Journal of
Mathematical
Inequalities
Volume 18, Number 2 (2024), 591–604 doi:10.7153/jmi-2024-18-32

LOWER BOUNDS FOR THE SMALLEST SINGULAR VALUE VIA PERMUTATION MATRICES

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(*Communicated by T. Burić*)

Abstract. We in this paper improve the well-known C. R. Johnson's lower bound for the smallest singular value via permutation matrices. A direct algorithm is also given to compute the new lower bound.

1. Introduction

Given a complex matrix $A \in \mathbb{C}^{n\times n}$, the singular values of A are the eigenvalues of $(AA^*)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A [2]. Denoted by $\sigma(A)$ the set which consists of all singular values of A, that is, $\sigma(A) = \{\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A)\}$ with

$$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A) \geqslant 0.$$

Bounding the smallest singular value of a matrix is an important topic in matrix analysis and matrix computation [2, 3]. One of the well-known lower bounds is presented by C. R. Johnson in 1989 [4]. It is stated that for a given matrix $A \in \mathbb{C}^{n\times n}$,

$$\sigma_n(A) \geqslant Bnd_J(A) := \min_{i\in N} \left\{ |a_{ii}| - \frac{1}{2}\left(r_i(A) + c_i(A)\right) \right\}, \tag{1.1}$$

where $r_i(A) = \sum\limits_{k\in N,\ k\neq i} |a_{ik}|$, $c_i(A) = r_i(A^\top) = \sum\limits_{k\in N,\ k\neq i} |a_{ki}|$ and $N = \{1,2,\dots,n\}$. It is pointed out here that the C. R. Johnson's bound $Bnd_J(A)$ is obtained by using the Gersgorin circle theorem [12] in a certain way. The other two lower bounds we would like to introduce are provided by C. R. Johnson and T. Szulc [5] in 1998. The first is

$$\sigma(A) \geqslant Bnd_{JS_1}(A), \tag{1.2}$$

where

$$Bnd_{JS_1}(A) := \min_i \frac{1}{2}\left(\left(4|a_{ii}|^2 + (r_i(A) - c_i(A))^2\right)^{\frac{1}{2}} - (r_i(A) + c_i(A)) \right);$$

Mathematics subject classification (2020): 15A18, 15A42, 15A60.
Keywords and phrases: Lower bound, smallest singular value, permutation matrices.
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The second is

$$\sigma(A) \geqslant Bnd_{JS_2}(A), \tag{1.3}$$

where

$$Bnd_{JS_2}(A) := \min_{\substack{i,j \\ i\neq j}} \frac{1}{2}\left(\mathbf{Re}a_{ii} + \mathbf{Re}a_{jj} - \left((\mathbf{Re}a_{ii} - \mathbf{Re}a_{jj})^2 + r_i(A+A^*)r_j(A+A^*)\right)^{\frac{1}{2}}\right)$$

and $\mathbf{Re}a_{ii}$ is the real part of a_{ii}.

In addition, there were many lower bounds for the smallest singular values in recent years, for example, L. M. Zou [14] in 2012 gave the following lower bound via the determinant and the Frobenius norm of a given matrix A, that is,

$$\sigma_n(A) \geqslant Bnd_Z(A) := |\det A| \left(\frac{n-1}{\|A\|_F^2 - l_1^2}\right)^{\frac{n-1}{2}}, \tag{1.4}$$

where $l_1 = |\det A| \left(\frac{n-1}{\|A\|_F^2}\right)^{\frac{n-1}{2}}$. Based on the bound (1.4), M. H. Lin and M. Y. Xie [7] in 2021 obtained another one lower bound for the smallest singular values, that is,

$$\sigma_n(A) \geqslant Bnd_{LX_1}(A) := \lim_{k\to\infty} l_k, \tag{1.5}$$

where $l_k = |\det A| \left(\frac{n-1}{\|A\|_F^2 - l_{k-1}^2}\right)^{\frac{n-1}{2}}$, $k = 2,3,\ldots$. It was also showed in [7] that

$$\sigma_n(A) \geqslant Bnd_{LX_2}(A) := |\det A| \left(\frac{n-1}{\|A\|_F^2 - b^2}\right)^{\frac{n-1}{2}} \tag{1.6}$$

holds for any nontrivial lower bound b of the minimum singular value of matrix A. Also based on the bound (1.4), X. Shun [10] in 2022 gave another two lower bounds for the smallest singular values. One is

