Journal of
Mathematical
Inequalities
Volume 18, Number 2 (2024), 631–642
doi:10.7153/jmi-2024-18-35

OPERATOR INEQUALITIES VIA THE TRIANGLE INEQUALITY

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(*Communicated by M. Krnić*)

Abstract. This article improves the triangle inequality for complex numbers, using the Hermite-Hadamard inequality for convex functions. Then, applications of the obtained refinement are presented to include some operator inequalities. The operator applications involve numerical radius inequalities and operator mean inequalities.

1. Introduction

In the field of mathematical inequalities, interest in refining existing inequalities or sharpening them has been at the center of researchers' attention; see [6] for example.

One of the most powerful tools in obtaining new inequalities or sharpening existing ones is the use of convex functions. Recall that a function $f : J \to \mathbb{R}$ is said to be convex on the interval J if it satisfies the basic inequality

$$f((1-t)a+tb) \leqslant (1-t)a+tb, \tag{1.1}$$

for all $a,b \in J$ and $0 \leqslant t \leqslant 1$. This inequality was refined in [5], where it has been shown that

$$f((1-t)a+tb)+2r_t\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right) \leqslant (1-t)f(a)+tf(b), \tag{1.2}$$

for $r_t = \min\{t, 1-t\}$. Applications of this inequality and related discussion can be found in [16, 19, 20, 23].

One of the most useful inequalities in convex analysis is the so called Hermite-Hadamard inequality, which states

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}; \tag{1.3}$$

as a refinement of (1.1) when $t = \frac{1}{2}$.

In this paper we employ the inequality (1.3) to refine the well known triangle inequality

$$|c+d| \leqslant |c|+|d|, \ \ c,d \in \mathbb{C}.$$

Mathematics subject classification (2020): Primary 47A63; Secondary 46L05, 47A60.

Keywords and phrases: Triangle inequality, numerical radius, operator geometric mean, convex function.

Paper JMI-18-35

More precisely, we show that

$$|c+d| \leqslant 2\int_0^1 |sc+(1-s)d|\,ds \leqslant |c|+|d|. \tag{1.4}$$

This will enable us to present a refinement of the Cauchy-Schwarz inequality for the inner product. Then applications that include refined forms for some numerical radius inequalities and operator mean inequalities will be given.

For this, we need to recall some notions related to Hilbert space operators. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $A\in\mathcal{B}(\mathcal{H})$, the operator norm and the numerical radius are defined respectively as

$$\|A\| = \sup_{\|x\|=1}\|Ax\| \quad\text{and}\quad w(A) = \sup_{\|x\|=1}|\langle Ax,x\rangle|.$$

It is well known that

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|.$$

Refining these inequalities has occupied an adequate area of research in this field. For example, in [14] Kittaneh showed that

$$w(A) \leqslant \frac{1}{2}\|\,|A|+|A^*|\,\|, \tag{1.5}$$

where A^* is the adjoint operator of A. The fact that this refines the inequality $w(A)\leqslant\|A\|$ is due to the triangle inequality and the observation $\|\,|A^*|\,\| = \|\,|A|\,\| = \|A\|$.

Using (1.4) we will be able to present a refined form of (1.5), where we find a scalar α such that $\frac{1}{2}\leqslant\alpha\leqslant 1$ and

$$w(A) \leqslant \frac{\alpha}{2}\|\,|A|+|A^*|\,\|,$$

for a certain class of operators. We should remark that finding better bounds for the numerical radius has received a renowned interest in the last few years, as one can see in [2, 3, 4, 8, 10, 11, 14, 15, 18, 25].

Using the same approach, we will be able to find an inequality that relates the geometric mean of $|A|^{2v}$ and $|A|^{2(1-v)}$ with the numerical radius. For this, we recall that the weighted geometric mean of the two strictly positive operators $A,B\in\mathcal{B}(\mathcal{H})$ is [1]

$$A\sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}};\ \ 0\leqslant t\leqslant 1.$$

When $t=\frac{1}{2}$, we write $\sharp$ instead of $\sharp_{\frac{1}{2}}$. Operator means and their inequalities have received a considerable attention in the literature, as one can find in [7, 9, 17, 21, 22, 24].

