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ESTIMATES FOR THE NORM OF THE SPHERICAL MAXIMAL OPERATOR ON FINITE GRAPHS

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Abstract. For a simple, finite, and connected graph G, the spherical maximal operator is defined as

$$\mathcal{M}_G^\circ h(t) = \sup_{r \geqslant 0} \frac{1}{|S(t,r)|} \sum_{u \in S(t,r)} |h(u)|,$$

where $S(t,r) = \{w \in V \mid d_G(w,t) = r\}$ is the sphere with center at t and having radius r. In this paper, we consider the spherical maximal operator $\mathcal{M}_G^\circ$ on ℓ^p spaces and calculate the $\|\mathcal{M}_G^\circ\|_{\ell^p}$ for $0 < p \leqslant 1$ and estimate the $\|\mathcal{M}_G^\circ\|_{\ell^p}$ for $1 < p < \infty$, when G is K_m. Furthermore, We establish the maximum and minimum bounds for the spherical maximum operator on finite graphs and indicate the graphs that achieve these bounds.

1. Introduction

Let $G = (V,E)$ be a finite, undirected and simple graph having m vertices. A geodesic metric space is defined on the vertices of the graph which is the distance between two vertices $l, z \in V$ is denote as $d_G(l,z)$ and defined as the count of edges in the shortest path between l and z. For a function $h : V \to \mathbb{R}$, the spherical maximal operator [8] $\mathcal{M}_G^\circ : \ell^p \to \ell^p$ such as

$$\mathcal{M}_G^\circ h(t) = \sup_{r \geqslant 0} \frac{1}{|S(t,r)|} \sum_{u \in S(t,r)} |h(u)|, \tag{1}$$

where $S(t,r) = \{w \in V \mid d_G(w,t) = r\}$ is the sphere with center at t and having radius r. $S(t,r)$ is defined only for those r for which $|S(t,r)| \neq 0$. Note that $S(t,1) = \{w \in V \mid d_G(w,t) = 1\}$ is the neighborhood of t and $|S(t,1)| = d_G(t)$ is the degree of t in graph G. As distance takes only natural numbers so we can change supremum in (1) with maximum. Clearly for the complete graph K_m we have $S(t,0) = \{t\}$ and $S(t,1) = V \backslash \{t\}$. Spherical maximal operator for complete graph K_m is

$$\mathcal{M}_{K_m}^\circ h(t) = \max\left\{ |h(t)|, \frac{1}{m-1} \sum_{q \in V \backslash \{t\}} |h(q)| \right\}. \tag{2}$$

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Similarly for a star graph S_m, we have

$$\mathcal{M}^{\circ}_{S_m}h(t)=\begin{cases}\max\left\{|h(t)|,\frac{1}{m-1}\sum\limits_{q\in V\setminus\{t\}}|h(q)|\right\}, & \text{for } t=c,\\ \max\left\{|h(t)|,|h(c)|,\frac{1}{m-2}\sum\limits_{w\in V\setminus\{t,c\}}|h(w)|\right\}, & \text{for } t\neq c,\end{cases} \tag{3}$$

where $c\in V$ be a central vertex of the star graph. For $0<p<\infty$, the norm of spherical maximal operator for a graph G is:

$$\|\mathcal{M}^{\circ}_{G}\|_p := \sup_{h\neq 0}\frac{\|\mathcal{M}^{\circ}_{G}h\|_p}{\|h\|_p},$$

where $\|h\|_p=\left(\sum\limits_{t\in V}|h(t)|^p\right)^{\frac{1}{p}}$.

LEMMA 1.1. ([7]) *Let G_m be the graph, and $\Lambda:\ell^p\to\ell^p$ is a sublinear operator with $0<p\leqslant 1$, then*

$$\|\Lambda\|_p=\max_{t\in V}\|\Lambda\delta_t\|_p.$$

EXAMPLE 1.1. Let $G=K_4$, $p=2$ and $f=\delta_i$ then $\|\mathcal{M}^{\circ}_{K_4}\|_2=1.1547$.