$$\sigma_n(A) \geqslant Bnd_{S_1}(A) := \left(l_2^2 + |\det(l_2^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - nl_2^2}\right)^{n-1}\right)^{\frac{1}{2}}, \tag{1.7}$$

where $l_2 = Bnd_Z(A)$, I_n denotes the identity matrix and A^H is the conjugate transpose of A. The other one is

$$\sigma_n(A) \geqslant Bnd_{S_2}(A) := \lim_{k\to\infty} b_{k+1}, \tag{1.8}$$

where

$$b_{k+1} = \left(l_2^2 + |\det(l_2^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)l_2^2 - b_k^2}\right)^{n-1}\right)^{\frac{1}{2}}, \quad k = 1,2,\cdots,$$

and

$$b_1 = \left(l_2^2 + |\det(l_2^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)l_2^2}\right)^{n-1}\right)^{\frac{1}{2}}.$$

We also refer to [1, 6, 8, 9, 11] and references therein for other lower bounds for the smallest singular value.

In this paper, based on permutation matrices some new lower bounds are given for the smallest singular value. These new bounds involving permutation matrices improve bounds $Bnd_J(A)$, $Bnd_{JS_1}(A)$, $Bnd_{JS_2}(A)$, $Bnd_Z(A)$, $Bnd_{LX_1}(A)$, $Bnd_{LX_2}(A)$, $Bnd_{S_1}(A)$ and $Bnd_{S_2}(A)$. We also give an algorithm to determine the exact value. Some numerical examples are also given to show the theoretical results.

2. Main results

To begin with, we introduce the permutation matrix and some facts.

DEFINITION 2.1. [2] A square matrix $P \in \mathbb{R}^{n\times n}$ is a permutation matrix if exactly one entry in each row and column is equal to 1 and all other entries are 0. Denote $\mathbb{P}^{n\times n}$ the set which consists of all $n\times n$ permutation matrices.

As is well known that the eigenvalues of $A \in \mathbb{C}^{n\times n}$ are the same as those of $P^\top AP$ for any permutation matrix $P \in \mathbb{P}^{n\times n}$. But this is not true for A and AP in general. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \tag{2.9}$$

all eigenvalues of A are $-1,\ 3$. For the permutation matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.10}$$

the matrix AP is equal to PA, i.e,

$$PA = AP = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

and all its eigenvalues are $1,\ 3$. However, for singular values case, it brings a different result, that is,

$$\sigma(A) = \sigma(AP) = \sigma(QA) = \sigma(QAP) \tag{2.11}$$

holds for any permutation matrix $P,\ Q \in \mathbb{P}^{n\times n}$. In fact, since

$$AP(AP)^* = APP^*A^* = APP^\top A^* = AA^*$$

and

$$QA(QA)^* = QAA^*Q^* = QAA^*Q^\top,$$

hence the eigenvalues of $AP(AP)^*$ are the same as those of AA^*. This is also true for $QA(QA)^*$ and AA^*. So $\sigma(A) = \sigma(AP) = \sigma(QA)$. Furthermore

$$\sigma(QAP) = \sigma(Q(AP)) = \sigma((QA)P) = \sigma(AP) = \sigma(QA) = \sigma(A).$$

This shows that left multiplication or right multiplication of a matrix A by a permutation matrix doesn't change the singular values of A, also see Remark (a) in [4]. However, left multiplication or right multiplication of a matrix A by a permutation matrix permutes the rows or the columns of matrix A, respectively. This may make some lower bound for the smallest singular value changed. Consider again the matrix A in (2.9), the C. R. Johnson's bound (1.1) for the smallest singular value $\sigma_n(A)$ is

$$\sigma_n(A) = 1 \geqslant Bnd_J(A) = -1.$$

It is trivial because the singular value is always nonnegative. However, by PA and AP we have

$$\sigma_n(A) = \sigma_n(AP) = \sigma_n(PA) \geqslant Bnd_J(AP) = Bnd_J(PA) = 1,$$

where P is defined as (2.10). It is sharp! In fact, for a given matrix $A \in \mathbb{C}^{n\times n}$, we can improve the C. R. Johnson's bound via permutation matrices.

