $n \in \mathbb{N}_0$.

Case $c = 0$. In this case equation (4) is

$$z_{n+1} = \frac{a}{d}z_n + \frac{b}{d}, \quad n \in \mathbb{N}_0. \tag{21}$$

Note that the conditions $c = 0$ and $c^2+d^2 \neq 0$ imply $d \neq 0$.

Now we consider two subcases.

Case $c = 0,\ a = d$. In this case equation (21) is

$$z_{n+1} = z_n + \frac{b}{d}, \quad n \in \mathbb{N}_0. \tag{22}$$

It is easy to see that the general solution to equation (22) is

$$z_n = z_0 + \frac{b}{d}n, \quad n \in \mathbb{N}_0. \tag{23}$$

Hence

$$z_n = x_n y_n = x_0 y_0 + \frac{b}{d} n, \quad n \in \mathbb{N}_0. \tag{24}$$

Case $c = 0,\ a \neq d$. In this case the general solution to equation (21) is given by the formula

$$z_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^n - 1\right) + \left(\frac{a}{d}\right)^n z_0, \quad n \in \mathbb{N}_0, \tag{25}$$

Hence

$$z_n = x_n y_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^n - 1\right) + \left(\frac{a}{d}\right)^n x_0 y_0, \quad n \in \mathbb{N}_0. \tag{26}$$

From above consideration we see that the following result holds.

THEOREM 1. *Consider system (5), where* $a,b,c,d \in \mathbb{R}$*,* $c^2 + d^2 \neq 0$ *and* $x_0, y_0 \in \mathbb{R}$*. Then the following statements hold.*

(a) If $c \neq 0$ *and* $(a+d)^2 \neq 4(ad-bc)$*, then the general solution to system (5) is given by formulas (16) and (17), where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{(x_0y_0 - \lambda_2 + \frac{d}{c})\lambda_1^{n+1} - (x_0y_0 - \lambda_1 + \frac{d}{c})\lambda_2^{n+1}}{(x_0y_0 - \lambda_2 + \frac{d}{c})\lambda_1^{n} - (x_0y_0 - \lambda_1 + \frac{d}{c})\lambda_2^{n}} - \frac{d}{c}, \quad n \in \mathbb{N}_0.$$

(b) If $c \neq 0$ *and* $(a+d)^2 = 4(ad-bc)$*, then the general solution to system (5) is given by formulas (16) and (17), where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{((x_0y_0 - \lambda_1 + \frac{d}{c})(n+1) + \lambda_1)\lambda_1}{(x_0y_0 - \lambda_1 + \frac{d}{c})n + \lambda_1} - \frac{d}{c}, \quad n \in \mathbb{N}_0.$$

(c) If $c = 0$ *and* $a = d$*, then the general solution to system (5) is given by formulas (16) and (17), where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = x_0 y_0 + \frac{b}{d} n, \quad n \in \mathbb{N}_0.$$

(d) If $c = 0$ *and* $a \neq d$*, then the general solution to system (5) is given by formulas (16) and (17), where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^n - 1\right) + \left(\frac{a}{d}\right)^n x_0 y_0, \quad n \in \mathbb{N}_0.$$

2.2. Second system

Here we consider the following two-dimensional system of difference equations

$$x_{n+1} = \frac{x_n y_n}{bx_n + ay_n}, \quad y_{n+1} = \frac{x_n y_n}{dx_n + cy_n}, \quad n \in \mathbb{N}_0, \tag{27}$$

where $a, b, c, d \in \mathbb{R}$, $a^2 + b^2 \neq 0$, $c^2 + d^2 \neq 0$ and $x_0, y_0 \in \mathbb{R}$.

If $x_{n_1} = 0$ for some $n_1 \in \mathbb{N}_0$, then from (27) we have

$$x_{n_1+1} = y_{n_1+1} = 0,$$

from which along with (27) it follows that x_{n_1+2} and y_{n_1+2} are not defined. If $y_{n_2} = 0$ for some $n_2 \in \mathbb{N}_0$, then from (27) we have $x_{n_2+1} = y_{n_2+1} = 0$, from which along with (27) it follows that x_{n_2+2} and y_{n_2+2} are not defined.