It is easy for the reader to check that

$$\mathcal{M}^{\circ}_{K_4}\delta_t(q)=\begin{cases}1, & \text{for } t=q,\\ \frac{1}{3}, & \text{for } t\neq q.\end{cases}$$

So, $\|\mathcal{M}^{\circ}_{K_4}\|_2=\|\mathcal{M}^{\circ}_{K_4}\delta_i\|_2=\left(1+\frac{3}{9}\right)^{\frac{1}{2}}=1.1547$.

The authors of paper [7] introduced the Hardy-Littlewood maximal operator defined on finite graphs, established the bounds for complete graph K_m, and demonstrated that these bounds are not the best possible. Additionally, they identified certain optimal bounds for the Hardy-Littlewood maximal operator defined on finite graphs. The authors of the works [2, 3] discovered the precise inequality of the Hardy-Littlewood maximal operator as applied to finite graphs. The work cited as [5] is a more specific and comprehensive application of the paper cited as [7]. In this paper, the authors established the concept of the generalized maximal operator on graphs. Also, they defined a general graph sequence G^m_n that includes the extreme boundaries of the generalized maximal operator as its endpoint. Furthermore, it was also determined that the bounds exhibit optimality. The publication referenced as [1] has extensively researched graph inequalities. The author of the work [4] computed the generalized maximal operator on the strong product of graphs. The Hardy-Littlewood maximal operator for infinite graphs is defined in the paper [8]. The paper also examines the geometric structure of infinite graphs by establishing the graphs' dilation index and overlapping index. The authors of the study [6] provided a definition of the fractional maximal operator on graphs and subsequently identified the extreme boundaries of graphs. In their study, the paper's authors [9] presented the Lipschitz, Hölder, Campanato, and Morrey spaces on infinite graphs. They also demonstrated the finitude of the Hardy-Littlewood maximal operator and its minuscule versions on these function spaces, subject to specific requirements on graphs. Additionally, the study identified the connections between the Hölder and Campanato spaces.

The objective of this study is to analyze the spherical maximal operator $\mathcal{M}_G^\circ$ on ℓ^p spaces and ascertain the value of $\|\mathcal{M}_G^\circ\|_{\ell^p}$ for $1 \geqslant p > 0$. Additionally, it will offer an approximation for the value of $\|\mathcal{M}_G^\circ\|_{\ell^p}$ for a given $1 < p$, given that G is a complete graph K_m. Furthermore, we will also obtain general and optimal outcomes for certain values of $0 < p \leqslant 1$.

2. Bounds for the complete graph

It is very important to estimate the norm of spherical maximal operator for complete graph K_m because it is the least bound for all graphs which we will prove in the Section 3.

THEOREM 2.1. *For a complete graph K_m, when $0 < p \leqslant 1$,*

$$\|\mathcal{M}_{K_m}^\circ\|_p = \left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}},$$

and, when $1 < p < \infty$,

$$\left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}} \leqslant \|\mathcal{M}_{K_m}^\circ\|_p \leqslant 2^{\frac{1}{p}}.$$

Proof. To prove this, we suppose that $\|h\|_p = 1$, for all $p > 0$ and $t \in V$ define δ_t, then

$$\|\mathcal{M}_{K_m}^\circ \delta_t\|_p = \left(\sum_{q \in V} \left(\mathcal{M}_{K_m}^\circ \delta_t(q)\right)^p\right)^{\frac{1}{p}} = \left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}.$$

As $\|\delta_t\|_p = 1$, so we get

$$\|\mathcal{M}_{K_m}^\circ\|_p \geqslant \left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}.$$

It follows from Lemma 1.1, for $0 < p \leqslant 1$, we have

$$\|\mathcal{M}_{K_m}^\circ\|_p = \left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}.$$