THEOREM 2.2. *Let $A \in \mathbb{C}^{n\times n}$. Then for any permutation matrix $P \in \mathbb{P}^{n\times n}$,*

$$\sigma_n(A) \geqslant Bnd_J(A,P) := \min_{i\in N}\left\{ |(AP)_{ii}| - \frac{1}{2}(r_i(AP) + c_i(AP)) \right\}, \tag{2.12}$$

and

$$\sigma_n(A) \geqslant Bnd_J(P,A) := \min_{i\in N}\left\{ |(PA)_{ii}| - \frac{1}{2}(r_i(PA) + c_i(PA)) \right\}, \tag{2.13}$$

where $(AP)_{ii}$ and $(PA)_{ii}$ are the (i,i)-entry of AP and PA respectively. Furthermore,

$$\sigma_n(A) \geqslant \max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) \geqslant Bnd_J(A), \tag{2.14}$$

and

$$\sigma_n(A) \geqslant \max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A) \geqslant Bnd_J(A), \tag{2.15}$$

Proof. It is easy to see that (2.12) and (2.13) hold from (2.11) and the C. R. Johnson's bound (1.1). Furthermore, taking $P = I$ we have

$$Bnd_J(A) = Bnd_J(A,I) \leqslant \max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A),$$

and

$$Bnd_J(A) = Bnd_J(P,A) \leqslant \max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A),$$

where I is the identity matrix. The conclusion follows. $\square$

Remark here that Theorem 2.2 tells us that by the permutation matrix, the well-known C. R. Johnson's bound $Bnd_J(A)$ can be improved further. However, it is not true for the L. M. Zou's bound $Bnd_Z(A)$, the M. H. Lin and M. Y. Xie's bound $Bnd_{LX_1}(A)$, and the X. Shun's bound $Bnd_{S_1}(A)$, i.e., these three bounds cannot be improved by permutation matrices because $|\det(A)| = |\det(AP)| = |\det(PA)|$ and $||AP||_F = ||PA||_F = ||A||_F$ hold for any permutation matrix P.

As is well known that $\mathbb{P}^{n\times n}$ consists of $n!$ permutation matrices, that is, the cardinality of $\mathbb{P}^{n\times n}$ is $n!$. So we can determine the exact value of

$$\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) \tag{2.16}$$

and

$$\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A) \tag{2.17}$$

in general by computing $n!$ $Bnd_J(A,P)$ and $Bnd_J(P,A)$, respectively. Next, we give an approach to determining (2.16) and (2.17) with less computation. Before that we introduce some notations. For a given matrix $A=[a_{ij}]\in\mathbb{C}^{n\times n}$, let

$$D(A)=[d_{ij}(A)]\in\mathbb{R}^{n\times n}, \tag{2.18}$$

where

$$d_{ij}(A)=|a_{ij}|-\frac{1}{2}\left(\sum_{k\in N,k\neq j}|a_{ik}|+\sum_{k\in N,k\neq i}|a_{kj}|\right).$$

Obviously,

$$Bnd_J(A)=\min_{i\in N} d_{ii}(A).$$

Let $P[i,j]$ be the permutation matrix with

$$(P[i,j])_{kq}=\begin{cases}1, & k=q\in N\setminus\{i,j\},\\ 1, & k=i, q=j,\\ 1, & k=j, q=i,\\ 0, & \text{others.}\end{cases}$$

Obviously, $d_{ii}(AP[i,j])=d_{ij}(A)$ and $d_{ii}(P[i,j]A)=d_{ji}(A)$.

LEMMA 2.3. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ and $D(A)$ be defined as (2.18). If*

$$\min\{d_{ij}(A),\ d_{ji}(A)\}\geqslant\min\{d_{ii}(A),\ d_{jj}(A)\},$$

then

$$\min\{d_{ii}(AP[i,j]),\ d_{jj}(AP[i,j])\}\geqslant\min\{d_{ii}(A),d_{jj}(A)\} \tag{2.19}$$

and

$$\min\{d_{ii}(P[i,j]A),\ d_{jj}(P[i,j]A)\}\geqslant\min\{d_{ii}(A),d_{jj}(A)\}. \tag{2.20}$$

