Journal of
Mathematical
Inequalities
Volume 18, Number 2 (2024), 643–684 doi:10.7153/jmi-2024-18-36

CHARACTERIZATIONS OF SLICE BESOV–TYPE AND SLICE TRIEBEL—LIZORKIN–TYPE SPACES AND APPLICATIONS

YUAN LU AND JIANG ZHOU*

(*Communicated by Y. Sawano*)

Abstract. Let $\alpha\in\mathbb{R}$, $\tau\in[0,\infty)$, $q\in(0,\infty]$ and $t,\ r,\ p\in(0,\infty)$. In this paper, we introduce the slice Besov-type space $(\dot{B}E^{\alpha,\tau}_{r,p,q})(\mathbb{R}^n)$ and the slice Triebel–Lizorkin-type space $(\dot{F}E^{\alpha,\tau}_{r,p,q})(\mathbb{R}^n)$, and establish their φ-transform characterizations in the sense of Frazier and Jawerth. The embedding properties, characterizations via the Peetre maximal function, the Lusin area function, smooth atomic and molecular decompositions of these spaces are also obtained. As applications, we obtain the boundedness on these spaces of Fourier multipliers with symbols satisfying some generalized Hörmander condition.

1. Introduction

The classical Besov space $B^s_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ were introduced between 1959 and 1975 (see, for example, [22]). These spaces form a very general unifying scale of many vital classical concrete function spaces such as Lebesgue spaces, Hölder–Zygmund spaces, Sobolev spaces, Bessel-potential spaces, Hardy spaces and $\mathrm{BMO}(\mathbb{R}^n)$, which have their own history. We refer the readers to Triebel's monographes [19–22]. Recently, to clarify the relations among Besov spaces, Triebel-Lizorkin spaces and Q spaces, Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $\tau,s\in\mathbb{R}$ and $p,q\in(0,\infty]$ and $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $\tau,s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty]$ and their inhomogeneous counterparts, $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$, for all admissible parameters, were introduced and studied in [27–29]. Some of real-variable characterizations of Besov-type and Triebel–Lizorkin-type spaces, via smooth atoms, molecules, wavelets, differences, oscillations, the Peetre maximal function, the Lusin area function and g^*_λ functions, have been established in [13, 28, 29, 31, 33]. Moreover, the Besov-type and the Triebel–Lizorkin-type spaces, including some of their special cases related to Q spaces, have been used to study the existence and the regularity of solutions of some partial differential equations such as (fractional) Navier-Stokes equations; see, for instance, [10–12, 36, 37]. In recent years, ones also generalize Besov and Triebel-Lizorkin spaces by replacing the fundamental space $L^p(\mathbb{R}^n)$ by something more general, like a Lebesgue space with variable exponents (see, for instances, [25, 26, 34, 35]) or, more

Mathematics subject classification (2020): Primary 42B30; Secondary 42B25, 42B20, 42B35, 46E30.
Keywords and phrases: Slice space, Besov space, Triebel–Lizorkin space, atomic, molecular, Peetre maximal function, Lusin area function, Fourier multiplier.

This project is supported by the National Natural Science Foundation of China (Grant No. 12061069).
* Corresponding author.

© ELEMENT, Zagreb
Paper JMI-18-36

generally, a Musielak–Orlicz space [33]. We may consult the reader to [14, 30, 32] for more details.

Rencently, the slice space $E_t^p(\mathbb{R}^n)$ with $p,\, t \in (0,\infty)$ was originally introduced by Auscher and Mourgoglou [2] and has been applied to study the classification of weak solutions in the natural classes for the boundary value problems of a t-independent elliptic system in the upper plane. In 2017, Auscher and Prisuelos-Arribas [3] introduced a more general slice space $(E_r^p)_t(\mathbb{R}^n)$ with $q,\, r,\, t \in (0,\infty)$, and has been applied to study the boundedness of operators such as the Hardy–Littlewood maximal operator, the Calderón–Zygmund operator and the Riesz potential.

In this paper, we develop a theory of generalized Besov-type and Triebel–Lizorkin-type spaces which are built on slice spaces. Molecular and atomic characterizations, the Peetre maximal function characterizations of these spaces are also established in this article. As applications, we study the boundedness of Fourier multipliers on these new spaces.

We first introduce some basic notation. In what follow, let $\mathbb{N} := \{1,\, 2,\, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ with the classical topology and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual spaces, namely, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ equipped with the weak-$*$ topology. For any $N \in \mathbb{Z}_+$, the space $\mathcal{S}_N(\mathbb{R}^n)$ is defined to be the set of all Schwartz functions satisfying that, for all multi-indices $\gamma := (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n \leqslant N$, $\int_{\mathbb{R}^n} \varphi(x) x^\gamma\, dx = 0$, where, for all $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. We also let $\mathcal{S}_{-1}(\mathbb{R}^n) := \mathcal{S}(\mathbb{R}^n)$, for $N \in \mathbb{Z}_+ \cup \{-1\}$. Let $\mathcal{S}_N'(\mathbb{R}^n)$ be the topological dual space of $\mathcal{S}_N(\mathbb{R}^n)$. Similarly, the space $\mathcal{S}_\infty(\mathbb{R}^n)$ is defined to be the set of all Schwartz functions satisfying that $\int_{\mathbb{R}^n} \varphi(x) x^\gamma\, dx = 0$ for all multi-indices $\gamma \in \mathbb{Z}_+^n$, and $\mathcal{S}_\infty'(\mathbb{R}^n)$ its topological dual space. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all polynomials on $\mathbb{R}^n$. For all $M \in \mathbb{Z}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\|\varphi\|_{\mathcal{S}_M(\mathbb{R}^n)} := \sup_{|\gamma| \leqslant M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma \varphi(x)|(1+|x|)^{n+M+|\gamma|}$, where, for any $\gamma := (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n$, $\partial^\gamma := (\frac{\partial}{\partial x_1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_n})^{\gamma_n}$.

Let ϕ and ψ be Schwartz functions on $\mathbb{R}^n$ satisfying that

$$\operatorname{supp} \widehat{\phi},\ \operatorname{supp} \widehat{\psi} \subset \{\xi \in \mathbb{R}^n :\ 1/2 \leqslant |\xi| \leqslant 2\}, \tag{1.1}$$

$$|\widehat{\phi}(\xi)|,\, |\widehat{\psi}(\xi)| \geqslant C > 0 \quad \text{if } 3/5 \leqslant |\xi| \leqslant 5/3 \tag{1.2}$$

and

$$\sum_{j \in \mathbb{Z}} \overline{\widehat{\phi}(2^j \xi)}\, \widehat{\psi}(2^j \xi) = 1 \quad \text{if } \xi \neq 0, \tag{1.3}$$

where, for any $f \in \mathcal{S}(\mathbb{R}^n)$ and for any $\xi \in \mathbb{R}^n$, $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$. Throughout this paper, for any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, we put

$$\psi_j(x) := 2^{jn} \psi(2^j x). \tag{1.4}$$

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Throughout this whole article, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The *symbols* $A \lesssim B$ means $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the *dyadic cube* $2^{-j}([0,1)^n + k)$, $x_{Q_{jk}} := 2^{-j}k$ its left corner and $\ell(Q_{jk})$ its *side length*. Let $\mathscr{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ and $j_Q := -\log_2 \ell(Q)$ for all $Q \in \mathscr{Q}$. If E is a subset of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its *characteristic function*.

Let $t, r, p \in (0,\infty)$, $q \in (0,\infty]$, $\alpha \in \mathbb{R}$ and $\tau \in [0,\infty)$. In Section 2, we introduce the slice Besov-type space $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and the slice Triebel–Lizorkin space $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ (see Definition 2.2 below), and establish their φ-transform T_ψ characterizations, which consequently shows that the spaces $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ are independent of the choice of admissible function ψ satisfying (1.1) and (1.2).

Section 3 is devoted to characterizing the spaces $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ via the Peetre maximal function in both discrete and continuous types. As applications of these characterizations, we obtain some embedding relations (see Proposition 3.1 below) among these spaces and show that slice Triebel–Lizorkin spaces include slice-Hardy spaces in [38] as special cases in Corollary 3.1 below.

In Section 4, we present some equivalent norm characterizations of these spaces for some special τ and obtain the generalized g^*_λ-function equivalent characterizations of $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ (see Theorem 4.2 below).

In Section 5, smooth atomic and molecular decompositions of these spaces are established by first considering the boundedness of almost diagonal operators on corresponding sequence spaces.

In Section 6, as applications, we study the mapping property of Fourier multipliers, with symbols satisfying some generalized Hörmander condition, on slice Besov-type and slice Triebel–Lizorkin-type spaces.

2. Slice Besov-type and Triebel–Lizorkin-type spaces

In this section, we first introduce slice Besov-type and slice Triebel–Lizorkin-type spaces and then establish their φ-transform characterizations. We begin with the notions of slice spaces [3], which is a generalization of the classical amalgam space $(L^p, \ell^q)(\mathbb{R})$ defined by Wiener [24] in 1926, in the formulation of his generalized harmonic analysis.

DEFINITION 2.1. Let $t, r, p \in (0,\infty)$. The *slice space* $(E^p_r)_t(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{(E^p_r)_t(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{B(x,t)}\|_{L^r(\mathbb{R}^n)}}{\|\mathbf{1}_{B(x,t)}\|_{L^r(\mathbb{R}^n)}} \right]^p dx \right\}^{\frac{1}{p}} < \infty.$$

REMARK 2.1. Let $t, r, p \in (0,\infty)$.

(i) Let $t = 1$. Then $(E^p_r)_t(\mathbb{R}^n)$ is the Wiener amalgam space $(L^r, L^q)(\mathbb{R}^n)$ [24].

(ii) If $r = p$, from [38, Proposition 2.11(iii)], we know that $(E^p_r)_t(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ coincide with the same quasi-norms.

DEFINITION 2.2. Let $t, r, p \in (0,\infty)$, $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$. Let ψ be a Schwartz function satisfying (1.1) and (1.2).

(i) The *slice Triebel–Lizorkin-type* space $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left[\sum_{j=j_P}^\infty \left(2^{j\alpha} \left|\psi_j * f\right|\right)^q \right]^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)}$$

with suitable modification made when $q = \infty$, where the supremum is taken over all dyadic cubes P.