For $1 < p < \infty$, we have to prove upper bound

$$\|\mathcal{M}^{\circ}_{K_m}h(t)\|_p = \left(\sum_{t=1}^{m} \max\left\{|h(t)|^p, \left(\sum_{q\in V\setminus\{t\}} \frac{1}{m-1}|h(q)|\right)^p\right\}\right)^{\frac{1}{p}},$$

after apply Jensen's inequality on the second term $\left(\sum_{q\in V\setminus\{t\}} \frac{1}{m-1}|h(q)|\right)^p$, we have

$$\|\mathcal{M}^{\circ}_{K_m}h(t)\|_p \leqslant \left(\sum_{t=1}^{m} \max\left\{|h(t)|^p, \frac{1}{m-1}\right\}\right)^{\frac{1}{p}}.$$

If $|h(t)|^p \leqslant \frac{1}{m-1}$ for all vertices, then

$$\|\mathcal{M}^{\circ}_{K_m}\|_p = \sup \|\mathcal{M}^{\circ}_{K_m}h\|_p \leqslant \sup\left(\frac{m}{m-1}\right)^{\frac{1}{p}}.$$

If $|h(t_\circ)|^p > \frac{1}{m-1}$ for some $t_\circ \in V$, then we have

$$\|\mathcal{M}^{\circ}_{K_m}\|_p = \sup \|M^{\circ}_{K_m}h\|_p \leqslant \sup\left(\sum_{|h(t_\circ)|^p > \frac{1}{m-1}} |h(t_\circ)|^p, \sum_{|h(t)|^p \leqslant \frac{1}{m-1}} \frac{1}{m-1}\right)^{\frac{1}{p}}$$
$$\leqslant 2^{\frac{1}{p}}. \quad \square$$

The graphs for the result of Theorem 2.1 is shown in the graphs of Figure 1 and Figure 2, when $p = 2$ and $p = 3$ respectively.

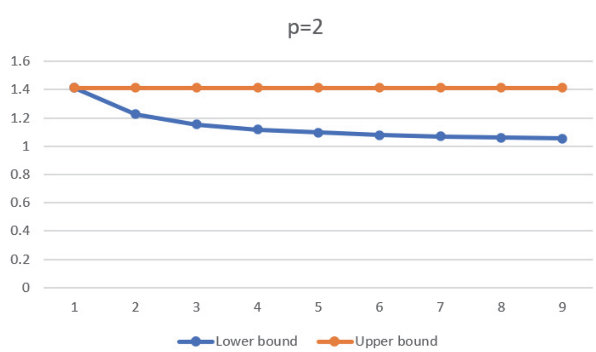

Figure 1: *Estimation for* $p = 2$

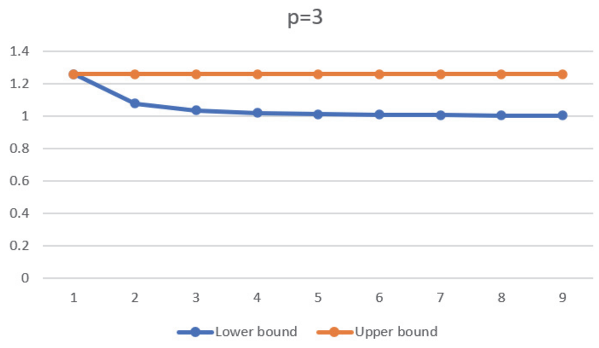

Figure 2: *Estimation for* $p = 3$

The 3D graph is shown in Figure 3. Graph is drawn in maple with the help of following command:

$$\text{shadebetween}\left(\left(1+\frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}, 2^{\frac{1}{p}}, m = 2..10, p = 1..10\right);$$

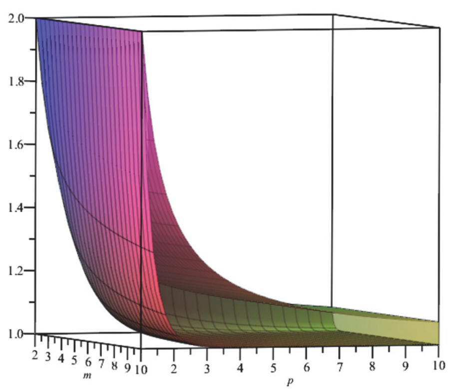

Figure 3: *A 3D graph for the estimation for* $2 \leqslant m \leqslant 10$ *and* $1 \leqslant p \leqslant 10$

The shaded region of the three-dimensional graph represents the extent between the upper and lower boundaries. Based on the graphs above, it can be concluded that the outcome of Theorem 2.1 is suboptimal for a given value of $1 < p < \infty$.