Proof. Note that

$$d_{ii}(AP[i,j])=d_{ij}(AP[i,j]P[i,j])=d_{ij}(A)$$

and

$$d_{jj}(AP[i,j])=d_{ji}(AP[i,j]P[i,j])=d_{ji}(A).$$

This implies Inequality (2.19). Similarly, we can obtain Inequality (2.20) because

$$d_{ii}(P[i,j]A) = d_{ji}(P[i,j]P[i,j]A) = d_{ji}(A)$$

and

$$d_{jj}(P[i,j]A) = d_{ij}(P[i,j]P[i,j]A) = d_{ij}(A).$$

The conclusion follows. $\square$

THEOREM 2.4. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $D(A)$ be defined as (2.18), and*

$$d_{i_0 i_0}(A) = \min_{i\in N}\{d_{ii}(A)\}.$$

If there is an index $j_0 \in N$ and $j_0 \neq i_0$ such that

$$\min\{d_{i_0 j_0}(A),\ d_{j_0 i_0}(A)\} \geqslant d_{i_0 i_0}(A),$$

then

$$\min_{i\in N}\{d_{ii}(AP[i_0,j_0])\} \geqslant \min_{i\in N}\{d_{ii}(A)\} \tag{2.21}$$

and

$$\min_{i\in N}\{d_{ii}(P[i_0,j_0]A)\} \geqslant \min_{i\in N}\{d_{ii}(A)\}. \tag{2.22}$$

Proof. We only prove Inequality (2.21) (Inequality (2.22) can be proved similarly). In fact, from Lemma 2.3 we have

$$\min\{d_{i_0 j_0}(A),\ d_{j_0 i_0}(A)\} = \min\{d_{i_0 i_0}(AP[i_0,j_0]), d_{j_0 j_0}AP[i_0,j_0])\} \geqslant d_{i_0 i_0}(A).$$

Furthermore, for any $i \in N$ with $i \neq i_0$ and $i \neq j_0$,

$$d_{ii}(AP[i_0,j_0]) = d_{ii}(A).$$

This implies

$$\begin{aligned}
\min_{i\in N}\{d_{ii}(AP[i_0,j_0])\} &= \min\left\{ d_{i_0 i_0}(AP[i_0,j_0]), d_{j_0 j_0}AP[i_0,j_0]),\ \min_{\substack{i\in N,\\ i\neq i_0,j_0}}\{d_{ii}(AP[i_0,j_0])\}\right\}\\
&\geqslant \min\left\{ d_{i_0 i_0}(A),\ \min_{\substack{i\in N,\\ i\neq i_0,j_0}}\{d_{ii}(A)\}\right\}\\
&= d_{i_0 i_0}(A)\\
&= \min_{i\in N}\{d_{ii}(A)\},
\end{aligned}$$

i.e., Inequality (2.21) holds. The conclusion follows. $\square$

THEOREM 2.5. *Let* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $D(A)$ *be defined as* (2.18)*, and*

$$d_{i_0i_0}(A) = \min_{i\in N}\{d_{ii}(A)\}.$$

If for any $j \in N$ *and* $j \neq i_0$ *such that*

$$d_{i_0i_0}(A) \geqslant \max\{d_{i_0j}(A),\ d_{ji_0}(A)\},$$

then

$$\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) = \max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A) = d_{i_0i_0}(A). \tag{2.23}$$

Proof. We first prove $\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) = d_{i_0i_0}(A)$. From (2.12) and (2.14), we have

$$\begin{aligned}\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) &\geqslant Bnd_J(A)\\ &= \min_{i\in N}\left\{|a_{ii}| - \frac{1}{2}\left(r_i(A)+c_i(A)\right)\right\}\\ &= \min_{i\in N}\{d_{ii}(A)\}\\ &= d_{i_0i_0}(A).\end{aligned}$$

Hence, we next only show that

$$\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) > d_{i_0i_0}(A) \tag{2.24}$$

cannot happen.

Suppose on the contrary that Inequality (2.24) holds. Without loss of generality, suppose $P' \in \mathbb{P}^{n\times n}$ such that

$$Bnd_J(A,P') = \max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P),$$

then

$$Bnd_J(A,P') := \min_{i\in N}\left\{|(AP')_{ii}| - \frac{1}{2}\left(r_i(AP')+c_i(AP')\right)\right\} = \min_{i\in N}\{d_{ii}(AP')\} > d_{i_0i_0}(A).$$

This implies that for any $j \in N$,

$$d_{jj}(AP') > d_{i_0i_0}(A).$$

In particular,

$$d_{i_0i_0}(AP') > d_{i_0i_0}(A).$$

This contradicts $d_{i_0i_0}(A) \geqslant d_{i_0j}(A)$ for any $j \neq i_0$ because right multiplication of a matrix A by a permutation matrix P' doesn't change the row index i_0.

Similarly, $\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A) = d_{i_0i_0}(A)$ can be proved easily. The conclusion follows. $\square$

Based on the result above, a direct algorithm is established to determine the exact value of $\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P)$, see Algorithm 1. Remark here that like Algorithm 1, we can give the algorithm for computing $\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A)$ by a similar way.