(ii) The *slice Besov-type* space $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left[\sum_{j=j_P}^\infty 2^{j\alpha q} \left\|\psi_j * f \mathbf{1}_P\right\|^q_{(E^p_r)_t(\mathbb{R}^n)} \right]^{1/q}$$

with suitable modification made when $q = \infty$, where the supremum is taken over all dyadic cubes P.

We also introduce their corresponding sequence spaces as follows.

DEFINITION 2.3. Let $t, r, p \in (0,\infty)$, $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$.

(i) The *slice sequence* space $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is defined to be the space of all sequences $u := \{u_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$ such that $\|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left[\sum_{Q\subset P, Q\in\mathcal{Q}} \left(|Q|^{-\alpha/n-1/2}|u_Q|\mathbf{1}_Q\right)^q \right]^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)}$$

where the supremum is taken over all dyadic cubes P.

(ii) The *slice sequence* space $(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is defined to be the space of all sequences $u := \{u_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$ such that $\|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left[\sum_{j=j_P}^\infty \left\| \sum_{\substack{\ell(Q)=2^{-j} \\ Q\subset P, Q\in\mathcal{Q}}} |Q|^{-\alpha/n-1/2}|u_Q|\mathbf{1}_Q\mathbf{1}_P \right\|^q_{(E^p_r)_t(\mathbb{R}^n)} \right]^{1/q}$$

where the supremum is taken over all dyadic cubes P.

For simplicity, in what follows, we always use $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ to denote either $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ or $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, and $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ to denote either $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ or $(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

Let us recall the notion on the φ-transform and its inverse (see, for example, [9]). Let φ and ψ satisfy (1.1) through (1.3). For $j\in\mathbb{Z}$ and $k\in\mathbb{Z}^n$, let $\psi_{Q_{jk}}(x):=|Q_{jk}|^{-1/2}\psi(2^jx-k)$, $x\in\mathbb{R}^n$. The *φ-transform S_φ* is the map taking each $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ to the sequence $S_\varphi f:=\{(S_\varphi f)_Q\}_{Q\in\mathcal{Q}}$ defined by $(S_\varphi f)_Q:=\langle f,\varphi_Q\rangle$ for any dyadic cubes Q. The *inverse φ-transform T_ψ* is the map taking a sequence $u:=\{u_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{C}$ to $T_\psi u:=\sum_{Q\in\mathcal{Q}}u_Q\psi_Q$. Then, we have the following φ-transform characterization.

THEOREM 2.1. *Let $t,r,p\in(0,\infty)$, $\alpha\in\mathbb{R}$, $\tau\in[0,\infty)$ and $q\in(0,\infty]$. Let φ and ψ be Schwartz functions satisfying (1.1) through (1.3). Then the operators $S_\varphi:(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\to(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $T_\psi:(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\to(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ are bounded. Furthermore $T_\psi\circ S_\varphi$ is the identity on $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.*

REMARK 2.2. Let t, r, p in Remark 2.1. Then Theorem 2.1 in this case is just [28, Theorem 3.1]; in particular, when $\tau=0$, Theorem 2.1 in this case is just [9, Theorem 3.3] and [8, Theorem 2.6].

To prove Theorem 2.1, we need some technical lemmas.

LEMMA 2.1. *Let $t,r,p\in(0,\infty)$. There exist two positive constants C_1 and C_2 such that, for any sequence $\{f_j\}_{j\in\mathbb{N}}\subset(E^p_r)_t(\mathbb{R}^n)$ with $\sum_{j\in\mathbb{N}}|f_j|\in(E^p_r)_t(\mathbb{R}^n)$,*

$$C_1\left[\sum_{j=1}^{\infty}\|f_j\|^{\frac{p}{\min\{1,p\}}}_{(E^p_r)_t(\mathbb{R}^n)}\right]^{\frac{\min\{1,,p\}}{p}}\leqslant\left\|\sum_{j=1}^{\infty}|f_j|\right\|_{(E^p_r)_t(\mathbb{R}^n)}\leqslant C_2\left[\sum_{j=1}^{\infty}\|f_j\|^{\frac{p}{\max\{1,p\}}}_{(E^p_r)_t(\mathbb{R}^n)}\right]^{\frac{\max\{1,p\}}{p}}. \tag{2.1}$$

Proof. Without loss of generality, we may assume that $\|f_j\|_{(E^p_r)_t(\mathbb{R}^n)}\neq0$ for all $j\in\mathbb{N}$. We first prove the second inequality of (2.1). Denote by $\vartheta:=\frac{p}{\max\{1,p\}}$. For $\vartheta\in(0,\min\{1,p\}]$. Using [38, Lemma 4.2], we write

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}|f_j|\right\|_{(E^p_r)_t(\mathbb{R}^n)}&=\left\|\left[\sum_{j=1}^{\infty}|f_j|\right]^{\vartheta}\right\|^{1/\vartheta}_{(E^{p/\vartheta}_{r/\vartheta})_t(\mathbb{R}^n)}\leqslant\left\|\sum_{j=1}^{\infty}|f_j|^{\vartheta}\right\|^{1/\vartheta}_{(E^{p/\vartheta}_{r/\vartheta})_t(\mathbb{R}^n)}\\&\leqslant\left[\sum_{j=1}^{\infty}\left\||f_j|^{\vartheta}\right\|_{(E^{p/\vartheta}_{r/\vartheta})_t(\mathbb{R}^n)}\right]^{1/\vartheta}=\left[\sum_{j=1}^{\infty}\|f_j\|^{\vartheta}_{(E^p_r)_t(\mathbb{R}^n)}\right]^{1/\vartheta},\end{aligned}$$

which completes the proof of the second inequality of (2.1).

Then we turn to show the first inequality of (2.1). Similarly denote by $\kappa := \frac{p}{\min\{1,p\}}$. Since $\kappa \in [\max\{1,p\},\infty)$, from [38, Lemma 5.4], we see

$$\left\|\sum_{j=1}^{\infty}|f_j|\right\|_{(E_r^p)_t(\mathbb{R}^n)} = \left\|\left[\sum_{j=1}^{\infty}|f_j|\right]^{\kappa}\right\|^{1/\kappa}_{(E_{r/\kappa}^{p/\kappa})_t(\mathbb{R}^n)} \geqslant \left\|\sum_{j=1}^{\infty}|f_j|^{\kappa}\right\|^{1/\kappa}_{(E_{r/\kappa}^{p/\kappa})_t(\mathbb{R}^n)}$$
$$\geqslant \left[\sum_{j=1}^{\infty}\left\||f_j|^{\kappa}\right\|_{(E_{r/\kappa}^{p/\kappa})_t(\mathbb{R}^n)}\right]^{1/\kappa} = \left[\sum_{j=1}^{\infty}\|f_j\|^{\kappa}_{(E_r^p)_t(\mathbb{R}^n)}\right]^{1/\kappa},$$

which yields the first inequality of (2.1) and hence completes the proof of Lemma 2.1. □

REMARK 2.3. Let $t, r, p \in (0,\infty)$. There exists a positive constant C such that, for any each other disjoint cubes $\{Q_j\}_{j\in\mathbb{N}}$,

$$\frac{1}{C}\left(\sum_{j=1}^{\infty}\|\mathbf{1}_{Q_j}\|^{p}_{(E_r^p)_t(\mathbb{R}^n)}\right)^{\frac{1}{p}} \leqslant \left\|\sum_{j=1}^{\infty}\mathbf{1}_{Q_j}\right\|_{(E_r^p)_t(\mathbb{R}^n)} \leqslant C\left(\sum_{j=1}^{\infty}\|\mathbf{1}_{Q_j}\|^{p}_{(E_r^p)_t(\mathbb{R}^n)}\right)^{\frac{1}{p}}$$

From Lemma 2.1, we can deduce the following properties. In what follows, the symbol $\subset$ stands for continuous embedding.

PROPOSITION 2.1. *Let $t, r, p \in (0,\infty)$. Let $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$ and $q, q_1, q_2 \in (0,\infty]$.*

(i) *If $q_1 \leqslant q_2$, then $(\dot{A}E^{\alpha,\tau}_{r,p,q_1})_t(\mathbb{R}^n) \subset (\dot{A}E^{\alpha,\tau}_{r,p,q_2})_t(\mathbb{R}^n)$.*

(ii)
$$\left(\dot{B}E^{\alpha,\tau}_{r,p,\frac{qp}{\max\{p,q\}}}\right)_t(\mathbb{R}^n) \subset (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \subset \left(\dot{B}E^{\alpha,\tau}_{r,p,\frac{qp}{\min\{p,q\}}}\right)_t(\mathbb{R}^n)$$

and

$$\left(\dot{b}E^{\alpha,\tau}_{r,p,\frac{qp}{\max\{p,q\}}}\right)_t(\mathbb{R}^n) \subset \left(\dot{f}E^{\alpha,\tau}_{r,p,q}\right)_t(\mathbb{R}^n) \subset \left(\dot{b}E^{\alpha,\tau}_{r,p,\frac{qp}{\min\{p,q\}}}\right)_t(\mathbb{R}^n).$$

Proof. As a special case, Liang, Yang, Yuan, Sawano and Ullrich obtained this property (i) in [14, Lemma 3.8]. It is pointed out here that the property (i) is a simple consequence of the inequality that, for any $\theta \in (0,1]$ and $\{a_j\}_j \subset \mathbb{C}$, $\left(\sum_j |a_j|\right)^{\theta} \leqslant \sum_j |a_j|^{\theta}$.

For property (ii), since the proof is similar, we only need to prove the first embedding $(\dot{B}E^{\alpha,\tau}_{r,p,\frac{qp}{\max\{p,q\}}})_t(\mathbb{R}^n) \subset (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Let $f \in (\dot{B}E^{\alpha,\tau}_{r,p,\frac{qp}{\max\{p,q\}}})_t(\mathbb{R}^n)$. From

Lemma 2.1, it follows that

$$\begin{aligned}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} &= \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \sum_{j=j_P}^{\infty} \left(2^{j\alpha}\left|\psi_j * f\right|\right)^q \right\|^{1/q}_{(E^{p/q}_{r/q})_t(P)} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} \left\| \left(2^{j\alpha}\left|\psi_j * f\right|\right)^q \right\|^{\frac{p/q}{\max\{1,p/q\}}}_{(E^{p/q}_{r/q})_t(P)} \right\}^{\frac{\max\{1,p/q\}}{p/q}\frac{1}{q}} \\ &= \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} \left\|2^{j\alpha}\left|\psi_j * f\right|\right\|^{\frac{qp}{\max\{p,q\}}}_{(E^p_r)_t(P)} \right\}^{\frac{\max\{p,q\}}{qp}} \\ &\sim \|f\|_{(\dot{B}E^{\alpha,\tau}_{r,p,\frac{qp}{\max\{p,q\}}})_t(\mathbb{R}^n)},\end{aligned}$$

which shows that $(\dot{B}E^{\alpha,\tau}_{r,p,\frac{qp}{\max\{p,q\}}})_t(\mathbb{R}^n) \subset (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Thus, we complete the proof of Proposition 2.1. □

The following lemma is a key tool used in this present article. We can get from [16, Lemma 2.9] with similar argument.