Hence, from now on we only consider the solutions to system (27) satisfying the condition

$$x_n \neq 0 \neq y_n, \tag{28}$$

for $n \in \mathbb{N}_0$.

Besides, we also assume that

$$bx_n + ay_n \neq 0 \neq dx_n + cy_n, \quad n \in \mathbb{N}_0,$$

so that the sequences x_n and y_n are defined for each $n \in \mathbb{N}_0$.

Dividing the corresponding sides of the equations in (27) we obtain

$$\frac{y_{n+1}}{x_{n+1}} = \frac{bx_n + ay_n}{dx_n + cy_n}, \quad n \in \mathbb{N}_0. \tag{29}$$

Using the change of variables

$$z_n = \frac{y_n}{x_n}, \quad n \in \mathbb{N}_0, \tag{30}$$

equation (29) is transformed to equation (4) (the change is allowed since (28) holds), from which together with the analysis conducted in the case of system (5) it follows that the formulas for the sequence $(z_n)_{n\in\mathbb{N}_0}$ obtained above also hold for the sequence defined in (30).

Now note that system (27) can be written in the form

$$x_{n+1} = \frac{y_n}{b + az_n}, \quad y_{n+1} = \frac{y_n}{d + cz_n}, \quad n \in \mathbb{N}_0. \tag{31}$$

From the second equation in (31) we easily get

$$y_n = y_0 \prod_{j=0}^{n-1} \frac{1}{cz_j + d}, \quad n \in \mathbb{N}_0. \tag{32}$$

Using (32) in the first equation in system (31) we get

$$x_n = \frac{y_0}{az_{n-1}+b}\prod_{j=0}^{n-2}\frac{1}{cz_j+d}, \quad n \in \mathbb{N}. \tag{33}$$

From (13), (19), (23), (25) and the above analysis it follows that the following result holds for the case of system (27).

THEOREM 2. *Consider system* (27)*, where* $a,b,c,d \in \mathbb{R}$*,* $a^2+b^2 \neq 0$*,* $c^2+d^2 \neq 0$ *and* $x_0, y_0 \in \mathbb{R}$*. Then the following statements hold.*

(a) If $c \neq 0$ *and* $(a+d)^2 \neq 4(ad-bc)$*, then the general solution to system* (27) *is given by formulas* (32) *and* (33)*, where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{(\frac{y_0}{x_0}-\lambda_2+\frac{d}{c})\lambda_1^{n+1}-(\frac{y_0}{x_0}-\lambda_1+\frac{d}{c})\lambda_2^{n+1}}{(\frac{y_0}{x_0}-\lambda_2+\frac{d}{c})\lambda_1^{n}-(\frac{y_0}{x_0}-\lambda_1+\frac{d}{c})\lambda_2^{n}} - \frac{d}{c}, \quad n \in \mathbb{N}_0.$$

(b) If $c \neq 0$ *and* $(a+d)^2 = 4(ad-bc)$*, then the general solution to system* (27) *is given by formulas* (32) *and* (33)*, where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{((\frac{y_0}{x_0}-\lambda_1+\frac{d}{c})(n+1)+\lambda_1)\lambda_1}{(\frac{y_0}{x_0}-\lambda_1+\frac{d}{c})n+\lambda_1} - \frac{d}{c}, \quad n \in \mathbb{N}_0.$$

(c) If $c = 0$ *and* $a = d$*, then the general solution to system* (27) *is given by formulas* (32) *and* (33)*, where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{y_0}{x_0} + \frac{b}{d}n, \quad n \in \mathbb{N}_0.$$

(d) If $c = 0$ *and* $a \neq d$*, then the general solution to system* (27) *is given by formulas* (32) *and* (33)*, where the sequence* $(z_n)_{n\in\mathbb{N}_0}$ *is given by*

$$z_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^n - 1\right) + \left(\frac{a}{d}\right)^n \frac{y_0}{x_0}, \quad n \in \mathbb{N}_0.$$

2.3. Third system

Here we consider the following two-dimensional system of difference equations

$$x_{n+1} = \frac{x_n(ax_n+by_nx_{n-1})}{x_{n-1}}, \quad y_{n+1} = \frac{cx_n+dy_nx_{n-1}}{x_{n-1}}, \quad n \in \mathbb{N}, \tag{34}$$

where $a,b,c,d \in \mathbb{R}$, $a^2+b^2 \neq 0$, $c^2+d^2 \neq 0$ and $x_0, x_1, y_1 \in \mathbb{R}$.