Algorithm 1 (Algorithm for computing $\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P)$)

Input: A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$.
Output: The exact value of $\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P)$.
Step 1: Compute $d_{ii}(A)$, $i \in N$;
Step 2: Determine $\min_{i\in N}\{d_{ii}(A)\}$. Let $d_{i_0i_0}(A) = \min_{i\in N}\{d_{ii}(A)\}$;
Step 3: Compute $d_{i_0j}(A)$ and $d_{ji_0}(A)$, $j \in N$ and $j \neq i_0$. If there is an index $j_0 \in N$ and $j_0 \neq i_0$ such that
$\min\{d_{i_0j_0}(A),\ d_{j_0i_0}(A)\} = \max_{j\in\Omega(i_0)} \min\{d_{i_0j}(A),\ d_{ji_0}(A)\} > d_{i_0i_0}(A)$,
where $\Omega(i_0) := \left\{ j\in N \mid \min\{d_{i_0j}(A),\ d_{ji_0}(A)\} \geqslant d_{i_0i_0}(A)\right\}$, then go to Step 4, otherwise go to Step 5;
Step 4: Determine $\min_{i\in N}\{d_{ii}(AP[i_0,j_0])\}$. If $d_{i_0i_0}(A) = \min_{i\in N}\{d_{ii}(AP[i_0,j_0])\}$, then go to Step 5; otherwise, $A = AP[i_0,j_0]$, and go to Step 2;

Step 5: Output $d_{i_0i_0}(A)$ as the exact value.

Next two examples are given to show the theoretical result provided above.

EXAMPLE 2.6. Consider the matrix

$$A = \begin{bmatrix} 3 & 0.1 & 6 & -0.1 \\ 6 & 0.1 & 3 & 0.1 \\ 0.1 & 1 & 0.1 & 4 \\ 0.1 & 5 & -0.1 & 0.5 \end{bmatrix}.$$

The smallest singular value $\sigma_4(A)$ of matrix A is 2.9967. By the C. R. Johnson's lower bound (1.1) we have

$$\sigma_4(A) \geqslant Bnd_J(A) = -7.5.$$

By the C. R. Johnson and T. Szulc's bound (1.2) we have

$$\sigma_4(A) \geqslant Bnd_{JS_1}(A) = -6.0967.$$

By the C. R. Johnson and T. Szulc's bound (1.3) we have

$$\sigma_4(A) \geqslant Bnd_{JS_2}(A) = -7.1938.$$

It is invalid because each singular value for matrices is always nonnegative. However, by Algorithm 1 we have

$$\sigma_4(A) \geqslant \max_{P \in \mathbb{P}^{n\times n}} Bnd_J(A,P) = 2.8.$$

By the L. M. Zou's bound (1.4) we have

$$\sigma_4(A) \geqslant Bnd_Z(A) = 1.8658.$$

By the M. H. Lin and M. Y. Xie's bound (1.5) we have

$$\sigma_4(A) \geqslant Bnd_{LX_1}(A) = 1.8717.$$

By the X. Shun's bound (1.7) we have

$$\sigma_4(A) \geqslant Bnd_{S_1}(A) = 2.2778.$$

By the X. Shun's bound (1.8) we have

$$\sigma_4(A) \geqslant Bnd_{S_2}(A) = 2.2952.$$

It is shown by this example that Algorithm 1 could provide a positive lower bound for the smallest singular value in some cases when the C. R. Johnson's lower bound (1.1), C. R. Johnson and T. Szulc's bounds (1.2) and (1.3) don't work. This example also shows that in some cases, although the L. M. Zou's bound (1.4), the M. H. Lin and M. Y. Xie's bound (1.5), and the X. Shun's bounds (1.7) and (1.8) are all positive, they are smaller than the lower bound obtained by Algorithm 1.