Our results will make use of the angle $\angle_{x,y}$ between two vectors $x,y \in \mathbb{C}^n$ or $x,y \in \mathcal{H}$. For such x,y, the Cauchy-Schwarz inequality states that $|\langle x,y\rangle| \leqslant \|x\|\,\|y\|$. From this, the angle between the non-zero vectors x,y can be defined by

$$\angle_{x,y} = \cos^{-1}\left(\frac{|\langle x,y\rangle|}{\|x\|\,\|y\|}\right).$$

The organization of the subsequent section will be as follows. In the first part, we discuss possible refinement and reverse of the scalar triangle inequality, with a detailed treatment. After that, applications towards the Cauchy-Schwarz inequality, numerical radius and operator geometric mean will be presented.

2. Main results

In this section, we present our main results. We begin by showing the scalar inequalities, which will lead to the desired operator versions.

2.1. Scalar inequalities

To establish our results in this section, we need the following well-known refinement of the triangle inequality. Notice that although this result can also be proven by the Hermite-Hadamard inequality [13, p. 38], we consider another way to prove it.

THEOREM 2.1. *Let c and d be two complex numbers. Then*

$$\left|\frac{c+d}{2}\right| \leqslant \int_0^1 |sc+(1-s)d|\,ds \leqslant \frac{|c|+|d|}{2}.$$

Proof. Let $a,b \in \mathbb{C}$, and define the function $f:\mathbb{R}\to[0,\infty)$ by $f(t)=|a+tb|$. Using the triangle inequality, it follows immediately that f is convex on $\mathbb{R}$. By the Hermite-Hadamard inequality (1.3), we have

$$\left|a+\left(\frac{x+y}{2}\right)b\right| \leqslant \frac{1}{x-y}\int_y^x |a+tb|\,dt \leqslant \frac{|a+xb|+|a+yb|}{2},$$

for real numbers $x>y$. Now, if $c,d\in\mathbb{C}$, let $a=\dfrac{dx-cy}{x-y}$ and $b=\dfrac{c-d}{x-y}$. Then $c=a+xb$ and $d=a+yb$. This implies

$$\left|\frac{c+d}{2}\right| \leqslant \frac{1}{x-y}\int_y^x \left|\frac{dx-cy+(c-d)t}{x-y}\right| dt \leqslant \frac{|c|+|d|}{2}.$$

Now, if we set $\frac{dx-cy+(c-d)t}{x-y}=sc+(1-s)d$, then we have $\frac{1}{x-y}dt=ds$. So for complex numbers c and d,

$$\left|\frac{c+d}{2}\right| \leqslant \int_0^1 |sc+(1-s)d|\,ds \leqslant \frac{|c|+|d|}{2},$$

as desired. □

As a direct consequence of Theorem 2.1, we can improve the celebrated Cauchy-Schwarz inequality, as follows.

COROLLARY 2.1. Let $x,y\in\mathcal{H}$. Then

$$|\langle x,y\rangle| \leqslant \int_0^1 \left|te^{i\theta}+(1-t)e^{-i\theta}\right| dt\,\|x\|\,\|y\| \leqslant \|x\|\,\|y\|,$$

where $\theta=\angle_{x,y}$.