If a solution to system (34) is defined, then the condition

$$x_n \neq 0, \quad n \in \mathbb{N}_0, \tag{35}$$

must hold.

Dividing the corresponding sides of the equations in (34) we obtain

$$\frac{x_{n+1}}{y_{n+1}}=\frac{x_n(ax_n+by_nx_{n-1})}{cx_n+dy_nx_{n-1}},\quad n\in\mathbb{N},$$

from which along with (35) it follows that

$$\frac{x_{n+1}}{y_{n+1}x_n}=\frac{ax_n+by_nx_{n-1}}{cx_n+dy_nx_{n-1}},\quad n\in\mathbb{N}. \tag{36}$$

Using the change of variables

$$z_n=\frac{x_n}{y_nx_{n-1}},\quad n\in\mathbb{N}, \tag{37}$$

equation (36) is transformed to equation (4), from which together with the analysis conducted in the case of system (5) it follows that the formulas for the sequence $(z_n)_{n\in\mathbb{N}_0}$ obtained therein also hold for the sequence defined in (37), but with the initial value z_1.

Now note that system (34) can be written in the form

$$x_{n+1}=x_ny_n(az_n+b),\quad y_{n+1}=y_n(cz_n+d),\quad n\in\mathbb{N}. \tag{38}$$

From the second equation in (38) we easily get

$$y_n=y_1\prod_{j=1}^{n-1}(cz_j+d),\quad n\in\mathbb{N}. \tag{39}$$

Using (39) in the first equation in system (38) we get

$$x_n=x_{n-1}(az_{n-1}+b)y_1\prod_{j=1}^{n-2}(cz_j+d),\quad n\geqslant 2,$$

from which it follows that

$$x_n=x_1y_1^{n-1}\prod_{k=2}^{n}(az_{k-1}+b)\prod_{j=1}^{k-2}(cz_j+d),\quad n\in\mathbb{N}. \tag{40}$$

From this we have that the following result holds.

THEOREM 3. *Consider system (34), where $a,b,c,d\in\mathbb{R}$, $a^2+b^2\neq 0$, $c^2+d^2\neq 0$ and $x_0,x_1,y_1\in\mathbb{R}$. Then the following statements hold.*

(a) If $c\neq 0$ and $(a+d)^2\neq 4(ad-bc)$, then the general solution to system (34) is given by formulas (39) and (40), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n=\frac{(\frac{x_1}{y_1x_0}-\lambda_2+\frac{d}{c})\lambda_1^n-(\frac{x_1}{y_1x_0}-\lambda_1+\frac{d}{c})\lambda_2^n}{(\frac{x_1}{y_1x_0}-\lambda_2+\frac{d}{c})\lambda_1^{n-1}-(\frac{x_1}{y_1x_0}-\lambda_1+\frac{d}{c})\lambda_2^{n-1}}-\frac{d}{c},\quad n\in\mathbb{N}.$$

(b) If $c \neq 0$ and $(a+d)^2 = 4(ad-bc)$, then the general solution to system (34) is given by formulas (39) and (40), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{((\frac{x_1}{y_1x_0} - \lambda_1 + \frac{d}{c})n + \lambda_1)\lambda_1}{(\frac{x_1}{y_1x_0} - \lambda_1 + \frac{d}{c})(n-1) + \lambda_1} - \frac{d}{c}, \quad n \in \mathbb{N}.$$

(c) If $c = 0$ and $a = d$, then the general solution to system (34) is given by formulas (39) and (40), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{x_1}{y_1x_0} + \frac{b}{d}(n-1), \quad n \in \mathbb{N}.$$

(d) If $c = 0$ and $a \neq d$, then the general solution to system (34) is given by formulas (39) and (40), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^{n-1} - 1\right) + \left(\frac{a}{d}\right)^{n-1}\frac{x_1}{y_1x_0}, \quad n \in \mathbb{N}.$$

2.4. Fourth system

Here we consider the following two-dimensional system of difference equations

$$x_{n+1} = \frac{x_n^2}{ax_{n-1} + bx_ny_n}, \quad y_{n+1} = \frac{cx_{n-1} + dx_ny_n}{x_n}, \quad n \in \mathbb{N}, \tag{41}$$

where $a,b,c,d \in \mathbb{R}$, $a^2+b^2 \neq 0$, $c^2+d^2 \neq 0$ and $x_0, x_1, y_1 \in \mathbb{R}$.