EXAMPLE 2.7. Consider the matrix

$$A = \begin{bmatrix} 3 & 6 & 0.1 & 0.1 \\ 5 & 2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 1 & 4 \\ 0.1 & 0.1 & 5 & 0.5 \end{bmatrix}.$$

The smallest singular value $\sigma_4(A)$ of matrix A is 2.9730. By the L. M. Zou's bound (1.4) we have

$$\sigma_4(A) \geqslant Bnd_Z(A) = 2.0338.$$

By the M. H. Lin and M. Y. Xie's bound (1.5) we have

$$\sigma_4(A) \geqslant Bnd_{LX_1}(A) = 2.0457.$$

By Algorithm 1 we have

$$\sigma_4(A) \geqslant \max_{P \in \mathbb{P}^{4\times 4}} Bnd_J(A,P) = 2.3. \tag{2.25}$$

Furthermore by (2.25) and the M. H. Lin and M. Y. Xie's bound (1.6) we have

$$\sigma_4(A) = |\det A| \left(\frac{4-1}{\|A\|_F^2 - \left(\max\limits_{P \in \mathbb{P}^{4\times 4}} Bnd_J(A,P) \right)^2} \right)^{\frac{4-1}{2}} = 2.0763.$$

It is shown by this example that in some cases, the lower bound obtained by Algorithm 1 is larger than the bounds of the L. M. Zou's bound (1.4), and the M. H. Lin and M. Y. Xie's bound (1.5). This example also shows that in some cases, the M. H. Lin and M. Y. Xie's bound (1.6) does not increase for the lower bound obtained by Algorithm 1.

We next will show that if $d_{i_0i_0}(A)=\min\limits_{i\in N}\{d_{ii}(A)\}\geqslant 0$, Theorem 2.5 can be improved further. Recall

$$d_{ij}(A)=|a_{ij}|-\frac{1}{2}\left(\sum_{k\in N,k\neq j}|a_{ik}|+\sum_{k\in N,k\neq i}|a_{kj}|\right).$$

Then $d_{ii}(A)$, $d_{jj}(A)$, $d_{ij}(A)$ and $d_{ji}(A)$ can be respectively rewritten as

$$d_{ii}(A)=|a_{ii}|-\frac{1}{2}\left(r_i^j(A)+|a_{ij}|+c_i^j(A)+|a_{ji}|\right),$$

$$d_{jj}(A)=|a_{jj}|-\frac{1}{2}\left(r_j^i(A)+|a_{ji}|+c_j^i(A)+|a_{ij}|\right),$$

$$d_{ij}(A)=|a_{ij}|-\frac{1}{2}\left(r_i^j(A)+|a_{ii}|+c_j^i(A)+|a_{jj}|\right),$$

and

$$d_{ji}(A)=|a_{ji}|-\frac{1}{2}\left(r_j^i(A)+|a_{jj}|+c_i^j(A)+|a_{ii}|\right),$$

where $r_i^j(A):=r_i(A)-|a_{ij}|=\sum\limits_{\substack{k\in N,\\ k\neq i,j}}|a_{ik}|$ and $c_i^j(A):=c_i(A)-|a_{ji}|=\sum\limits_{\substack{k\in N,\\ k\neq i,j}}|a_{ki}|$.

LEMMA 2.8. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ and $D(A)$ be defined as (2.18). If*

$$\min\{d_{ii}(A),d_{jj}(A)\}\geqslant 0,$$

then

$$\min\{d_{ij}(A),d_{ji}(A)\}\leqslant 0.$$

Proof. Without loss of generality, suppose $d_{ii}(A)=\min\{d_{ii}(A),d_{jj}(A)\}$, then $d_{jj}(A)\geqslant d_{ii}(A)\geqslant 0$, i.e.,

$$2|a_{ii}|-(|a_{ij}|+|a_{ji}|)-\left(r_i^j(A)+c_i^j(A)\right)\geqslant 0$$

and

$$2|a_{jj}|-(|a_{ji}|+|a_{ij}|)-\left(r_j^i(A)+c_j^i(A)\right)\geqslant 0.$$

Thus

$$2(|a_{ii}|+|a_{jj}|)-2(|a_{ij}|+|a_{ji}|)-\left(r_i^j(A)+c_i^j(A)+r_j^i(A)+c_j^i(A)\right)\geqslant 0,$$

equivalently,

$$2(|a_{ij}|+|a_{ji}|) \leqslant 2(|a_{ii}|+|a_{jj}|) - \left(r_i^j(A)+c_i^j(A)+r_j^i(A)+c_j^i(A)\right). \tag{2.26}$$

Suppose on the contrary that $\min\{d_{ij}(A), d_{ji}(A)\} > 0$. Then $2(d_{ij}(A)+d_{ji}(A)) > 0$, i.e,

$$2(|a_{ij}|+|a_{ji}|) - 2(|a_{ii}|+|a_{jj}|) - \left(r_i^j(A)+c_j^i(A)+r_j^i(A)+c_i^j(A)\right) > 0,$$

equivalently,

$$2(|a_{ij}|+|a_{ji}|) > 2(|a_{ii}|+|a_{jj}|) + \left(r_i^j(A)+c_j^i(A)+r_j^i(A)+c_i^j(A)\right),$$

which contradicts (2.26). Therefore $\min\{d_{ij}(A), d_{ji}(A)\} \leqslant 0$. The proof is completed. □