LEMMA 2.2. *Let $t, r, p \in (0,\infty)$. Then there exists a positive constant C such that, for all cubes $B_1 \subset B_2$,*

$$\frac{\|\mathbf{1}_{B_2}\|_{(E^p_r)_t(\mathbb{R}^n)}}{\|\mathbf{1}_{B_1}\|_{(E^p_r)_t(\mathbb{R}^n)}} \leqslant C\left(\frac{|B_2|}{|B_1|}\right)^{\frac{1}{p}} \quad \text{and} \quad \frac{\|\mathbf{1}_{B_1}\|_{(E^p_r)_t(\mathbb{R}^n)}}{\|\mathbf{1}_{B_2}\|_{(E^p_r)_t(\mathbb{R}^n)}} \leqslant C\left(\frac{|B_1|}{|B_2|}\right)^{\frac{1}{p}}.$$

COROLLARY 2.1. *Let $t, r, p \in (0,\infty)$. Then there exist positive constants C_1, C_2, C_3 and C_4 such that, for all dyadic cubes Q_{jk}, if $j \in \mathbb{Z}_+$, it holds that*

$$C_1 2^{-jn/p} \leqslant \|\mathbf{1}_{Q_{jk}}\|_{(E^p_r)_t(\mathbb{R}^n)} \leqslant C_2 2^{-jn/p} \tag{2.2}$$

and, if $j \in \mathbb{Z}\setminus\mathbb{Z}_+$, it holds that

$$C_3 2^{-jn/p} \leqslant \|\mathbf{1}_{Q_{jk}}\|_{(E^p_r)_t(\mathbb{R}^n)} \leqslant C_4 2^{-jn/p}. \tag{2.3}$$

REMARK 2.4. Let t, r, p be as in Remark 2.1(ii). We know that $(E^p_r)_t(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. In this case, since $\|\mathbf{1}_E\|_{(E^p_r)_t(\mathbb{R}^n)} = |E|^{1/p}$ for any measurable set $E \subset \mathbb{R}^n$, we find that Lemma 2.2 holds true immediately, and hence (2.2) and (2.3) in Corollary 2.1 are just $\|\mathbf{1}_{Q_{jk}}\|_{(E^p_r)_t(\mathbb{R}^n)} = 2^{-jn/p}$ for any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

Next we show that the inverse φ-transform T_ψ is well defined for any $u \in (\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

LEMMA 2.3. *Let $t, r, p \in (0,\infty)$. Let $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$ and $q, q_1, q_2 \in (0,\infty]$. If ψ satisfies (1.1) through (1.3), then for any $u \in (\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $T_\psi u := \sum_{Q\in\mathcal{Q}} u_Q \psi_Q$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, $T_\psi : (\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \to \mathcal{S}'_\infty(\mathbb{R}^n)$ is continuous.*

Proof. To prove Lemma 2.3, by Proposition 2.1(ii), it suffices to show that T_ψ is well defined on $(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

Let $u \in (\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. We will prove that there exists an $M \in \mathbb{Z}_+$ such that, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, $|T_\psi u(f)| \lesssim \|f\|_{\mathcal{S}_M(\mathbb{R}^n)}$. By Definition 2.3 (ii), it is easy to know that, for any cube $Q \in \mathcal{Q}$, $|u_Q| \leqslant \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} |Q|^{\alpha/n+1/2} \|\mathbf{1}_Q\|^{\tau-1}_{(E^p_r)_t(\mathbb{R}^n)}$. Then

$$\begin{aligned}
\left|T_\psi u(f)\right| &\leqslant \sum_{Q\in\mathcal{Q}} |u_Q|\,|\langle\psi_Q,f\rangle| \leqslant \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sum_{Q\in\mathcal{Q}} |Q|^{\alpha/n+1/2} \|\mathbf{1}_Q\|^{\tau-1}_{(E^p_r)_t(\mathbb{R}^n)} |\langle\psi_Q,f\rangle| \\
&\leqslant \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sum_{j\in\mathbb{Z}_+} \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)=2^{-j}}} |Q|^{\alpha/n+1/2} \|\mathbf{1}_Q\|^{\tau-1}_{(E^p_r)_t(\mathbb{R}^n)} |\langle\psi_Q,f\rangle| \\
&\quad + \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sum_{j\in\mathbb{Z}\setminus\mathbb{Z}_+} \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)=2^{-j}}} |Q|^{\alpha/n+1/2} \|\mathbf{1}_Q\|^{\tau-1}_{(E^p_r)_t(\mathbb{R}^n)} |\langle\psi_Q,f\rangle| \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

To estimate the first term I_1, we need the following inequality proved in [28, p. 459]: for any large enough $L \in (0,\infty)$, there exists $M \in \mathbb{N}$ such that, for any $Q = Q_{jk} \in \mathcal{Q}$,

$$|\langle\psi_Q,f\rangle| \lesssim \|f\|_{\mathcal{S}_M(\mathbb{R}^n)} \left(1+\frac{|x_Q|^n}{\max\{1,|Q|\}}\right)^{-L} \left(\min\{2^{-jn},2^{jn}\}\right)^L,$$

where x_Q denotes the lower left-corner $2^{-j}k$ of $Q := Q_{jk}$. Then, by (2.3), we know that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \|f\|_{\mathcal{S}_M(\mathbb{R}^n)} \sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathbb{Z}^n} 2^{-j(\alpha+\frac{n}{2})} 2^{-jn\left(\frac{\tau}{p}-\frac{1}{p}\right)} \left(2^{jn}+|k|^n\right)^{-L} \\
&\lesssim \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \|f\|_{\mathcal{S}_M(\mathbb{R}^n)} \\
&\quad \times \left\{ \sum_{j\in\mathbb{Z}_+} 2^{-j(\alpha+\frac{n}{2})-jn\left(\frac{\tau}{p}-\frac{1}{p}\right)-jnL} \sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathbb{Z}^n\setminus\{\vec{0}_n\}} 2^{-j(\alpha+\frac{n}{2})-jn\left(\frac{\tau}{p}-\frac{1}{p}\right)-jnL/2} |k|^{-nL/2} \right\} \\
&\lesssim \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \|f\|_{\mathcal{S}_M(\mathbb{R}^n)},
\end{aligned}$$

where L is chosen large enough such that the above series converge. By (2.3) and an argument similar to the above, we also conclude that $\mathrm{I}_2 \lesssim \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \|f\|_{\mathcal{S}_M(\mathbb{R}^n)}$, which, together with the estimate for I_1, implies that

$$\left|T_\psi u(f)\right| \lesssim \|u\|_{(\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \|f\|_{\mathcal{S}_M(\mathbb{R}^n)}.$$

Therefore, $T_\psi u = \sum_{Q\in\mathcal{Q}} u_Q \psi_Q$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$, which completes the proof of Lemma 2.3. $\square$

Let $t,\ s,\ r,\ p \in (0,\infty)$. The space $((E^p_r)_t(\mathbb{R}^n),\ell^s)$ is defined to be the set of all $\{f_j\}_{j\in\mathbb{Z}}$ such that $\|\{f_j\}_{j\in\mathbb{Z}}\|_{\ell^s} \in (E^p_r)_t(\mathbb{R}^n)$ endowed with the quasi-norm

$$\left\|\{f_j\}_{j\in\mathbb{Z}}\right\|_{((E^p_r)_t(\mathbb{R}^n),\ell^s)} := \left\| \left[\sum_{j\in\mathbb{Z}} \left|f_j\right|^s\right]^{1/s} \right\|_{(E^p_r)_t(\mathbb{R}^n)} < \infty.$$

Recall that, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Hardy–Littlewood maximal operator $\mathcal{M}$ is defined by setting

$$\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\,dy,$$

where the supremum is taken over all cubes B of $\mathbb{R}^n$.

The following slice Fefferman–Stein vector-valued inequality was proved in [38].

LEMMA 2.4. *Let $t \in (0,\infty)$ and $s, r, p \in (1,\infty)$. There exists a positive constant C such that, for any $\{f_j\}_{j\in\mathbb{Z}} \in ((E_r^p)_t(\mathbb{R}^n), \ell^s)$,*

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} [\mathcal{M}(f_j)]^s \right\}^{1/s} \right\|_{(E_r^p)_t(\mathbb{R}^n)} \leqslant C \left\| \left[\sum_{j\in\mathbb{Z}} |f_j|^s \right]^{1/s} \right\|_{(E_r^p)_t(\mathbb{R}^n)}.$$

For $u := \{u_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$, $\theta \in (0,\infty)$ and $\lambda \in (n,\infty)$, let $u^*_{\theta,\lambda} := \{(u^*_{\theta,\lambda})_Q\}_{Q\in\mathcal{Q}}$, where, for $Q \in \mathcal{Q}$,

$$(u^*_{\theta,\lambda})_Q := \left[\sum_{\{R\in\mathcal{Q}:\ \ell(R)=\ell(Q)\}} \frac{|u_R|^\theta}{(1+[\ell(Q)]^{-1}|x_R - x_Q|)^\lambda} \right]^{1/\theta}.$$

Next we establish the following technical lemma.

LEMMA 2.5. *Let t, r, $p \in (0,\infty)$, $\alpha \in \mathbb{R}$, $q \in (0,\infty]$ and $\lambda \in (n,\infty)$. Then there exists a positive constant C such that, for any $u := \{u_Q\}_{Q\in\mathcal{Q}} \in (\dot{a}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)$,*

$$\|u\|_{(\dot{a}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \left\| u^*_{\min\{p,q\},\lambda} \right\|_{(\dot{a}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C \|u\|_{(\dot{a}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)}.$$

Proof. By similarity, we only give the proof of Lemma 2.5 for the space $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Since $|u_Q| \leqslant (u^*_{\min\{p,q\},\lambda})_Q$ for any dyadic cube Q, it immediately deduces that $\|u\|_{(\dot{f}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|u^*_{\min\{p,q\},\lambda}\|_{(\dot{f}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)}$.