Proof. We have

$$|\langle x,y\rangle| = |\cos\theta|\,\|x\|\,\|y\|. \tag{2.1}$$

Since

$$\cos\theta=\mathcal{R}\left(e^{i\theta}\right)=\frac{e^{i\theta}+e^{-i\theta}}{2},$$

Theorem 2.1 applied for $c=e^{i\theta}$ and $d=e^{-i\theta}$ implies that

$$|\langle x,y\rangle| \leqslant \int_0^1 \left|te^{i\theta}+(1-t)e^{-i\theta}\right| dt\,\|x\|\,\|y\| \leqslant \|x\|\,\|y\|,$$

as desired. □

REMARK 2.1. Corollary 2.1 means

$$|\cos\theta| \leqslant \int_0^1 \left|te^{i\theta}+(1-t)e^{-i\theta}\right| dt \leqslant 1.$$

To calculate the constant that appears in Corollary 2.1, notice that for an arbitrary $\theta\in\mathbb{R}$,

$$\begin{aligned}\left|te^{i\theta}+(1-t)e^{-i\theta}\right| &= |t(\cos\theta+i\sin\theta)+(1-t)(\cos\theta-i\sin\theta)|\\ &= |\cos\theta+i(2t-1)\sin\theta|\\ &= \sqrt{\cos^2\theta+(2t-1)^2\sin^2\theta}.\end{aligned}$$

For the case $\sin\theta = 0$, we have $\int_0^1 \sqrt{\cos^2\theta}dt = |\cos\theta| = 1$, because $\sin\theta = 0$. If we assume $\sin\theta \neq 0$, then

$$\begin{aligned}
&\int_0^1 \sqrt{\cos^2\theta + (2t-1)^2\sin^2\theta}dt \\
&= \frac{|\sin\theta|}{2}\int_{-1}^1 \sqrt{s^2+\cot^2\theta}ds \\
&= \frac{|\sin\theta|}{4}\left[s\sqrt{s^2+\cot^2\theta} + \cot^2\theta\log\left|s+\sqrt{s^2+\cot^2\theta}\right|\right]_{-1}^1 \\
&= \frac{1}{2} + \frac{1}{4}|\sin\theta|\cot^2\theta\log\left|\frac{1+|\sin\theta|}{1-|\sin\theta|}\right| =: \mu(\theta).
\end{aligned}$$

Treating the cases $\sin\theta > 0$ or $\sin\theta < 0$ implies that

$$\mu(\theta) = \frac{1}{2} + \frac{1}{4}\sin\theta\cot^2\theta\log\left|\frac{1+\sin\theta}{1-\sin\theta}\right|.$$

Thus we have

$$\mu(\theta) = \frac{1}{4}\left(2+\cos\theta\cot\theta\log\frac{1+\sin\theta}{1-\sin\theta}\right), \quad \theta \neq n\pi, \quad \text{where} \quad n = 0,1,2,\ldots.$$

Since $\mu(\theta) \to 1$ when $\theta \to n\pi$ and $|\cos\theta| = 1$ for $\theta = n\pi$, where $n = 0,1,2,\ldots$, we have

$$\mu(\theta) = \frac{1}{4}\left(2+\cos\theta\cot\theta\log\frac{1+\sin\theta}{1-\sin\theta}\right).$$

We study the properties of the function $\mu(\theta)$. It is sufficient to consider $0 \leqslant \theta < \pi$, since $\mu(\theta+\pi) = \mu(\theta)$. Also, we notice that by definition of the angle $\angle_{x,y}$, we must have $0 \leqslant \angle_{x,y} \leqslant \pi$. For this purpose, we prepare the following lemma.

LEMMA 2.1. *If* $0 < x < 1$, *then*

$$\frac{2x}{x^2+1} \leqslant \log\frac{1+x}{1-x}. \tag{2.2}$$

If $-1 < x < 0$, *then the reversed inequality holds.*

Proof. We firstly prove the first statement. Putting $t := \frac{1+x}{1-x}$ for $0 < x < 1$, the inequality (2.2) is equivalent to the inequality $\frac{t^2-1}{t^2+1} \leqslant \log t$ for $t > 1$. Letting $a := t^2 > 1$ and using the inequality $\frac{a-1}{\log a} \leqslant \frac{a+1}{2}$ for $a > 0$, we get the first inequality.