A sequence $(x_n)_{n\in\mathbb{N}_0}$ satisfying (41) is well-defined if condition (35) and the following one

$$ax_{n-1} + bx_ny_n \neq 0, \quad n \in \mathbb{N},$$

hold.

Multiplying the corresponding sides of the equations in (41) we obtain

$$x_{n+1}y_{n+1} = \frac{x_n(cx_{n-1} + dx_ny_n)}{ax_{n-1} + bx_ny_n}, \quad n \in \mathbb{N},$$

from which along with (35) it follows that

$$\frac{x_n}{x_{n+1}y_{n+1}} = \frac{ax_{n-1} + bx_ny_n}{cx_{n-1} + dx_ny_n}, \quad n \in \mathbb{N}. \tag{42}$$

Using the change of variables

$$z_n = \frac{x_{n-1}}{x_ny_n}, \quad n \in \mathbb{N}, \tag{43}$$

equation (42) is transformed to equation (4), from which together with the analysis conducted in the case of system (5) it follows that the formulas for the sequence z_n

obtained therein also holds for the sequence defined in (43), but with the initial value z_1.

Now note that system (41) can be written in the form

$$x_{n+1}=\frac{x_n}{y_n(az_n+b)},\quad y_{n+1}=y_n(cz_n+d),\quad n\in\mathbb{N}. \tag{44}$$

From the second equation in (44) we easily get

$$y_n=y_1\prod_{j=1}^{n-1}(cz_j+d),\quad n\in\mathbb{N}. \tag{45}$$

Using (45) in the first equation in system (44) we get

$$x_n=\frac{x_{n-1}}{y_1(az_{n-1}+b)\prod_{j=1}^{n-2}(cz_j+d)},\quad n\in\mathbb{N},$$

from which it follows that

$$x_n=\frac{x_1}{y_1^{n-1}\prod_{k=2}^{n}(az_{k-1}+b)\prod_{j=1}^{k-2}(cz_j+d)},\quad n\in\mathbb{N}. \tag{46}$$

From this we have that the following result holds.

THEOREM 4. *Consider system (41), where $a,b,c,d\in\mathbb{R}$, $a^2+b^2\neq 0$, $c^2+d^2\neq 0$ and $x_0,x_1,y_1\in\mathbb{R}$. Then the following statements hold.*

(a) If $c\neq 0$ and $(a+d)^2\neq 4(ad-bc)$, then the general solution to system (41) is given by formulas (45) and (46), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n=\frac{(\frac{x_0}{x_1y_1}-\lambda_2+\frac{d}{c})\lambda_1^n-(\frac{x_0}{x_1y_1}-\lambda_1+\frac{d}{c})\lambda_2^n}{(\frac{x_0}{x_1y_1}-\lambda_2+\frac{d}{c})\lambda_1^{n-1}-(\frac{x_0}{x_1y_1}-\lambda_1+\frac{d}{c})\lambda_2^{n-1}}-\frac{d}{c},\quad n\in\mathbb{N}.$$

(b) If $c\neq 0$ and $(a+d)^2=4(ad-bc)$, then the general solution to system (41) is given by formulas (45) and (46), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n=\frac{((\frac{x_0}{x_1y_1}-\lambda_1+\frac{d}{c})n+\lambda_1)\lambda_1}{(\frac{x_0}{x_1y_1}-\lambda_1+\frac{d}{c})(n-1)+\lambda_1}-\frac{d}{c},\quad n\in\mathbb{N}.$$

(c) If $c=0$ and $a=d$, then the general solution to system (41) is given by formulas (45) and (46), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n=\frac{x_0}{x_1y_1}+\frac{b}{d}(n-1),\quad n\in\mathbb{N}.$$