Conversely, let $\eta := \min\{p,q\}$ and $a := \frac{1}{2}\eta(n/\lambda+1)$. Then $a \in (\eta/2,\eta)$ and $\lambda \in (n\eta/a,\infty)$. Hence, by [9, Lemma A.2], we know that, for any $j \in \mathbb{Z}$,

$$\sum_{\substack{\ell(Q)=2^{-j}\\ Q\in\mathcal{Q}}} (u^*_{\eta,\lambda})_Q |Q|^{-\alpha/n-1/2} \mathbf{1}_Q \lesssim \left[\mathcal{M}\left(\left[\sum_{\substack{\ell(P)=2^{-j}\\ P\in\mathcal{Q}}} |u_P||P|^{-\alpha/n-1/2} \mathbf{1}_P \right]^a \right) \right]^{1/a}.$$

From this, $a\in(0,\eta)$ and Lemma 2.4, it follows that

$$
\begin{aligned}
&\left\|\left[\sum_{j\in\mathbb{Z}}\left(\sum_{\substack{\ell(Q)=2^{-j}\\ Q\in\mathcal{Q}}}|Q|^{-\alpha/n-1/2}(u^*_{\eta,\lambda})_Q\mathbf{1}_Q\right)^q\right]^{1/q}\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\left[\sum_{\substack{\ell(P)=2^{-j}\\ P\in\mathcal{Q}}}|u_P||P|^{-\alpha/n-1/2}\mathbf{1}_P\right]^a\right)\right]^{q/a}\right\}^{a/q}\right\|^{1/a}_{(E^{p/a}_{r/a})_t(\mathbb{R}^n)}\\
&\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\sum_{\substack{\ell(P)=2^{-j}\\ P\in\mathcal{Q}}}|u_P||P|^{-\alpha/n-1/2}\mathbf{1}_P\right]^q\right\}^{a/q}\right\|^{1/a}_{(E^{p/a}_{r/a})_t(\mathbb{R}^n)}\\
&=\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{\substack{\ell(P)=2^{-j}\\ P\in\mathcal{Q}}}\left[|u_P||P|^{-\alpha/n-1/2}\mathbf{1}_P\right]^q\right\}^{1/q}\right\|_{(E^p_r)_t(\mathbb{R}^n)},
\end{aligned}
$$

which implies that $\left\|u^*_{\min\{p,q\},\lambda}\right\|_{(\dot{a}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)}\leqslant C\|u\|_{(\dot{a}E^{\alpha,0}_{r,p,q})_t(\mathbb{R}^n)}$ and hence completes the proof of Lemma 2.5. □

By Lemmas 2.5 and 2.2, we conclude the following result.

LEMMA 2.6. *Let* $t,r,\ p\in(0,\infty)$, $\alpha\in\mathbb{R}$, $\tau\in[0,\infty)$ *and* $q\in(0,\infty]$. *If* $\lambda\in(n,\infty)$, *then there exists a positive constant* C *such that, for any* $u:=\{u_Q\}_{Q\in\mathcal{Q}}\in(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$,

$$\|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant\left\|u^*_{\min\{p,q\},\lambda}\right\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant C\|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}. \tag{2.4}$$

Proof. Our proof of this lemma is similar to the proof of [28, Lemma 3.3]. We only prove Lemma 2.6 for the space $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. By the fact that $|u_Q|\leqslant(u^*_{\min\{p,q\},\lambda})_Q$, for any dyadic cube Q, it is easy to know that $\|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant\|u^*_{\min\{p,q\},\lambda}\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.

To prove the second inequality of (2.4), for any given dyadic cube P, let $v:=\{v_Q\}_{Q\in\mathcal{Q}}$ and $w:=\{w_Q\}_{Q\in\mathcal{Q}}$, where $v_Q:=u_Q$ if $Q\subset 3P$ and $v_Q:=0$ otherwise, and, for any cube Q, $w_Q:=u_Q-v_Q$. Then, for any dyadic cube Q, we have

$$\left(u^*_{\min\{p,q\},\lambda}\right)_Q\lesssim\left(v^*_{\min\{p,q\},\lambda}\right)_Q+\left(w^*_{\min\{p,q\},\lambda}\right)_Q. \tag{2.5}$$

By Lemmas 2.5 and 2.2, we find that

$$\begin{aligned}
\mathrm{I_P} &:= \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{\substack{Q\subset P\\ Q\in\mathscr{Q}}} \left[|Q|^{-\alpha/n-1/2} \left(v^*_{\min\{p,q\},\lambda} \right)_Q \mathbf{1}_Q \right]^q \right\}^{1/q} \right\|_{(E^p_r)_t(\mathbb{R}^n)} \\
&\lesssim \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{\substack{Q\subset 3P\\ Q\in\mathscr{Q}}} \left[|Q|^{-\alpha/n-1/2} |v_Q| \mathbf{1}_Q \right]^q \right\}^{1/q} \right\|_{(E^p_r)_t(\mathbb{R}^n)} \\
&\lesssim \|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.
\end{aligned}$$

Now it remains to deal with $w^*_{\min\{p,q\},\lambda}$. For any $i\in\mathbb{Z}_+$, $k\in\mathbb{Z}^n$ with $|k|\geqslant 2$ and dyadic cube P, let $A(i,k,P) := \{R\in\mathscr{Q}:\ \ell(R)=2^{-i}\ell(P),\ R\subset P+k\ell(P),\ R\cap(3P)=\emptyset\}$. Note that, for any dyadic cube $Q\subset P$ and $R\in A(i,k,P)$, $1+[\ell(R)]^{-1}|x_Q-x_R|\sim 2^i|k|$. Then, by an argument similar to that used in the proof of [28, Lemma 3.3] (see also [9, Lemma A.2]), we know that, for any $x\in P$ and $a\in(0,\min\{p,q\}]$,

$$\begin{aligned}
&\sum_{R\in A(i,k,P)} \frac{(|R|^{-\alpha/n-1/2}|u_R|)^{\min\{p,q\}}}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{\lambda}} \\
&\lesssim (2^i)^{-\lambda+n\min\{p,q\}/a}|k|^{-\lambda} \left[\mathcal{M}\left(\sum_{\substack{\ell(R)=2^{-i}\ell(P)\\ R\subset P+k\ell(P)}} \left[|R|^{-\alpha/n-1/2}|u_R|\mathbf{1}_R \right]^a \right)(x+k\ell(P)) \right]^{\frac{\min\{p,q\}}{a}},
\end{aligned}$$

which further implies that

$$\begin{aligned}
\mathrm{J_P} &:= \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{\substack{Q\subset P\\ Q\in\mathscr{Q}}} \left[|Q|^{-\alpha/n-1/2} \left(w^*_{\min\{p,q\},\lambda} \right)_Q \mathbf{1}_Q \right]^q \right\}^{1/q} \right\|_{(E^p_r)_t(\mathbb{R}^n)} \\
&\lesssim \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{i=0}^{\infty} \left[\sum_{\substack{k\in\mathbb{Z}^n\\ |k|\geqslant 2}} \sum_{R\in A(i,k,P)} \frac{(|R|^{-\alpha/n-1/2}|w_Q|)^{\min\{p,q\}}}{[1+\ell(R)^{-1}|x_Q-x_R|]^{\lambda}} \right]^{\frac{q}{\min\{p,q\}}} \right\}^{\frac{1}{q}} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)}
\end{aligned}$$

$$\lesssim \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{i=0}^{\infty} \left[\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geqslant 2}} \left(2^i\right)^{-\lambda+\frac{n\min\{p,q\}}{a}} |k|^{-\lambda} \right.\right.\right.$$

$$\left.\left.\left. \times \left\{ \mathcal{M}\left(\sum_{\substack{\ell(R)=2^{-i}\ell(P) \\ R\subset P+k\ell(P)}} \left[|R|^{-\alpha/n-1/2}|u_R|\mathbf{1}_R \right]^a \right) \right\}^{\frac{\min\{p,q\}}{a}} \right]^{\frac{q}{\min\{p,q\}}} \right\}^{\frac{1}{q}} \right\|_{(E^p_r)_t(\mathbb{R}^n)}.$$

Choosing $a := \frac{2n\min\{p,q\}}{n+\lambda}$, we easily see that $a \in (0,\min\{p,q\})$. Then, from this, Lemma 2.4, we further deduce that

$$\mathrm{J_P} \lesssim \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\{ \sum_{i=0}^{\infty} \left\| \left[\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geqslant 2}} \left(2^i\right)^{-\lambda+\frac{n\min\{p,q\}}{a}} |k|^{-\lambda} \right.\right.\right.$$

$$\left.\left.\left. \times \left\{ \sum_{\substack{\ell(R)=2^{-i}\ell(P) \\ R\subset P+k\ell(P)}} \left[|R|^{-\alpha/n-1/2}|u_R|\mathbf{1}_R \right]^a \right\}^{\frac{\min\{p,q\}}{a}} \right]^{\frac{a}{\min\{p,q\}}} \right\|^{\frac{qp}{a\max\{p,q\}}}_{(E^{p/a}_{r/a})_t(\mathbb{R}^n)} \right\}^{\frac{\max\{p,q\}}{qp}}.$$