To prove the reversed inequality for $-1 < x < 0$, we set $s := \frac{1-x}{1+x}$. Then the desired inequality is equivalent to $\dfrac{s^2-1}{s^2+1} \geqslant \log s$ for $0 < s < 1$. Letting $s^2 = a$, this is equivalent to $\dfrac{a-1}{\log a} \leqslant \dfrac{a+1}{2}$, $0 < a < 1$. This completes the proof. $\square$

Now we present the monotonicity of the function $\mu(\theta)$. As we mentioned earlier, this function has period π, so we study it only on the interval $[0,\pi]$.

PROPOSITION 2.1. The function $\mu(\theta)$ is decreasing on the interval $\left[0,\frac{\pi}{2}\right]$ and is increasing in $\left[\frac{\pi}{2},\pi\right]$.

Proof. By elementary calculations, we have

$$\mu'(\theta) = \frac{\cos\theta}{8\sin^2\theta}\nu(\theta), \quad \text{where} \quad \nu(\theta) := 4\sin\theta - 2(\sin^2\theta + 1)\log\frac{1+\sin\theta}{1-\sin\theta}.$$

Since we have $\lim\limits_{\theta\to n\pi/2}\mu'(\theta) = 0$ for $n = 0,1,2$, we consider the values $\theta \neq n\pi/2$, where $n = 0,1,2$. Putting $x := \sin\theta$ for $0 < \theta < \pi$, we consider the function $\hat{\nu}(x) = 4x - 2(x^2+1)\log\dfrac{1+x}{1-x}$ for $0 < x < 1$. We have $\hat{\nu}(x) < 0$ by Lemma 2.1. Therefore we have $\nu(\theta) \leqslant 0$ for $0 \leqslant \theta \leqslant \pi$.

Taking account that $\cos\theta$ is positive when $0 < \theta < \dfrac{\pi}{2}$ and is negative when $\dfrac{\pi}{2} < \theta < \pi$, we have $\mu'(\theta) \leqslant 0$ when $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$ and $\mu'(\theta) \geqslant 0$ when $\dfrac{\pi}{2} \leqslant \theta \leqslant \pi$. This completes the proof. $\square$

COROLLARY 2.2. The inequality $\dfrac{1}{2} \leqslant \mu(\theta) \leqslant 1$ holds for $\theta \geqslant 0$.

Proof. It suffices to consider the values $0 \leqslant \theta \leqslant \pi$. We have $\lim\limits_{\theta\to 0}\mu(\theta) = \lim\limits_{\theta\to\pi}\mu(\theta)$ $= 1$ and $\lim\limits_{\theta\to\pi/2}\mu(\theta) = \dfrac{1}{2}$. By Proposition 2.1, we infer that $\dfrac{1}{2} \leqslant \mu(\theta) \leqslant 1$. $\square$

The following is a straightforward consequence from Proposition 2.1.

COROLLARY 2.3. The following holds.

(i) If $0 \leqslant \theta_1 < \theta < \theta_2 \leqslant \frac{\pi}{2}$, then

$$\mu(\theta) \leqslant \mu(\theta_1).$$

(ii) If $\frac{\pi}{2} \leqslant \theta_1 < \theta < \theta_2 \leqslant \pi$, then

$$\mu(\theta) \leqslant \mu(\theta_2).$$

In the above discussion, we have treated certain refinements of the triangle inequality. In the following, we present a reverse of the triangle inequality. In the next subsection, we present applications of both forms.

THEOREM 2.2. *Let c and d be two complex numbers. Then for any $0<t<1$,*

$$\frac{|c|+|d|}{2}-\frac{1}{2r_t}\left((1-t)|c|+t|d|-|(1-t)c+td|\right)\leqslant\left|\frac{c+d}{2}\right|,$$

where $r_t=\min\{t,1-t\}$.