(d) If $c=0$ and $a\neq d$, then the general solution to system (41) is given by formulas (45) and (46), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n=\frac{b}{a-d}\left(\left(\frac{a}{d}\right)^{n-1}-1\right)+\left(\frac{a}{d}\right)^{n-1}\frac{x_0}{x_1y_1},\quad n\in\mathbb{N}.$$

2.5. Fifth system

Here we consider the following two-dimensional system of difference equations

$$x_{n+1}=x_n\frac{ax_ny_{n-1}+bx_{n-1}y_n}{x_{n-1}y_{n-1}},\quad y_{n+1}=y_n\frac{cx_ny_{n-1}+dx_{n-1}y_n}{x_{n-1}y_{n-1}},\quad n\in\mathbb{N},\tag{47}$$

where $a,b,c,d\in\mathbb{R}$, $a^2+b^2\neq 0$, $c^2+d^2\neq 0$ and $x_0,y_0,x_1,y_1\in\mathbb{R}$.

If a solution to system (47) is well-defined, then it is easily seen that the condition (28) must hold.

Dividing the corresponding sides of equations in (47) we obtain

$$\frac{x_{n+1}}{y_{n+1}}=\frac{x_n(ax_ny_{n-1}+bx_{n-1}y_n)}{y_n(cx_ny_{n-1}+dx_{n-1}y_n)},\quad n\in\mathbb{N},$$

from which along with (28) it follows that

$$\frac{x_{n+1}y_n}{y_{n+1}x_n}=\frac{ax_ny_{n-1}+bx_{n-1}y_n}{cx_ny_{n-1}+dx_{n-1}y_n},\quad n\in\mathbb{N}.\tag{48}$$

Using the change of variables

$$z_n=\frac{x_ny_{n-1}}{y_nx_{n-1}},\quad n\in\mathbb{N},\tag{49}$$

equation (48) is transformed to equation (4), from which together with the analysis conducted in the case of system (5) it follows that the formulas for the sequence z_n obtained therein also holds for the sequence defined in (49), but with the initial value z_1.

Now note that system (47) can be written in the form

$$x_{n+1}=\frac{x_ny_n}{y_{n-1}}(az_n+b),\quad y_{n+1}=\frac{y_n^2}{y_{n-1}}(cz_n+d),\quad n\in\mathbb{N}.\tag{50}$$

From the second equation in (50) we have

$$\frac{y_{n+1}}{y_n}=\frac{y_n}{y_{n-1}}(cz_n+d),\quad n\in\mathbb{N},$$

from which it follows that

$$\frac{y_n}{y_{n-1}}=\frac{y_1}{y_0}\prod_{j=1}^{n-1}(cz_j+d),\quad n\in\mathbb{N},\tag{51}$$

that is,

$$y_n=y_{n-1}\frac{y_1}{y_0}\prod_{j=1}^{n-1}(cz_j+d),\quad n\in\mathbb{N},$$

from which it follows that

$$y_n=\frac{y_1^n}{y_0^{n-1}}\prod_{k=1}^{n}\prod_{j=1}^{k-1}(cz_j+d),\quad n\in\mathbb{N}.\tag{52}$$

Using (51) in the first equation in system (50) we get

$$x_n = x_{n-1}\frac{y_1}{y_0}(az_{n-1}+b)\prod_{j=1}^{n-2}(cz_j+d), \quad n \geqslant 2,$$

from which it follows that

$$x_n = x_1\left(\frac{y_1}{y_0}\right)^{n-1}\prod_{k=2}^{n}(az_{k-1}+b)\prod_{j=1}^{k-2}(cz_j+d), \quad n\in\mathbb{N}. \tag{53}$$

From this we have that the following result holds.