From Lemmas 2.1 and 2.2, we can deduce that

$$\mathrm{J_P} \lesssim \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left[\sum_{i=0}^{\infty} \left\{ \sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geqslant 2}} \left[\left(2^i\right)^{-\lambda+\frac{n\min\{p,q\}}{a}} |k|^{-\lambda} \right] \right.\right.$$

$$\left.\left. \times \left\| \left\{ \sum_{\substack{\ell(R)=2^{-i}\ell(P) \\ R\subset P+k\ell(P)}} \left[|R|^{-\alpha/n-1/2}|u_R|\mathbf{1}_R \right]^q \right\}^{\frac{1}{q}} \right\|^{\min\{p,q\}}_{(E^p_r)_t(\mathbb{R}^n)} \right\}^{\frac{qp}{\min\{p,q\}\max\{p,q\}}} \right]^{\frac{\max\{p,q\}}{qp}}$$

$$\lesssim \|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \left[\sum_{i=0}^{\infty} \left\{ \sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geqslant 2}} \left[\left(2^i\right)^{-\lambda+\frac{n\min\{p,q\}}{a}} |k|^{-\lambda} \right] \right.\right.$$

$$\left.\left. \times \left[\frac{\|\mathbf{1}_{P+k\ell(P)}\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \right]^{\min\{p,q\}} \right\}^{\frac{qp}{\min\{p,q\}\max\{p,q\}}} \right]^{\frac{\max\{p,q\}}{qp}}$$

$$\lesssim \|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \left[\sum_{i=0}^{\infty} \left\{ \sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geqslant 2}} \left(2^i\right)^{-\lambda+\frac{n\min\{p,q\}}{a}} |k|^{-\lambda} \right\}^{\frac{qp}{\min\{p,q\}\max\{p,q\}}} \right]^{\frac{\max\{p,q\}}{qp}}$$

$$\sim \|u\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

Finally, by (2.5), we obtain that

$$\left\| u^*_{\min\{p,q\},\lambda} \right\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \sup_{P\in\mathscr{Q}} (\mathrm{I_P} + \mathrm{J_P}) \lesssim \|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

Therefore, we complete the proof of Lemma 2.6. □

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1.1) through (1.3). For any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $Q \in \mathscr{Q}$ with $\ell(Q) = 2^{-j}$, define the *sequence* $\sup(f) := \{\sup_Q(f)\}_{Q\in\mathscr{Q}}$ by setting $\sup_Q(f) := |Q|^{1/2} \sup_{y\in Q} |\psi_j * f(y)|$ and, for any $\gamma \in \mathbb{Z}_+$, the sequence $\inf_\gamma(f) := \{\inf_{Q,\gamma}(f)\}_{Q\in\mathscr{Q}}$ by setting $\inf_{Q,\gamma}(f) := |Q|^{1/2} \max\{\inf_{y\in\widetilde{Q}} |\psi_j * f(y)| :\ \ell(\widetilde{Q}) = 2^{-\gamma}\ell(Q),\ \widetilde{Q} \subset Q\}$. As an argument similar to that used in the proof of [28, Lemma 3.4], we have the following lemma, the details being omitted.

LEMMA 2.7. *Let $t, r, p \in (0,\infty)$. Let $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$, $q \in (0,\infty]$ and $\gamma \in \mathbb{Z}_+$ be sufficiently large. Then there exists a constant $C \in [1,\infty)$ such that, for any $f \in (\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$,*

$$C^{-1} \left\|\inf_\gamma(f)\right\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|\sup(f)\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C \left\|\inf_\gamma(f)\right\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

With Lemmas 2.6 and 2.7, the proof of Theorem 2.1 follows the method pioneered by Frazier and Jawerth (see [9, pp. 50–51]). We omit the details.

In consequence of Theorem 2.1, we immediately obtain the following conclusion.

COROLLARY 2.2. *With all the notation as in Definition 2.2, the space $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is independent of the choice of ψ satisfying satisfying (1.1) and (1.2).*

3. Characterizations via Peetre maximal functions

In this section, we characterize the space $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ in terms of Peetre maximal functions in both continuous and discrete types. The characterization of $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ by means of the Lusin-area function is also obtained. As an application, we prove that $\mathcal{S}_\infty(\mathbb{R}^n) \subset (\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\phi * f$ makes sense. For any $s \in (0,\infty)$, $j \in \mathbb{Z}$, $a \in (0,\infty)$ and $x \in \mathbb{R}^n$, the Peetre maximal function $(\phi_t^* f)_a$ and $(\phi_j^* f)_a$ are defined by setting,

$$(\phi_t^* f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\phi_t * f(x+y)|}{(1+|y|/t)^a} \quad \text{and} \quad \left(\phi_j^* f\right)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\phi_j * f(x+y)|}{(1+2^j|y|)^a}.$$

where $\phi_s(\cdot) := s^{-n}\phi(s^{-1}\cdot)$ and ϕ_k is as in (1.4). Observing the above notation, we know that $(\phi_k^* f)_a(x) = (\phi_{2^{-k}}^* f)_a(x)$. Since this difference is always made clear in the context, we do not take care of this abuse of notation.

THEOREM 3.1. *Let $t, r, p \in (0,\infty)$, $\alpha + n\tau < R+1$, $R \in \mathbb{Z}_+ \cup \{-1\}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$. Let ψ be a Schwartz function satisfying (1.1) and (1.2). If*

$$a \in \left(\frac{n}{\min\{p,q\}}, \infty\right), \tag{3.1}$$

then the space $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is characterized by

$$(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'_\infty(\mathbb{R}^n) : \left\| f | (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \right\|_i < \infty \right\}, \quad i \in \{1,2,3,4\},$$

where

$$\left\| f | (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \right\|_1 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \int_0^{\ell(P)} s^{-\alpha q} |\psi_s * f|^q \frac{ds}{s} \right\}^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)},$$

$$\left\| f | (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \right\|_2 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \int_0^{\ell(P)} s^{-\alpha q} [(\psi_s^* f)_a]^q \frac{ds}{s} \right\}^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)},$$

$$\left\| f | (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \right\|_3 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \int_0^{\ell(P)} s^{-\alpha q} \int_{|z|<s} |\psi_s * f(\cdot+z)|^q \, dz \frac{ds}{s} \right\}^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)}$$

and

$$\left\| f | (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \right\|_4 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{j=j_P}^\infty 2^{\alpha j q} \left[\left(\psi_j^* f\right)_a \right]^q \right\}^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)}$$

with usual modification made when $q = \infty$.

The following estimate plays an vital role in the proof of Theorem 3.1, which was proved in [33, Lemma 3.2].

LEMMA 3.1. *Let* $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *satisfy* (1.1) *and* (1.2). *Then, for any* $s \in [1,2]$, $a \leqslant N$, $l \in \mathbb{Z}$ *and* $x \in \mathbb{R}^n$, *it holds that*

$$\left[(\phi^*_{2^{-l}s} f)_a(x)\right]^\theta \leqslant C_{(\theta)} \sum_{k=0}^{\infty} 2^{-kN\theta} 2^{(k+l)n} \int_{\mathbb{R}^n} \frac{|(\phi_{k+l} * f(y))|^\theta}{(1+2^l|x-y|)^{a\theta}}\, dy,$$

where θ *is an arbitrary fixed positive number and* $C_{(\theta)}$ *a positive constant independent of* ϕ, f, l, x *and* t, *but may depend on* θ.

Our proof of Theorem 3.1 is similar to the proofs of [13, Theorem 3.2] and [33, Theorem 3.1]. For completeness, we give the details.

Proof of Theorem 3.1. We first show that, for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1 \sim \left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2 \sim \left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_4 \sim \|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}, \tag{3.2}$$

where the implicit positive constants are independent of f.

Obviously, for any a, $s \in (0,\infty)$ and $x \in \mathbb{R}^n$, $|\psi_s * f(x)| \leqslant (\psi^*_s f)_a(x)$ and hence

$$\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1 \leqslant \left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2$$

and

$$\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_4.$$

Next we prove that $\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_2 \lesssim \|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_1$. To this end, by (3.1), we can choose a positive number θ such that

$$\frac{n}{a} < \theta < \min\{p,q\}. \tag{3.3}$$

Then from Lemma 3.1 and the Minkowski inequality, we deduce that

$$\begin{aligned}
\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2 &\leqslant \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{l=j_P}^{\infty} \int_1^2 2^{l\alpha q} \left[\sum_{k=0}^{\infty} 2^{-kN\theta} 2^{(k+l)n} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \int_{\mathbb{R}^n} \frac{|(\psi_{k+l})_s * f(y)|^\theta}{(1+2^l|\cdot - y|)^{a\theta}}\, dy \right]^{q/\theta} \frac{ds}{s} \right\}^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)} \\
&\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left\{ \sum_{l=j_P}^{\infty} \int_1^2 2^{l\alpha q} \left[\sum_{k=0}^{\infty} 2^{-kN\theta} 2^{(k+l)n} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \int_{\mathbb{R}^n} \frac{[\int_1^2 |(\psi_{k+l})_s * f(y)|^q \frac{ds}{s}]^{\theta/q}}{(1+2^l|\cdot - y|)^{a\theta}}\, dy \right]^{q/\theta} \right\}^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)},
\end{aligned}$$

where the natural number $N\in[a,\infty)$ is determined later. From [13, (3.6)], we know that, for any $P\in\mathcal{Q}$ and $x\in P$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{[\int_1^2|(\psi_{k+l})_s*f(y)|^q\frac{ds}{s}]^{\theta/q}}{(1+2^l|\cdot-y|)^{a\theta}}\,dy \\
&\quad\leqslant 2^{-ln}\mathcal{M}\left(\left[\int_1^2|(\psi_{k+l})_s*f|^q\frac{ds}{s}\right]^{\theta/q}\mathbf{1}_{3P}\right)(x)\\
&\qquad+\sum_{i\in\mathbb{Z}^n,\|i\|_{\ell^1}\geqslant 2}\|i\|_{\ell^1}^{-a\theta}2^{-la\theta}2^{j_P(a\theta-n)}\\
&\qquad\times\mathcal{M}\left(\left[\int_1^2|(\psi_{k+l})_s*f|^q\frac{ds}{s}\right]^{\theta/q}\mathbf{1}_{P+i\ell(P)}\right)(x)\\
&\quad=:\mathrm{I}_1+\mathrm{I}_2.
\end{aligned}\tag{3.4}$$