Proof. We know that if f is a convex function on $\mathbb{R}$, then for $x,y\in\mathbb{R}$ and any $0<t<1$,

$$f((1-t)x+ty)\leqslant(1-t)f(x)+tf(y)-2r_t\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right),$$

where $r_t=\min\{t,1-t\}$, see (1.2). Since for $a,b\in\mathbb{C}$, $f(t)=|a+tb|$ is convex on $\mathbb{R}$, we get

$$\begin{aligned}|a+((1-t)x+ty)b| &\leqslant (1-t)|a+xb|+t|a+yb|\\ &\qquad -2r_t\left(\frac{|a+xb|+|a+yb|}{2}-\left|a+\frac{x+y}{2}b\right|\right).\end{aligned}$$

Now, applying the same method as in Theorem 2.1, we infer

$$|(1-t)c+td|\leqslant(1-t)|c|+t|d|-2r_t\left(\frac{|c|+|d|}{2}-\left|\frac{c+d}{2}\right|\right),$$

as desired □

REMARK 2.2. Let $x,y\in\mathcal{H}$, and let $r_t=\min\{t,1-t\}$ with $0<t<1$. By Theorem 2.2,

$$\begin{aligned}0\leqslant 1-\frac{1}{2r_t}\left(1-\left|te^{i\theta}+(1-t)e^{-i\theta}\right|\right)\\ \leqslant\left|\frac{e^{i\theta}+e^{-i\theta}}{2}\right|\\ =|\cos\theta|,\end{aligned}$$

where $\theta=\angle_{x,y}$. Thus we have

$$0\leqslant\gamma_t(\theta)\,\|x\|\,\|y\|\leqslant|\langle x,y\rangle|,\tag{2.3}$$

where

$$\gamma_t(\theta):=1-\frac{1}{2r_t}\left(1-\left|te^{i\theta}+(1-t)e^{-i\theta}\right|\right)=1-\frac{1}{2r_t}\left(1-\sqrt{\cos^2\theta+(2t-1)^2\sin^2\theta}\right)$$

for $\theta \in [0,\pi]$ and $r_t = \min\{t, 1-t\}$ with $0 < t < 1$. This provides a reverse of the Cauchy-Schwarz inequality.

We notice that $\gamma_t(\theta) = \gamma_{1-t}(\theta)$ for $0 < t < 1$. Also, since we have the Cauchy-Schwarz inequality $|\langle x,y\rangle| \leqslant \|x\|\,\|y\|$, we can use (2.3) to obtain sufficient conditions on the equality $|\langle x,y\rangle| = \|x\|\,\|y\|$, as follows. We notice that $\gamma_t(\theta) = 1$ when $\theta = 0,\pi$. Thus, when $\angle_{x,y} = 0,\pi$, we have $|\langle x,y\rangle| = \|x\|\,\|y\|$.
Further when $\theta = \frac{\pi}{2}$, $\gamma_t(\theta) = 0$.

Also, we notice that $\max_t \gamma_t(\theta) = \cos\theta$, which is evident because

$$\langle x,y\rangle = \|x\|\,\|y\|\cos\theta.$$

Again, since $\gamma_t(\theta)$ is periodic in θ with period π, it is sufficient to study it on the interval $[0,\pi]$.

PROPOSITION 2.2. Let $0 < t < 1$ be fixed, and let γ_t be as above. Then $\gamma_t(\theta)$ is decreasing on $\left[0,\frac{\pi}{2}\right]$ and increasing on $\left[\frac{\pi}{2},\pi\right]$. In addition, we have $0 \leqslant \gamma_t(\theta) \leqslant 1$.

Proof. Elementary calculations show that

$$\frac{d\gamma_t(\theta)}{d\theta} = -\frac{t(1-t)\sin 2\theta}{r_t\sqrt{\cos^2\theta + (2t-1)^2\sin^2\theta}}.$$

This shows the assertion of monotonicity. Since we have $\lim\limits_{\theta\to n\pi/2}\gamma_t(\theta) = 1$ for $n = 0,2$ and $\lim\limits_{\theta\to n\pi/2}\gamma_t(\theta) = \dfrac{2r_t - 1 + \sqrt{(2t-1)^2}}{2r_t} = 0$ for $n = 1$, we have $0 \leqslant \gamma_t(\theta) \leqslant 1$ by monotonicity of $\gamma_t(\theta)$. $\square$

2.2. Hilbert space operator inequalities

In this part of the paper, we present some applications of the above scalar inequalities. These applications will treat operator inequalities, where numerical radius and operator means are discussed.