THEOREM 5. *Consider system* (47)*, where* $a,b,c,d\in\mathbb{R}$*,* $a^2+b^2\neq 0$*,* $c^2+d^2\neq 0$ *and* $x_0,y_0,x_1,y_1\in\mathbb{R}$*. Then the following statements hold.*

(a) If $c\neq 0$ *and* $(a+d)^2\neq 4(ad-bc)$*, then the general solution to system* (47) *is given by formulas* (52) *and* (53)*, where the sequence* $(z_n)_{n\in\mathbb{N}}$ *is given by*

$$z_n = \frac{(\frac{x_1y_0}{y_1x_0}-\lambda_2+\frac{d}{c})\lambda_1^n-(\frac{x_1y_0}{y_1x_0}-\lambda_1+\frac{d}{c})\lambda_2^n}{(\frac{x_1y_0}{y_1x_0}-\lambda_2+\frac{d}{c})\lambda_1^{n-1}-(\frac{x_1y_0}{y_1x_0}-\lambda_1+\frac{d}{c})\lambda_2^{n-1}}-\frac{d}{c}, \quad n\in\mathbb{N}.$$

(b) If $c\neq 0$ *and* $(a+d)^2 = 4(ad-bc)$*, then the general solution to system* (47) *is given by formulas* (52) *and* (53)*, where the sequence* $(z_n)_{n\in\mathbb{N}}$ *is given by*

$$z_n = \frac{((\frac{x_1y_0}{y_1x_0}-\lambda_1+\frac{d}{c})n+\lambda_1)\lambda_1}{(\frac{x_1y_0}{y_1x_0}-\lambda_1+\frac{d}{c})(n-1)+\lambda_1}-\frac{d}{c}, \quad n\in\mathbb{N}.$$

(c) If $c=0$ *and* $a=d$*, then the general solution to system* (47) *is given by formulas* (52) *and* (53)*, where the sequence* $(z_n)_{n\in\mathbb{N}}$ *is given by*

$$z_n = \frac{x_1y_0}{y_1x_0}+\frac{b}{d}(n-1), \quad n\in\mathbb{N}.$$

(d) If $c=0$ *and* $a\neq d$*, then the general solution to system* (47) *is given by formulas* (52) *and* (53)*, where the sequence* $(z_n)_{n\in\mathbb{N}}$ *is given by*

$$z_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^{n-1}-1\right)+\left(\frac{a}{d}\right)^{n-1}\frac{x_1y_0}{y_1x_0}, \quad n\in\mathbb{N}.$$

2.6. Sixth system

Here we consider the following two-dimensional system of difference equations

$$x_{n+1} = x_n\frac{ax_ny_n+bx_{n-1}y_{n-1}}{x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-1}y_n^2}{cx_ny_n+dx_{n-1}y_{n-1}}, \quad n\in\mathbb{N}, \tag{54}$$

where $a,b,c,d \in \mathbb{R}$, $a^2+b^2 \neq 0$, $c^2+d^2 \neq 0$ and $x_0,x_1,y_0,y_1 \in \mathbb{R}$.

If a two-dimensional sequence $(x_n,y_n)_{n\in\mathbb{N}}$ satisfying system (54) is well-defined, then it is clear that condition (28) must hold.

Multiplying the corresponding sides of the equations in (54) we obtain

$$x_{n+1}y_{n+1} = x_n y_n \frac{ax_ny_n + bx_{n-1}y_{n-1}}{cx_ny_n + dx_{n-1}y_{n-1}}, \quad n \in \mathbb{N},$$

from which along with (28) it follows that

$$\frac{x_{n+1}y_{n+1}}{x_ny_n} = \frac{ax_ny_n + bx_{n-1}y_{n-1}}{cx_ny_n + dx_{n-1}y_{n-1}}, \quad n \in \mathbb{N}. \tag{55}$$

Using the change of variables

$$z_n = \frac{x_ny_n}{x_{n-1}y_{n-1}}, \quad n \in \mathbb{N}, \tag{56}$$

equation (55) is transformed to equation (4), from which together with the analysis conducted in the case of system (5) it follows that the formulas for the sequence z_n obtained therein also holds for the sequence defined in (56), but with the initial value z_1.