3. Optimal bounds

In this section, we will find the bounds for the norm of spherical maximal operator for finite graphs. Moreover, we will prove that the norm of spherical maximal operator of complete graph and star graph are the end points of $\|\mathcal{M}_G\|_p$, for any graph of m vertices. For this section, we suppose $p \in (0,1]$.

THEOREM 3.1. *Let G be a graph of m vertices, then*

$$\left(1+\frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}} \leqslant \|\mathcal{M}_G^{\circ}\|_p \leqslant m^{\frac{1}{p}}.$$

Proof. Let G be a graph of m vertices. For a vertex $t \in V$, define δ_t, then

$$\begin{aligned}\|\mathcal{M}_G^{\circ}\delta_t\|_p &= \left((\mathcal{M}_G^{\circ}\delta_t(t))^p + \sum_{q\in V\setminus\{t\}} (\mathcal{M}_G^{\circ}\delta_t(q))^p\right)^{\frac{1}{p}} \\ &= \left(1+\sum_{q\in V\setminus\{t\}}\left(\frac{1}{|S(q,r)|}\sum_{w\in S(q,r)}\delta_t(w)\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Since $1 \leqslant |S(q,r)| \leqslant m-1$, for $r \geqslant 1$ so

$$\left(1+\sum_{q\in V\setminus\{t\}}\left(\frac{1}{(m-1)^p}\right)\right)^{\frac{1}{p}} \leqslant \|\mathcal{M}_G^{\circ}\delta_t\|_p \leqslant \left(1+\sum_{q\in V\setminus\{t\}}(1)\right)^{\frac{1}{p}}$$

$$\left(1+\frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}} \leqslant \|\mathcal{M}_G^{\circ}\delta_t\|_p \leqslant m^{\frac{1}{p}}.$$

It follows from Lemma 1.1, we have $\|\mathcal{M}_G^{\circ}\|_p = \max_{t\in V}\|\mathcal{M}_G^{\circ}\delta_t\|_p$, so we get

$$\left(1+\frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}} \leqslant \|\mathcal{M}_G^{\circ}\|_p \leqslant m^{\frac{1}{p}}.$$

Which is the required expression. $\square$

The graphs illustrating the outcomes of Theorem 3.1 are depicted in Figure 4.

Figure 5 displays a three-dimensional graph illustrating the outcome of Theorem 3.1.

The shaded area shows the region where the norm of any graph can be lie for $2 \leqslant m \leqslant 10$ and $0 \leqslant p \leqslant 1$. Now we have to find the extreme points of this estimation.