Let $\delta\in(0,\infty)$ and $N\in(\max\{a,\delta,\delta+n/\theta-\alpha\},\infty)$. By $\theta\in(0,\min\{p,q\})$, the Hölder inequality, Lemmas 2.4 and 2.2, we conclude that

$$\begin{aligned}
&\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{l=j_P}^{\infty}2^{l\alpha q}\left[\sum_{k=0}^{\infty}2^{-kN\theta}2^{(k+l)n}\mathrm{I}_1\right]^{q/\theta}\right\}^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{l=j_P}^{\infty}\sum_{k=0}^{\infty}2^{k[-(N-\delta)q+nq/\theta]-k\alpha q}\right.\right.\\
&\quad\left.\left.\times\left[\mathcal{M}\left(\left[\int_{2^{-k-l}}^{2^{-k-l+1}}s^{-\alpha q}|\psi_s*f|^q\frac{ds}{s}\right]^{\theta/q}\mathbf{1}_{3P}\right)\right]^{q/\theta}\right\}^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\\
&\quad\times\left\|\left\{\sum_{l=j_P}^{\infty}\sum_{k=0}^{\infty}2^{-k(N-\delta)q+knq/\theta}2^{-k\alpha q}\int_{2^{-k-l}}^{2^{-k-l+1}}s^{-\alpha q}|\psi_s*f|^q\frac{ds}{s}\right\}^{1/q}\mathbf{1}_{3P}\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\int_0^{2\ell(P)}s^{-\alpha q}|\psi_s*f|^q\frac{ds}{s}\right]^{1/q}\mathbf{1}_{3P}\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\lesssim\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1.
\end{aligned}\tag{3.5}$$

Similar to the estimate (3.5), by (3.1), Lemmas 2.1 and 2.2, we conclude that

$$\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{l=j_P}^{\infty}2^{l\alpha q}\left[\sum_{k=0}^{\infty}2^{-kN\theta}2^{(k+l)n}\mathrm{I}_2\right]^{q/\theta}\right\}^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)} \tag{3.6}$$

$$\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{i\in\mathbb{Z}^n,\|i\|_{\ell^1}\geqslant 2}\|i\|_{\ell^1}^{-a\theta}\left(\sum_{l=j_P}^{\infty}2^{l\alpha q}\right.\right.\right.$$

$$\left.\left.\left.\times\left[\sum_{k=0}^{\infty}2^{-kN\theta+kn}\mathcal{M}\left(\left[\int_1^2|(\psi_{k+l})_s*f|^q\frac{ds}{s}\right]^{\theta/q}\mathbf{1}_{P+i\ell(P)}\right)\right]^{q/\theta}\right)^{\theta/q}\right\}^{1/\theta}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}$$

$$\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\{\sum_{i\in\mathbb{Z}^n,\|i\|_{\ell^1}\geqslant 2}\|i\|_{\ell^1}^{-a\theta}\left\|\left[\sum_{l=j_P}^{\infty}2^{l\alpha q}\sum_{k=0}^{\infty}2^{-k(N-\delta)q+knq/\theta}\right.\right.\right.$$

$$\left.\left.\left.\times\left\{\mathcal{M}\left(\left[\int_1^2|(\psi_{k+l})_s*f|^q\frac{ds}{s}\right]^{\theta/q}\mathbf{1}_{P+i\ell(P)}\right)\right\}^{q/\theta}\right]^{\theta/q}\right\|_{(E^{p/\theta}_{r/\theta})_t(P)}\right\}^{\frac{1}{\theta}}.$$

By Lemma 2.4 again, we have

$$\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{l=j_P}^{\infty}2^{l\alpha q}\left[\sum_{k=0}^{\infty}2^{-kN\theta}2^{(k+l)n}\mathrm{I}_2\right]^{q/\theta}\right\}^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}$$

$$\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\{\sum_{i\in\mathbb{Z}^n,\|i\|_{\ell^1}\geqslant 2}\|i\|_{\ell^1}^{-a\theta}\left\|\left[\sum_{l=j_P}^{\infty}2^{l\alpha q}\sum_{k=0}^{\infty}2^{-k(N-\delta)q+knq/\theta}\right.\right.\right.$$

$$\left.\left.\left.\times\int_1^2|(\psi_{k+l})_s*f|^q\frac{ds}{s}\mathbf{1}_{P+i\ell(P)}\right]^{1/q}\mathbf{1}_P\right\|^{\theta}_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{\frac{1}{\theta}}$$

$$\lesssim\left\{\sum_{i\in\mathbb{Z}^n,\|i\|_{\ell^1}\geqslant 2}\|i\|_{\ell^1}^{-a\theta}\left[\sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_{P+i\ell(P)}\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\right.\right.$$

$$\left.\left.\times\left\|\left\{\int_0^{2\ell(P)}s^{-\alpha q}|\psi_s*f|^q\frac{ds}{s}\right\}^{1/q}\right\|_{(E^p_r)_t(P+i\ell(P))}\right]^{\theta}\right\}^{\frac{1}{\theta}}$$

$$\lesssim\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1.$$

Combining the estimates (3.5) and (3.6), we have

$$\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2 \lesssim \left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1 .$$

With slight modifications of the above argument, we also conclude that

$$\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2 \lesssim \|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$$

and

$$\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_4 \lesssim \left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1 ,$$

which yields (3.2).

Next we prove that $\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_2 \sim \|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_3$. In fact, we only need to prove that $\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_2 \lesssim \|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_3$, since the inverse inequality is trivial.

For any $x,\ y \in \mathbb{R}^n$, $k \in \mathbb{Z}_+$ and $l \in \mathbb{Z}$, we have $1+2^l|x-y| \lesssim 1+2^l|x-(y+z)|$, whenever $t \in [1,2]$ and $|z| < 2^{-(k+l)}t$. By this, Lemma 3.1 and the Minkowski inequality, it was proved in [33, p. 121] (see also, [13, (3.9)]) that

$$\begin{aligned}
&\int_1^2 \left[(\psi^*_{2^{-l}s}f)_a(x)\right]^q \frac{ds}{s}\\
&\lesssim \left\{\sum_{k=0}^{\infty} 2^{-kN\theta+(k+l)n}2^{(k+l)n\theta/q}\int_{\mathbb{R}^n}\frac{[\int_1^2\int_{|z|\leqslant 2^{-(k+l)}s}|(\psi_{k+l})_s*f(y+z)|^q\,dz\frac{ds}{s}]^{\theta/q}}{(1+2^l|x-y|)^{a\theta}}\,dy\right\}^{q/\theta},
\end{aligned}$$

which, together with

$$\begin{aligned}
&\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2\\
&\quad\lesssim \sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{l=j_P}^{\infty}2^{l\alpha q}\int_1^2\left[(\psi^*_{2^{-l}s}f)_a\right]^q\frac{ds}{s}\right\}^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)},
\end{aligned}$$

and the Hölder inequality, implies that

$$\begin{aligned}
&\left\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2\\
&\quad\lesssim \sup_{P\in\mathcal{Q}}\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{l=j_P}^{\infty}2^{l\alpha q+2lnq/\theta}\sum_{k=0}^{\infty}2^{-k(N-\delta)q+2knq/\theta}\left\{\int_{\mathbb{R}^n}\frac{1}{(1+2^l|\cdot-y|)^{a\theta}}\right.\right.\right.\\
&\quad\left.\left.\left.\times\left[\int_1^2\int_{|z|\leqslant 2^{-(k+l)}s}|(\psi_{k+l})_s*f(y+z)|^q\,dz\frac{ds}{s}\right]^{\theta/q}dy\right\}^{q/\theta}\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)},
\end{aligned}$$

where $\delta \in (0,\infty)$, $N \in (\max\{a,\delta\},\infty)$ and θ is as in (3.3).

From [13, pp. 1080–1081], we know that, for any $x \in P$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{(1+2^l|x-y|)^{a\theta}} \left[\int_1^2 \int_{|z|\leqslant 2^{-(k+l)}s} |(\psi_{k+l})_s * f(y+z)|^q\, dz \frac{ds}{s}\right]^{\theta/q} dy \\
&\lesssim 2^{-ln} \mathcal{M}\left(\left[\int_1^2 \int_{|z|<2^{-(k+l)}s} |(\psi_{k+l})_s * f(\cdot+z)|^q\, dz \frac{ds}{s}\right]^{\theta/q} \mathbf{1}_{3P}\right)(x) \\
&\quad + \sum_{i\in\mathbb{Z}^n, \|i\|_{\ell^1}\geqslant 2} \|i\|_{\ell^1}^{-a\theta} 2^{-(l-j_P)(a\theta-n)} 2^{-ln} \\
&\quad \times \mathcal{M}\left(\left[\int_1^2 \int_{|z|<2^{-(k+l)}s} |(\psi_{k+l})_s * f(\cdot+z)|^q\, dz \frac{ds}{s}\right]^{\theta/q} \mathbf{1}_{P+i\ell(P)}\right)(x).
\end{aligned}$$

Then, applying (3.1), (3.3), Lemmas 2.1, 2.2 and 2.4, by an argument similar to that used in the estimates (3.5) and (3.6), we further obtain that $\|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_2 \lesssim \|f|(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\|_3$. This finishes the proof of Theorem 3.1. □

We remark that the approach used in the proof of Theorem 3.1 is originated from Ullrich [23], which is further traced back to Bui, Paluszyński and Taibleson [4, 5] and, especially, Rychkov [17].

The slice Besov-type space $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ also have the following characterizations similar to those of $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ as in Theorem 3.1, whose proofs are also similar to that of Theorem 3.1. We omit the details.

THEOREM 3.2. *Let $t, r, p \in (0,\infty)$, $\alpha + n\tau < R+1$, $R \in \mathbb{Z}_+ \cup \{-1\}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$. Let ψ be a Schwartz function satisfying* (1.1) *and* (1.2)*. If*

$$a \in \left(\frac{n}{p}, \infty\right), \tag{3.7}$$

then the space $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ is characterized by

$$(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) = \left\{f \in \mathcal{S}'_\infty(\mathbb{R}^n) : \left\|f|(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_i < \infty\right\}, \quad i \in \{1,2,3\},$$

where

$$\left\|f|(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_1 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\{\int_0^{\ell(P)} s^{-\alpha q} \|\psi_s * f\mathbf{1}_P\|^q_{(E^p_r)_t(\mathbb{R}^n)} \frac{ds}{s}\right\}^{1/q},$$

$$\left\|f|(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_2 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\{\int_0^{\ell(P)} s^{-\alpha q} \|(\psi_s^* f)_a \mathbf{1}_P\|^q_{(E^p_r)_t(\mathbb{R}^n)} \frac{ds}{s}\right\}^{1/q},$$

and

$$\left\|f|(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)\right\|_3 := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\{\sum_{k=j_P}^\infty 2^{\alpha k q} \left\|(\psi_k^* f)_a \mathbf{1}_P\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{1/q}$$

with usual modification made when $q = \infty$.