Now note that system (54) can be written in the form

$$x_{n+1} = \frac{x_ny_{n-1}}{y_n}(az_n+b), \quad y_{n+1} = \frac{y_n^2}{y_{n-1}(cz_n+d)}, \quad n \in \mathbb{N}. \tag{57}$$

From the second equation in (57) we have

$$\frac{y_{n+1}}{y_n} = \frac{y_n}{y_{n-1}} \frac{1}{cz_n+d}, \quad n \in \mathbb{N},$$

from which it follows that

$$\frac{y_n}{y_{n-1}} = \frac{y_1}{y_0} \prod_{j=1}^{n-1} \frac{1}{cz_j+d}, \quad n \in \mathbb{N}, \tag{58}$$

that is,

$$y_n = y_{n-1} \frac{y_1}{y_0} \prod_{j=1}^{n-1} \frac{1}{cz_j+d}, \quad n \in \mathbb{N},$$

from which it follows that

$$y_n = \frac{y_1^n}{y_0^{n-1}} \prod_{k=1}^{n} \prod_{j=1}^{k-1} \frac{1}{cz_j+d}, \quad n \in \mathbb{N}. \tag{59}$$

Using (58) in the first equation in system (57) we get

$$x_n = x_{n-1} \frac{y_0}{y_1}(az_{n-1}+b) \prod_{j=1}^{n-2}(cz_j+d), \quad n \geqslant 2,$$

from which it follows that

$$x_n = x_1 \left(\frac{y_0}{y_1}\right)^{n-1} \prod_{k=2}^{n}(az_{k-1}+b)\prod_{j=1}^{k-2}(cz_j+d), \quad n \in \mathbb{N}. \tag{60}$$

From this we have that the following result holds.

THEOREM 6. *Consider system (54), where $a,b,c,d \in \mathbb{R}$, $a^2+b^2 \neq 0$, $c^2+d^2 \neq 0$ and $x_0,y_0,x_1,y_1 \in \mathbb{R}$. Then the following statements hold.*

(a) If $c \neq 0$ and $(a+d)^2 \neq 4(ad-bc)$, then the general solution to system (54) is given by formulas (59) and (60), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{(\frac{x_1y_1}{x_0y_0}-\lambda_2+\frac{d}{c})\lambda_1^n-(\frac{x_1y_1}{x_0y_0}-\lambda_1+\frac{d}{c})\lambda_2^n}{(\frac{x_1y_1}{x_0y_0}-\lambda_2+\frac{d}{c})\lambda_1^{n-1}-(\frac{x_1y_1}{x_0y_0}-\lambda_1+\frac{d}{c})\lambda_2^{n-1}}-\frac{d}{c}, \quad n\in\mathbb{N}.$$

(b) If $c \neq 0$ and $(a+d)^2 = 4(ad-bc)$, then the general solution to system (54) is given by formulas (59) and (60), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{((\frac{x_1y_1}{x_0y_0}-\lambda_1+\frac{d}{c})n+\lambda_1)\lambda_1}{(\frac{x_1y_1}{x_0y_0}-\lambda_1+\frac{d}{c})(n-1)+\lambda_1}-\frac{d}{c}, \quad n\in\mathbb{N}.$$

(c) If $c=0$ and $a=d$, then the general solution to system (54) is given by formulas (59) and (60), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{x_1y_1}{x_0y_0}+\frac{b}{d}(n-1), \quad n\in\mathbb{N}.$$

(d) If $c=0$ and $a\neq d$, then the general solution to system (54) is given by formulas (59) and (60), where the sequence $(z_n)_{n\in\mathbb{N}}$ is given by

$$z_n = \frac{b}{a-d}\left(\left(\frac{a}{d}\right)^{n-1}-1\right)+\left(\frac{a}{d}\right)^{n-1}\frac{x_1y_1}{x_0y_0}, \quad n\in\mathbb{N}.$$

REMARK 1. By using above formulas for the general solutions to systems (5), (27), (34), (41), (47) and (54), one can describe their well-defined solutions. This simple task we leave to the reader as an exercise.

3. Conclusion

Here we present six interesting classes of nonlinear two-dimensional systems of difference equations. We conduct detailed analysis of their solvability and present some methods for solving them as well as some formulas for their general solutions. Some modifications of the methods and ideas presented here could be used in dealing with the problem of solvability of some other classes of difference equations and systems of difference equations. They could serve also as a motivation for further investigations in the direction.

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Stevo Stević
Mathematical Institute of the Serbian Academy of Sciences
Knez Mihailova 36/III, 11000 Beograd, Serbia
and
Department of Medical Research, China Medical University Hospital
China Medical University
Taichung 40402, Taiwan, Republic of China
e-mail: sscite1@gmail.com