In the first result, we improve the mixed Cauchy-Schwarz inequality, which states that if $A \in \mathcal{B}(\mathcal{H})$, then [12]

$$|\langle Ax,y\rangle| \leqslant \sqrt{\left\langle |A|^{2v}x,x\right\rangle\left\langle |A^*|^{2(1-v)}y,y\right\rangle},\quad x,y\in\mathcal{H},\quad 0\leqslant v\leqslant 1.$$

This inequality has been used extensively in the literature when dealing with numerical radius inequalities, see [26] for example.

THEOREM 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ with the the polar decomposition $A = U|A|$ and let $x,y \in \mathcal{H}$. Then for any $0 \leqslant v \leqslant 1$,*

$$|\langle Ax,y\rangle| \leqslant \mu(\theta)\sqrt{\left\langle |A|^{2v}x,x\right\rangle\left\langle |A^*|^{2(1-v)}y,y\right\rangle},$$

where $\theta = \angle_{|A|^v x,|A|^{1-v}U^*y}$.

Proof. According to the assumptions and by employing Corollary 2.1, we have

$$\begin{aligned}
|\langle Ax,y\rangle| &= |\langle U\,|A|\,x,y\rangle| \\
&= \left|\left\langle U|A|^{1-v}|A|^{v}x,y\right\rangle\right| \\
&= \left|\left\langle |A|^{v}x,|A|^{1-v}U^{*}y\right\rangle\right| \\
&\leqslant \mu(\theta)\,\||A|^{v}x\|\left\||A|^{1-v}U^{*}y\right\| \\
&= \mu(\theta)\sqrt{\left\langle |A|^{2v}x,x\right\rangle\left\langle U|A|^{2(1-v)}U^{*}y,y\right\rangle} \\
&= \mu(\theta)\sqrt{\left\langle |A|^{2v}x,x\right\rangle\left\langle |A^{*}|^{2(1-v)}y,y\right\rangle},
\end{aligned}$$

as desired. $\square$

Now we are ready to present a new bound for the numerical radius. This form refines (1.5), when $v=\frac{1}{2}$ for a certain class of operators.

COROLLARY 2.4. Let $A\in\mathcal{B}(\mathcal{H})$ have the polar decomposition $A=U\,|A|$, $0\leqslant v\leqslant 1$ and let $\theta_x=\angle_{|A|^{v}x,|A|^{1-v}U^{*}x}$ where $x\in\mathcal{H}$ with $\|x\|=1$. If

(i) If $0\leqslant\theta_1<\theta_x<\theta_2\leqslant\frac{\pi}{2}$ for all unit vectors $x\in\mathcal{H}$, then

$$\omega(A)\leqslant\frac{\mu(\theta_1)}{2}\left\||A|^{2v}+|A^{*}|^{2(1-v)}\right\|.$$

(ii) If $\frac{\pi}{2}\leqslant\theta_1<\theta<\theta_2\leqslant\pi$ for all unit vectors $x\in\mathcal{H}$, then

$$\omega(A)\leqslant\frac{\mu(\theta_2)}{2}\left\||A|^{2v}+|A^{*}|^{2(1-v)}\right\|.$$

Proof. We prove the first inequality. Let $x\in\mathcal{H}$ be a unit vector. By Theorem 2.3,

$$\begin{aligned}
|\langle Ax,x\rangle| &\leqslant \mu(\theta_x)\sqrt{\left\langle |A|^{2v}x,x\right\rangle\left\langle |A^{*}|^{2(1-v)}x,x\right\rangle} \\
&\leqslant \mu(\theta_1)\sqrt{\left\langle |A|^{2v}x,x\right\rangle\left\langle |A^{*}|^{2(1-v)}x,x\right\rangle} \\
&\leqslant \mu(\theta_1)\left(\frac{\left\langle |A|^{2v}x,x\right\rangle+\left\langle |A^{*}|^{2(1-v)}x,x\right\rangle}{2}\right) \\
&= \frac{\mu(\theta_1)}{2}\left\langle\left(|A|^{2v}+|A^{*}|^{2(1-v)}\right)x,x\right\rangle \\
&\leqslant \frac{\mu(\theta_1)}{2}\left\||A|^{2v}+|A^{*}|^{2(1-v)}\right\|,
\end{aligned}$$