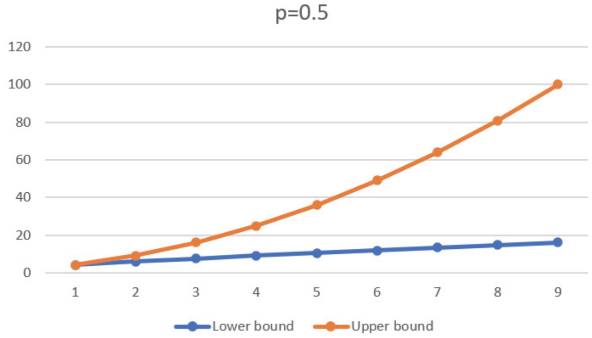

Figure 4: *Estimation for* $p = 0.5$

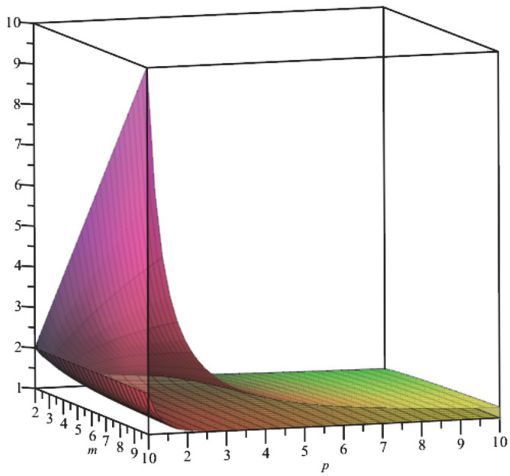

Figure 5: *A 3D graph for the estimation for* $2 \leqslant m \leqslant 10$ *and* $0 \leqslant p \leqslant 1$

THEOREM 3.2. *For a graph* G, $\left|\left|M_G^{\circ}\right|\right|_p = \left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}$ *if and only if* $G = K_m$.

Proof. If $G = K_m$ then $\|\mathcal{M}_G^{\circ}\|_p = \left(1 + \frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}$ is trivial. For the converse, we suppose that $G \neq K_m$, A $u \neq v$ exists where the distance between u and v is greater than 1. We consider two sets

$$J = S(u,1) = \{k \in V : d_G(u,k) = 1\}$$

and

$$K = S(v,1) = \{k \in V : d_G(v,k) = 1\}.$$

As G is connected, so $|J|, |K| \geqslant 1$. Now we will analyze two cases:

Case 1 $\min\{|J|,|K|\} \leqslant \frac{m-1}{2}$,

Case 2 $\min\{|J|,|K|\} > \frac{m-1}{2}$.

For Case 1, suppose that $|J| \leqslant \frac{m-1}{2}$. Let $t \in J$, then $d_G(t,u) = 1$, define δ_t, then we have

$$\begin{aligned}
\|\mathcal{M}_G^\circ \delta_t\|_p &= \left(\sum_{i \in V} (\mathcal{M}_G^\circ \delta_t(i))^p \right)^{\frac{1}{p}} \\
&= \left((\mathcal{M}_G^\circ \delta_t(t))^p + (\mathcal{M}_G^\circ \delta_t(u))^p + \sum_{x \in V \setminus \{t,u\}} (\mathcal{M}_G^\circ \delta_t(x))^p \right)^{\frac{1}{p}} \\
&\geqslant \left(1 + \frac{1}{|J|^p} + \frac{m-2}{(m-1)^p} \right)^{\frac{1}{p}} \\
&\geqslant \left(1 + \frac{2^p}{(m-1)^p} + \frac{m-2}{(m-1)^p} \right)^{\frac{1}{p}} \\
&> \left(1 + \frac{1}{(m-1)^{p-1}} \right)^{\frac{1}{p}} \\
\|\mathcal{M}_G^\circ\|_p &\geqslant \|\mathcal{M}_G^\circ \delta_t\|_p \\
&> \left(1 + \frac{1}{(m-1)^{p-1}} \right)^{\frac{1}{p}}.
\end{aligned}$$

Which finishes our proof of Case 1.

Now consider the Case 2 in which $\min\{|J|,|K|\} > \frac{m-1}{2}$. Clearly $J \bigcap K \neq \emptyset$. Let $t \in J \bigcap K$ and define δ_t, then we have

$$\begin{aligned}
\|\mathcal{M}_G^\circ \delta_t\|_p &= \left(\sum_{i \in V} (\mathcal{M}_G^\circ \delta_t(i))^p \right)^{\frac{1}{p}} \\
&= \left((\mathcal{M}_G^\circ \delta_t(u))^p + (\mathcal{M}_G^\circ \delta_t(v))^p + (\mathcal{M}_G^\circ \delta_t(t))^p + \sum_{x \in V \setminus \{u,v,t\}} (\mathcal{M}_G^\circ \delta_t(x))^p \right)^{\frac{1}{p}} \\
&\geqslant \left(1 + \frac{1}{|J|^p} + \frac{1}{|K|^p} + \frac{m-3}{(m-1)^p} \right)^{\frac{1}{p}},
\end{aligned}$$

using that $t \in J \bigcap K$ and $|J|, |K| \leqslant m-2$

$$\|\mathcal{M}_G^{\circ}\|_p \geqslant \|\mathcal{M}_G^{\circ}\delta_t\|_p \geqslant \left(1+\frac{2}{(m-2)^p}+\frac{m-3}{(m-1)^p}\right)^{\frac{1}{p}}$$
$$> \left(1+\frac{1}{(m-1)^{p-1}}\right)^{\frac{1}{p}}. \quad \square$$