Lemma 2.8 also tells us that if $\min\{d_{ii}(A), d_{jj}(A)\} \geqslant 0$, then

$$\min\{d_{ii}(A), d_{jj}(A)\} \geqslant \min\{d_{ij}(A), d_{ji}(A)\}.$$

It brings the following results for $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P)$ and $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A)$.

THEOREM 2.9. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$. If there is a permutation matrix $P'\in\mathbb{P}^{n\times n}$ such that*

$$\min_{i\in N}\{d_{ii}(AP')\} \geqslant 0,$$

then

$$\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P) = \min_{i\in N}\{d_{ii}(AP')\}. \tag{2.27}$$

If there is a permutation matrix P' such that

$$\min_{i\in N}\{d_{ii}(P'A)\} \geqslant 0,$$

then

$$\max_{P\in\mathbb{P}^{n\times n}} Bnd_J(P,A) = \min_{i\in N}\{d_{ii}(P'A)\}. \tag{2.28}$$

Proof. We only prove Equality (2.27) (Equality (2.28) can be proved similarly). Without loss of generality, suppose

$$d_{i_0i_0}(AP') = \min_{i\in N} d_{ii}(AP'),$$

then for any $j\in N$ and $j\neq i_0$, $d_{jj}(AP') \geqslant d_{i_0i_0}(AP') \geqslant 0$. From Lemma 2.8, it follows that for any $j\in N$ and $j\neq i_0$,

$$\min\{d_{i_0j}(AP'), d_{ji_0}(AP')\} \leqslant 0 \leqslant d_{i_0i_0}(AP').$$

Furthermore, from Theorem 2.5 Equality (2.27) holds. The conclusion follows. □

EXAMPLE 2.10. Also consider the matrix A in Example 2.6. Then the matrix

$$D(A) = \begin{bmatrix} -3.2 & -7.5 & 2.8 & -6.75 \\ 2.8 & -7.5 & -3.2 & -6.75 \\ -7 & -3.7 & -7 & 3.05 \\ -7.25 & 4.05 & -7.25 & -4.2 \end{bmatrix}.$$

Take the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$D(AP) = \begin{bmatrix} 2.8 & -3.2 & -6.75 & -7.5 \\ -3.2 & 2.8 & -6.75 & -7.5 \\ -7 & -7 & 3.05 & -3.7 \\ -7.25 & -7.25 & -4.2 & 4.05 \end{bmatrix}.$$

By Theorem 2.9, we have

$$\sigma_4(A) \geqslant \max_{P \in \mathbb{P}^{n\times n}} Bnd_J(A,P) = 2.8.$$

In particular, if the permutation matrix P' is the identity matrix. i.e., $P' = I$, then Theorem 2.9 reduces the following result.

COROLLARY 2.11. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $D(A)$ be defined as (2.18). If*

$$\min_{i \in N}\{d_{ii}(A)\} \geqslant 0,$$

then

$$\max_{P \in \mathbb{P}^{n\times n}} Bnd_J(A,P) = \max_{P \in \mathbb{P}^{n\times n}} Bnd_J(P,A) = \min_{i \in N}\{d_{ii}(A)\}. \tag{2.29}$$

EXAMPLE 2.12. Consider the matrix

$$A = \begin{bmatrix} 10 & 9 & 1 \\ 0 & 6 & 0 \\ 9 & 0 & 6 \end{bmatrix}.$$

Then the smallest singular value $\sigma_3(A)$ of matrix A is 2.2662, and

$$D(A) = \begin{bmatrix} 0.5 & 0.5 & -11.5 \\ -12.5 & 1.5 & -6.5 \\ 1 & -15 & 1 \end{bmatrix}.$$

By Corollary 2.11 we have

$$\sigma_3(A) \geqslant \max_{P \in \mathbb{P}^{n\times n}} Bnd_J(A,P) = \max_{P \in \mathbb{P}^{n\times n}} Bnd_J(P,A) = \min_{i \in N}\{d_{ii}(A)\} = 0.5,$$

which is the C. R. Johnson's lower bound exactly.