REMARK 3.1. In the case that t, r, p are as in Remark 2.1(ii), Theorems 3.1 and 3.2 were proved in [13, Theorems 3.1 and 3.2].

We apply Theorems 3.1 and 3.2 to obtain the following imbedding conclusion.

PROPOSITION 3.1. *Let* $t, r, p \in (0,\infty)$. *Let* $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$ *and* $q \in (0,\infty]$. *Then* $\mathcal{S}_\infty(\mathbb{R}^n) \subset (\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$.

REMARK 3.2. It is pointed out here that our Proposition 3.1 is included in [13, Theorem 3.14]. For completeness, we give the details.

Proof of Proposition 3.1. To prove Proposition 3.1, by Proposition 2.1, we only need to prove that $\mathcal{S}_\infty(\mathbb{R}^n) \subset (\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$.

We first prove $\mathcal{S}_\infty(\mathbb{R}^n) \subset (\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Let $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ and ψ be a Schwartz function satisfying (1.1) and (1.2). From [27, Lemma 2.2], we know that, for any $M \in \mathbb{N}$, there exists a positive constant $C = C_{(M,n)}$ such that, for any $i,\ j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\left|\psi_j * f(x)\right| \leqslant C\|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}2^{-|j|M}\frac{2^{\min\{0,j\}M}}{(2^{-\min\{0,j\}}+|x|)^{n+M}}. \tag{3.8}$$

Then we devote to showing $f \in (\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Fix the dyadic cube $P := P_{j_P k_P}$ with $j_P \in \mathbb{Z}$ and $k_P \in \mathbb{Z}^n$. We divide it into two case for j_P.

Case I: $j_P \geqslant 0$. In this case, for any $x \in P$, $1+|x| \sim 1+2^{-j_P}|k_P|$. Let $M \in \mathbb{N}\cap[\alpha,\infty)$ satisfying that $\frac{M}{2} > \alpha + \frac{n\tau}{p} - \frac{n}{p}$. Then, applying (3.8) and Corollary 2.1, we see

$$\begin{aligned}
\mathrm{J_P} &:= \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left\{\sum_{j=j_P}^\infty 2^{j\alpha q}\left\|\psi_j * f\mathbf{1}_P\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{1/q}\\
&\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\\
&\quad\times\left\{\sum_{j=j_P}^\infty 2^{j(\alpha-M)q}\left\|(1+|\cdot|)^{-(n+M)}\mathbf{1}_P\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{1/q}\\
&\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\left\{\sum_{j=j_P}^\infty 2^{j(\alpha-M)q}\right\}^{1/q}2^{-\frac{j_P M}{2}}(1+|k_P|)^{-\frac{M}{2}}\frac{\|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)}}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\\
&\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}2^{j_P[\alpha-\frac{M}{2}+\frac{n\tau}{p}-\frac{n}{p}]}(1+|k_P|)^{-\frac{M}{2}}\\
&\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}.
\end{aligned}$$

Case II: $j_P < 0$. If P is away from the original point, then $|k_P| \geqslant 1$ and, for any $x \in P$, $|x| \sim 2^{-j_P}|k_P| \gtrsim 1$. Hence it is easy to see that, for any $x \in P$ and $j \in \mathbb{Z}$ with $j_P \leqslant j \leqslant -1$,

$$\max\left\{\frac{1}{1+|x|}, \frac{1}{2^{-j}+|x|}\right\} \lesssim \frac{2^{j_P}}{|k_P|} \lesssim \frac{2^{j_P}}{1+|k_P|}.$$

By this, (3.8) and Corollary 2.1, we see

$$\begin{aligned} \mathrm{J_P} &\lesssim \frac{\|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\{\sum_{j=0}^{\infty} 2^{j(\alpha-M)q}\left\|\frac{1}{(1+|\cdot|)^{n+M}}\mathbf{1}_P\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right. \\ &\quad \left. + \sum_{j=j_P}^{-1} 2^{j\alpha q}\left\|\frac{1}{(2^{-j}+|\cdot|)^{n+M}}\mathbf{1}_P\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{1/q} \\ &\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)} \\ &\quad \times \left\{2^{j_P[M+n-\frac{n}{p}+\frac{n\tau}{p}]}(1+|k_P|)^{-M-n} + 2^{j_P[M+n-\frac{n}{p}+\frac{n\tau}{p}]}(1+|k_P|)^{-M-n}\right\} \\ &\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}, \end{aligned} \tag{3.9}$$

when we choose $M \in \mathbb{Z}\cap[\alpha,\infty)$ such that $M > \frac{n}{p} - \frac{n\tau}{p} - n$.

If the original point falls into the closure of P, then we can easily see that, for any $i \in \{1,\cdots,-j_P+1\}$, $P \subset \cup_{i=0}^{-j_P+1} S_i$, where $S_0 := B(\vec{0}_n,\sqrt{n})$ and $S_i := 2^i S_0 \setminus 2^{i-1}S_0$. Notice that $\{S_i\}_{i=0}^{-j_P+1}$ are disjoint. Then we have

$$\left\|\frac{1}{(1+|\cdot|)^{n+M}}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)} \lesssim \left\{\sum_{i=0}^{-j_P+1}\left\|\frac{1}{(1+|\cdot|)^{n+M}}\mathbf{1}_{S_i}\right\|^p_{(E^p_r)_t(S_i)}\right\}^{\frac{1}{p}}.$$

By this and an argument similar to that used in the estimate (3.9), we conclude that

$$\begin{aligned} \mathrm{J_P} &\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\left\{\left[1+\sum_{i=1}^{-j_P+1}\left\|\frac{1}{(1+|\cdot|)^{n+M}}\right\|^p_{(E^p_r)_t(S_i)}\right]^{\frac{q}{p}}\right. \\ &\quad \left. + \sum_{j=j_P}^{-1} 2^{jsq}\left[2^{j(n+M)} + \sum_{i=1}^{-j_P+1}\left\|\frac{1}{(2^{-j}+|\cdot|)^{n+M}}\right\|^p_{(E^p_r)_t(S_i)}\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \\ &\lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}, \end{aligned}$$

when M is chosen large enough. Thus,

$$\|f\|_{(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} = \sup_{P\in\mathcal{Q}} \mathrm{J_P} \lesssim \|f\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|\psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)},$$

which implies that $\mathcal{S}_\infty(\mathbb{R}^n) \subset (\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

The second step is the proof of $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$. It needs to prove that there exists an $M \in \mathbb{N}$ such that, for any $f \in (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $\Psi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$|\langle f,\Psi\rangle| \lesssim \|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\|\Psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}.$$

Let ψ and ϕ be two Schwartz functions satisfying (1.1) through (1.3). Then by [27, Lemma 2.1] and (3.8), we know that

$$\begin{aligned}|\langle f,\Psi\rangle| &\leqslant \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\left|\psi_j*\Psi(x)\right|\left|\phi_j*f(x)\right|dx\\ &\lesssim \|\Psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\left\{\sum_{j\in\mathbb{Z}_+}\int_{\mathbb{R}^n}\frac{2^{-jM}|\phi_j*f(x)|}{(1+|x|)^{n+M}}\,dx+\sum_{j\in\mathbb{Z}\setminus\mathbb{Z}_+}\int_{\mathbb{R}^n}\frac{|\phi_j*f(x)|}{(2^{-j}+|x|)^{n+M}}\,dx\right\}\\ &=: \|\Psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\,(\mathrm{I}_1+\mathrm{I}_2).\end{aligned}$$

We first estimate I_1. For any $j\in\mathbb{Z}_+$, $k\in\mathbb{Z}^n$, $a\in(0,\infty)$ and $y\in Q_{jk}$, by the definition of $(\phi_j^*)_a$, there exists a positive constant C independent of y such that

$$\int_{Q_{0k}}\left|\phi_j*f(x)\right|dx\leqslant\left(\phi_j^*f\right)_a(y)\int_{Q_{0k}}\left(1+2^j|x|+2^j|y|\right)^a dx\leqslant C2^{ja}\left(\phi_j^*f\right)_a(y)(1+|k|)^a,$$

which implies that

$$\int_{Q_{0k}}\left|\phi_j*f(x)\right|dx\lesssim 2^{ja}(1+|k|)^a\inf_{y\in Q_{jk}}\left(\phi_j^*f\right)_a(y). \tag{3.10}$$

Let $M\in\mathbb{Z}$ satisfying that

$$M>\max\left\{a-\alpha+\frac{n}{p}-\frac{n\tau}{p},a\right\}.$$

Then, by (3.10), Corollary 2.1 and Theorem 3.1, we conclude that,

$$\begin{aligned}\mathrm{I}_1&\lesssim\sum_{j\in\mathbb{Z}_+}2^{-jM}\sum_{k\in\mathbb{Z}^n}\int_{Q_{0k}}\frac{|\phi_j*f(x)|}{(1+|k|)^{n+M}}\,dx\lesssim\sum_{j\in\mathbb{Z}_+}2^{-jM+ja}\sum_{k\in\mathbb{Z}^n}\frac{\inf_{y\in Q_{jk}}(\phi_j^*f)_a(y)}{(1+|k|)^{n+M-a}}\\ &\lesssim\sum_{j\in\mathbb{Z}_+}2^{-j(M-a)}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+M)+a}\frac{\|(\phi_j^*f)_a\|_{(E_r^p)_t(Q_{jk})}}{\|\mathbf{1}_{Q_{jk}}\|_{(E_r^p)_t(\mathbb{R}^n)}}\\ &\lesssim\sum_{j\in\mathbb{Z}_+}2^{-j(M-a)}2^{\frac{jn}{p}}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+M)+a}\left\|\left(\phi_j^*f\right)_a\right\|_{(E_r^p)_t(Q_{jk})}\\ &\lesssim\sum_{j\in\mathbb{Z}_+}2^{-j(M-a)}2^{\frac{jn}{p}}2^{-j\alpha}\left[\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+M)+a}\|\mathbf{1}_{Q_{jk}}\|_{(E_r^p)_t(\mathbb{R}^n)}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\right]\\ &\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\sum_{j\in\mathbb{Z}_+}2^{j[-M+a-\alpha+\frac{n}{p}-\frac{n\tau}{p}]}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-M+a}\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.\end{aligned}$$