THEOREM 3.3. *For an undirected graph G, $\left||M_G^{\circ}|\right|_p = m^{\frac{1}{p}}$ if and only if $G \sim S_m$.*

Proof. Let $t \in V$ be the central vertex, define δ_t then

$$\|\mathcal{M}_{S_m}^{\circ}\delta_t\|_p = \left(\left(\mathcal{M}_{S_m}^{\circ}\delta_t(t)\right)^p + \sum_{x\in V\setminus\{t\}}\left(\mathcal{M}_{S_m}^{\circ}\delta_t(x)\right)^p\right)^{\frac{1}{p}}$$
$$= (1+m-1)^{\frac{1}{p}} = m^{\frac{1}{p}}$$
$$\|\mathcal{M}_{S_m}^{\circ}\|_p \geqslant m^{\frac{1}{p}},$$

it follows from Lemma 1.1, we get

$$\|\mathcal{M}_{S_m}^{\circ}\|_p = m^{\frac{1}{p}}.$$

In the case of the converse, let us assume that $G \nsim S_m$ and $m \geqslant 3$. In this case, two distinct vertices u, v in V guarantee that the degrees of u and v are more than 1. It should be noted that for any function $h : V \to \mathbb{R}$, where $|h|_1$ is less than 1, either $\mathcal{M}_G^{\circ}h(t) = h(t)$ or $\mathcal{M}_G^{\circ}h(t) \leqslant \frac{1}{d_G(t)}$. Let us consider a collection

$$J = \{t \in V : \mathcal{M}_G^{\circ}h(t) = h(t)\}.$$

Then, we have

$$\|\mathcal{M}_G^{\circ}h\|_p = \left(\sum_{t\in J}(\mathcal{M}_G^{\circ}h(t))^p + \sum_{x\in V\setminus J}(\mathcal{M}_G^{\circ}h(x))^p\right)^{\frac{1}{p}}$$
$$\leqslant \left(\sum_{t\in J}(h(t))^p + \sum_{x\in V\setminus J}\left(\frac{1}{d_G(x)}\right)^p\right)^{\frac{1}{p}},$$

now, if $u, v \in J$, then

$$\|\mathcal{M}_G^{\circ}h\|_p \leqslant (1+m-2)^{\frac{1}{p}}.$$

If $u \notin J$, then we have

$$\begin{aligned}\|\mathcal{M}_G^\circ h\|_p &\leqslant \left(1+\left(\frac{1}{d_G(u)}\right)^p+m-2\right)^{\frac{1}{p}}\\ &\leqslant \left(1+\frac{1}{2^p}+m-2\right)^{\frac{1}{p}}.\end{aligned}$$

Same calculation for $v \in J$, hence

$$\|\mathcal{M}_G^\circ\|_p \leqslant \max\left\{(m-1)^{\frac{1}{p}}, \left(1+\frac{1}{2^p}+m-2\right)^{\frac{1}{p}}\right\} < m^{\frac{1}{p}}.$$

Which completes our arguments. □

4. Conclusion

This work examined the spherical maximal operator $\mathcal{M}_G^\circ$ on ℓ^p spaces. We computed the $\|\mathcal{M}_G^\circ\|_{\ell^p}$ for $0 < p \leqslant 1$ and estimated the $\|\mathcal{M}_G^\circ\|_{\ell^p}$ for $1 < p < \infty$, assuming that G is a complete graph K_m. Moreover, some general optimal results are also calculated for $p \in (0,1]$.

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