3. Conclusions

We in this paper improve the C. R. Johnson's bound $Bnd_J(A)$ for the smallest singular value by permutation matrices. The new lower bound $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P)$ could be positive when $Bnd_J(A)$ is nonnegative in some cases. A direct algorithm is given for determining this new lower bound.

As shown in Example 2.6, the C. R. Johnson and T. Szulc's bounds $Bnd_{JS_1}(A)$ and $Bnd_{JS_2}(A)$ are all negative for this case. Hence, they are all invalid in some cases. In fact, like the bound $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P)$ in Theorem 2.2, the C. R. Johnson and T. Szulc's bounds can be improved by permutation matrices similarly, i.e., for the smallest singular value $\sigma_n(A)$ of matrix A,

$$\sigma_n(A) \geqslant \max_{P\in\mathbb{P}^{n\times n}} Bnd_{JS_1}(A,P) \geqslant Bnd_{JS_1}(A),$$

and

$$\sigma_n(A) \geqslant \max_{P\in\mathbb{P}^{n\times n}} Bnd_{JS_2}(A,P) \geqslant Bnd_{JS_2}(A),$$

where

$$Bnd_{JS_1}(A,P) := \min_i \frac{1}{2}\left(\left(4|(AP)_{ii}|^2 + (r_i(AP)-c_i(AP))^2\right)^{\frac{1}{2}} - (r_i(AP)+c_i(AP))\right)$$

and

$$\begin{aligned} Bnd_{JS_2}(A,P) := \min_{\substack{i,j\\ i\neq j}} \frac{1}{2}\Big(&\mathbf{Re}(AP)_{ii} + \mathbf{Re}(AP)_{jj} \\ &- \left((\mathbf{Re}(AP)_{ii} - \mathbf{Re}(AP)_{jj})^2 + r_i(AP+(AP)^*)r_j(AP+(AP)^*)\right)^{\frac{1}{2}}\Big). \end{aligned}$$

We conjecture here that by the technique for computing $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_J(A,P))$, the algorithms like Algorithm 1 can be given for determining $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_{JS_1}(A,P)$ and $\max\limits_{P\in\mathbb{P}^{n\times n}} Bnd_{JS_2}(A,P)$, respectively.

Acknowledgement. The authors would like to thank the editor and the anonymous referee for the valuable suggestions and comments which improve the overall presentation of this paper. This work was partly supported by National Natural Science Foundation of China (No. 12061087), Yunnan Fundamental Research Projects (No. 202401AT070479), and Yunnan Provincial Xingdian Talent Support Program.

REFERENCES

[1] Y. P. HONG, C.-T. PAN, *A lower bound for the smallest singular value*, Linear Algebra Appl. 1992; 172: 27–32.

[2] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, Cambridge, 1985, Cambridge University Press.

[3] R. A. HORN, C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge, 1991, Cambridge University Press.

[4] C. R. JOHNSON, *A Gersgorin-type lower bound for the smallest singular value*, Linear Algebra Appl. 1989; 112: 1–7.

[5] C. R. JOHNSON, T. SZULC, *Further lower bounds for the smallest singular value*, Linear Algebra Appl. 1998; 272: 169–179.

[6] A. KAUR, S. H. LUI, *New lower bounds on the minimum singular value of a matrix*, Linear Algebra Appl. 2023; 666: 62–95.

[7] M. H. LIN, M. Y. XIE, *On some lower bounds for smallest singular value of matrices*, Applied Mathematics Letters. 2021; 121: 107411.

[8] L. Q. QI, *Some simple estimates for singular values of a matrix*, Linear Algebra Appl. 1984; 56: 105–119.

[9] O. ROJO, *Further bounds for the smallest singular value and the spectral condition number*, Computers and Mathematics with Applications. 1999; 38: 215–228.

[10] X. SHUN, *Two new lower bounds for the smallest singular value*, J. Math. Inequal. 2022; 16: 63–68.

[11] J. M. VARAH, *A lower bound for the smallest singular value of a matrix*, Linear Algebra Appl. 1975; 11: 3–5.

[12] R. S. VARGA, *Geršgorin and his Circles*, Berlin (Bln), 2004, Springer-Verlag.

[13] L. M. ZOU, Y. JIANG, *Estimation of the eigenvalues and the smallest singular value of matrices*, Linear Algebra Appl. 2010; 433: 1203–1211.

[14] L. M. ZOU, *A lower bound for the smallest singular value*, J. Math. Inequal. 2012; 6: 625–629.

(Received September 15, 2023)

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