where the second inequality is obtained from Corollary 2.3, and the third inequality follows from the arithmetic-geometric mean inequality. Therefore,

$$|\langle Ax,x\rangle| \leqslant \frac{\mu(\theta_1)}{2}\left\||A|^{2v}+|A^*|^{2(1-v)}\right\|.$$

Now, we get the desired result by taking the supremum over all unit vector $x\in\mathcal{H}$. □

We conclude this work by presenting the following relation between the geometric mean and the numerical radius of the operator A.

COROLLARY 2.5. Let $A\in\mathcal{B}(\mathcal{H})$ have the polar decomposition $A=U|A|$, $0\leqslant v\leqslant 1$ and let $\theta_x=\angle_{|A|^v x,|A|^{1-v}U^*x}$ where $x\in\mathcal{H}$ with $\|x\|=1$.

(i) If $0\leqslant\theta_1<\theta_x<\theta_2\leqslant\frac{\pi}{2}$, then

$$\cos(\theta_2)\left\||A|^{2v}\sharp|A^*|^{2(1-v)}\right\|\leqslant\omega(A).$$

(ii) If $\frac{\pi}{2}\leqslant\theta_1<\theta_x<\theta_2\leqslant\pi$, then

$$\cos(\theta_1)\left\||A|^{2v}\sharp|A^*|^{2(1-v)}\right\|\leqslant\omega(A).$$

Proof. We prove case (i) since case (ii) can be proved similarly. In this case, by Proposition 2.2, we have $\gamma_t(\theta_2)\leqslant\gamma_t(\theta)$. Putting $x=|A|^{1-t}x$, $y=|A|^{1-v}U^*x$, where $x\in\mathcal{H}$ is a unit vector, in the inequality (2.3), we have

$$\begin{aligned}\cos(\theta_2)\left\langle|A|^{2v}\sharp|A^*|^{2(1-v)}x,x\right\rangle&\leqslant\cos(\theta_2)\sqrt{\left\langle|A|^{2v}x,x\right\rangle\left\langle|A^*|^{2(1-v)}x,x\right\rangle}\\&=\cos(\theta_2)\sqrt{\left\langle|A|^{2v}x,x\right\rangle\left\langle U|A|^{2(1-v)}U^*x,x\right\rangle}\\&=\cos(\theta_2)\,\||A|^v x,x\|\left\||A|^{1-v}U^*x,x\right\|\\&\leqslant\left|\left\langle|A|^v x,|A|^{1-v}U^*x\right\rangle\right|\\&=|\langle Ax,x\rangle|.\end{aligned}$$

Notice that the first inequality follows from the following fact for strictly positive operators A,B,

$$\langle A\sharp Bx,x\rangle\leqslant\sqrt{\langle Ax,x\rangle\langle Bx,x\rangle}.$$

Thus,

$$\cos(\theta_2)\left\langle|A|^{2v}\sharp|A^*|^{2(1-v)}x,x\right\rangle\leqslant|\langle Ax,x\rangle|,$$

and this implies

$$\cos(\theta_2)\left\||A|^{2v}\sharp|A^*|^{2(1-v)}\right\|\leqslant\omega(A),$$

as desired. □

REFERENCES

[1] T. ANDO AND F. HIAI, *Operator log-convex functions and operator means*, Math. Ann. **350** (2011), 611–630.

[2] M. BAKHERAD AND K. SHEBRAWI, *Upper bounds for numerical radius inequalities involving off-diagonal operator matrices*, Ann. Funct. Anal. **9** (3) (2018), 297–309.

[3] P. BHUNIA, A. BHANJA, S. BAG, AND K. PAUL, *Bounds for the Davis–Wielandt radius of bounded linear operators*, Ann. Funct. Anal. **12** (2021), Article 18.

[4] A. BOURHIM AND M. MABROUK, *Numerical radius and product of elements in C^*-algebras*, Linear Multilinear Algebra **65** (6) (2017), 1108–1116.

[5] S. S. DRAGOMIR, J. E. PEČARIĆ AND L. E. PERSSON, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungar. **70** (1996), 129–143.

[6] S. FURUICHI AND H. R. MORADI, *Advances in mathematical inequalities*, De Gruyter, 2020.

[7] S. FURUICHI, H. R. MORADI AND M. SABABHEH, *Some inequalities for interpolational operator means*, J. Math. Inequal. **15** (1) (2021), 107–116.

[8] H. L. GAU, K. Z. WANG, AND P. Y. WU, *Numerical radii for tensor products of matrices*, Linear Multilinear Algebra **63** (10) (2015), 1916–1936.

[9] I. H. GÜMÜŞ, H. R. MORADI, AND M. SABABHEH, *More accurate operator means inequalities*, J. Math. Anal. Appl. **465** (1) (2018), 267–280.

[10] M. HAJMOHAMADI, R. LASHKARIPOUR, AND M. BAKHERAD, *Some generalizations of numerical radius on off-diagonal part of 2×2 operator matrices*, J. Math. Inequal. **12** (2) (2018), 447–457.

[11] O. HIRZALLAH, F. KITTANEH, AND K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices*, Int. Eq. Oper. Theory **71** (2011), 129–147.

[12] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.

[13] E. KIKIANTY, *Hermite-Hadamard inequality in the geometry of Banach spaces*, PhD thesis, Victoria University, 2010.

[14] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (1) (2003), 11–17.

[15] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (1) (2005), 73–80.

[16] F. MITROI, *About the precision in Jensen–Steffensen inequality*, Ann. Univ. Craiova Math. Comput. Sci. Ser. **37** (4) (2010), 73–84.

[17] H. R. MORADI, S. FURUICHI, Z. HEYDARBEYGI AND M. SABABHEH, *Revisiting the Grüss inequality*, Oper. Matrices **15** (4) (2021), 1379–1392.

[18] M. S. MOSLEHIAN AND M. SATTARI, *Inequalities for operator space numerical radius of 2×2 block matrices*, J. Math. Phys. **57** (1) (2016), Article 015201.

[19] M. SABABHEH, *Improved Jensen's inequality*, Math. Inequal. Appl. **20** (2) (2017), 389-403.

[20] M. SABABHEH, *Means refinements via convexity*, Mediterr. J. Math. **14** (2017), Article 125.

[21] M. SABABHEH, *On the matrix harmonic mean*, J. Math. Inequal. **12** (4) (2018), 901–920.

[22] M. SABABHEH, S. FURUICHI, Z. HEYDARBEYGI, AND H. R. MORADI, *On the arithmetic-geometric mean inequality*, J. Math. Inequal. **15** (3) (2021), 1255–1266.

[23] M. SABABHEH, S. FURUICHI AND H. R. MORADI, *Improved matrix inequalities using radical convexity*, J. Math. Inequal. **17** (1) (2023), 349–364.

[24] M. SABABHEH, S. FURUICHI, S. SHEYBANI AND H. R. MORADI, *Singular values inequalities for matrix means*, J. Math. Inequal. **16** (1) (2022), 169–179.

[25] M. SABABHEH AND H. R. MORADI, *More accurate numerical radius inequalities*, Linear Multilinear Algebra. **69** (10) (2021), 1964–1973.

[26] S. SHEYBANI, M. SABABHEH, AND H. R. MORADI, *Weighted inequalities for the numerical radius*, Vietnam J. Math. **51** (2) (2023), 363–377.

(Received April 3, 2022)

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