Similarly, for I_2, we also have

$$\begin{aligned}\mathrm{I}_2&\lesssim\sum_{j\in\mathbb{Z}\setminus\mathbb{Z}_+}2^{j(n+M)}\sum_{k\in\mathbb{Z}^n}\int_{Q_{jk}}\frac{|\phi_j*f(x)|}{(1+|k|)^{n+M}}\,dx\\ &\lesssim\sum_{j\in\mathbb{Z}\setminus\mathbb{Z}_+}2^{jM}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-M}\inf_{y\in Q_{jk}}\left(\phi_j^*f\right)_a(y)\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.\end{aligned}$$

Together with the estimates for I_1 and I_2, one can deduce that

$$|\langle f,\Psi\rangle| \lesssim \|\Psi\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$$

and hence completes the proof of Proposition 3.1. □

As a consequence of Theorem 3.1, we shall show that the slice-Hardy space $(HE^p_r)_t(\mathbb{R}^n)$ in [38] are special cases of the slice Triebel–Lizorkin type spaces. Let $t,r,\ p\in(0,\infty)$. The *slice Hardy space* $(HE^p_r)_t(\mathbb{R}^n)$ is defined as the space of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $f^*\in(E^p_r)_t(\mathbb{R}^n)$ equipped with the quasi-norm $\|f\|_{(HE^p_r)_t(\mathbb{R}^n)} := \|f^*\|_{(E^p_r)_t(\mathbb{R}^n)}$, where, for any $x\in\mathbb{R}^n$,

$$f^*(x) := \sup_{\phi\in\mathcal{S}_m(\mathbb{R}^n)}\ \sup_{|x-y|<s} |f*\phi_s(y)|$$

and, for any $m\in\mathbb{N}$,

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{\phi\in\mathcal{S}(\mathbb{R}^n):\ \|\phi\|_{\mathcal{S}_m(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,|\alpha|\leqslant m+1}(1+|x|)^{(m+2)(n+1)}|\partial^\alpha_x\phi(x)|\leqslant 1\right\}.$$

We remark here that, the slice Hardy spaces introduced in [38] contains the Hardy-amalgam spaces of Z. V. de P. Ablé and J. Feuto [1] as special cases. The real-variable characterizations via the atom, the molecule, various maximal functions, the Poisson integral and the Littlewood–Paley functions are also obtained. Moreover, the finite atomic characterizations are also proved and applied to induce a description of their dual spaces.

Let $q\in(\max\{1,p\},\infty]$, $s\in[0,\min\{1,p\})$ and $d\in\mathbb{Z}\cap[\lfloor n(1/s-1)\rfloor,\infty)$. Denote, by $(HE^p_r)^{q,d}_t(\mathcal{S}'(\mathbb{R}^n))$, the *atomic slice Hardy spaces* defined as in [38, p. 22] and, by $(HE^p_r)^{q,d}_t(\mathcal{S}'_\infty(\mathbb{R}^n))$, the *atomic slice Hardy spaces* defined in the same way as $(HE^p_r)^{q,d}_t(\mathcal{S}'(\mathbb{R}^n))$ but with $\mathcal{S}'(\mathbb{R}^n)$ replaced by $\mathcal{S}'_\infty(\mathbb{R}^n)$.

By an argument similar to that used in the proof of [15, Theorem 1.7] (see also, [33, Proposition 3.6]), we obtain the following conclusion and omit the details of the proof.

PROPOSITION 3.2. *Let $t,r,\ p\in(0,\infty)$. Then $f\in(HE^p_r)_t(\mathbb{R}^n)$ if and only if $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ and $S_\psi(f)\in(E^p_r)_t(\mathbb{R}^n)$. Moreover, for any $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$, there exists a positive constant C such that*

$$C^{-1}\left\|S_\psi(f)\right\|_{(E^p_r)_t(\mathbb{R}^n)}\leqslant\|f\|_{(HE^p_r)_t(\mathbb{R}^n)}\leqslant C\left\|S_\psi(f)\right\|_{(E^p_r)_t(\mathbb{R}^n)},$$

where ψ is a Schwartz function satisfying (1.1) and (1.2) and, for any $x\in\mathbb{R}^n$,

$$S_\psi(f)(x) := \left\{\int_0^\infty\int_{\{y\in\mathbb{R}^n:\ |y-x|<s}|(\psi_s*f)(y)|^2\frac{dyds}{s^{n+1}}\right\}^{\frac12}.$$

From Proposition 3.2 and Theorem 3.1, we immediately obtain that the slice Hardy space $(HE^p_r)_t(\mathbb{R}^n)$ are special cases of the slice Triebel–Lizorkin-type spaces, which is formulated as the following corollary.

COROLLARY 3.1. *Let $t,r,\ p\in(0,\infty)$. Then $(\dot{F}E^{0,0}_{r,p,2})_t(\mathbb{R}^n)$ and $(HE^p_r)_t(\mathbb{R}^n)$ coincide with equivalent norms.*

4. Characterizations of $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ for some special τ

In this section, we will characterize the space $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ with some special τ, which will be used to study the boundedness of Fourier multipliers on $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ in Section 6.

Let $t, r, p \in (0,\infty)$. Let $\alpha \in \mathbb{R}$, $q \in (0,\infty]$, $\tau \in [0,\infty)$, $a \in (0,\infty)$, $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). Define

$$\|f\|^*_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left\| \left[\sum_{j\in\mathbb{Z}} (2^{j\alpha}|\psi_j * f|)^q\right]^{1/q} \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)},$$

$$\|f\|^*_{(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}} \left[\sum_{j\in\mathbb{Z}} 2^{j\alpha q} \|\psi_j * f\mathbf{1}_P\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right]^{1/q},$$

$$\|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} = \|f\|^{**}_{(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \sup_{x\in Q} |Q|^{-\alpha/n} \|\mathbf{1}_Q\|^{1-\tau}_{(E^p_r)_t(\mathbb{R}^n)} \left|\psi_{j_Q} * f(x)\right|$$

and

$$\|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} = \|f\|_{(\dot{B}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \inf_{x\in Q} |Q|^{-\alpha/n} \|\mathbf{1}_Q\|^{1-\tau}_{(E^p_r)_t(\mathbb{R}^n)} (\psi^*_{j_Q} f)_a(x).$$

THEOREM 4.1. *Let $t, r, p \in (0,\infty)$, $\alpha \in \mathbb{R}$ and $q \in (0,\infty]$.*

(i) *If $\tau \in [0,1)$, then $f \in (\dot{A}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|f\|^*_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} < \infty$. Moreover, there exists a positive constant C, independent of f, such that*

$$\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|f\|^*_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

(ii) *If*

$$\tau \in (1,\infty) \quad \text{and} \quad q \in (0,\infty), \tag{4.1}$$

or $q = \infty$ and $\tau = 1$, then $f \in (\dot{A}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|f\|^{**}_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} < \infty$. Moreover, there exists a positive constant C, independent of f, such that

$$\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|f\|^{**}_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

(iii) *Let τ be as in* (ii). *Then $f \in (\dot{A}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|f\|_{(\dot{A}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} < \infty$, where $a \in (0,\infty)$ is chosen large enough as in Theorem 3.1 for $(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)$ or as in Theorem 3.2 for $(\dot{B}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of f, such that*

$$\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|f\|_{(\dot{A}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

Proof. By similarity, we only prove Theorem 4.1 for the space $(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)$.

To show (i), for any $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$, it is easy to see that $\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant \|f\|^*_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$. It remains to prove that, for any $f\in(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $\|f\|^*_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant \|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$. To end this, for any given dyadic cube P, by Lemma 2.5, we show that

$$\begin{aligned}
&\frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j\in\mathbb{Z}}\left(2^{j\alpha}\left|\psi_j*f\right|\right)^q\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\quad\leqslant \frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j=-\infty}^{j_P-1}\left(2^{j\alpha}\left|\psi_j*f\right|\right)^q\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\qquad+\frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j=j_P}^{\infty}\left(2^{j\alpha}\left|\psi_j*f\right|\right)^q\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\quad=: \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

Obviously, $\mathrm{I}_2\leqslant\|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)}$. Next we estimate I_1. Notice that, for any $j\leqslant j_P-1$, there exists a unique dyadic cube P_j such that $P\subset P_j$ and $\ell(P_j)=2^{-j}$. Then for any $a\in(0,\infty)$, we have $|\psi_j*f(x)|\lesssim\inf_{y\in P_j}(\psi_j^*f)_a(y)$ for any $x\in P$. Thus, by Theorem 3.1 and choosing a as in Theorem 3.1, we find that

$$\begin{aligned}
\mathrm{I}_1&\lesssim\frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left\{\sum_{j=-\infty}^{j_P-1}2^{j\alpha q}\left[\inf_{y\in P_j}\left(\psi_j^*f\right)_a(y)\right]^q\right\}^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)} \qquad (4.2)\\
&\lesssim\frac{1}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j=-\infty}^{j_P-1}\left\|2^{j\alpha}\left(\psi_j^*f\right)_a\right\|^q_{(E^p_r)_t(P_j)}\|\mathbf{1}_{P_j}\|^{-q}_{(E^p_r)_t(P_j)}\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\
&\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\frac{\|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)}}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}\left[\sum_{j=-\infty}^{j_P-1}\|\mathbf{1}_{P_j}\|^{\tau q}_{(E^p_r)_t(\mathbb{R}^n)}\|\mathbf{1}_{P_j}\|^{-q}_{(E^p_r)_t(\mathbb{R}^n)}\right]^{1/q}.
\end{aligned}$$

Then, by Lemma 2.2, we know that

$$2^{-\frac{jn}{p}}|P|^{-\frac{1}{p}}\|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)}\lesssim\left\|\mathbf{1}_{P_j}\right\|_{(E^p_r)_t(\mathbb{R}^n)}\lesssim 2^{-\frac{jn}{p}}|P|^{-\frac{1}{p}}\|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)},$$

which, together with (4.2) and $\tau\in[0,1)$, implies that

$$\mathrm{I}_1\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}|P|^{-\frac{\tau q}{p}}|P|^{\frac{q}{p}}\left[\sum_{j=-\infty}^{j_P-1}2^{-\frac{jn\tau q}{p}}2^{\frac{jnq}{p}}\right]^{1/q}\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

Combining with the estimation of I_1 and I_2, we conclude that

$$\|f\|^*_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\lesssim\|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)}$$

and hence completes the proof of (i).

For (ii), we first prove that $\|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)}$ for $q \in (0,\infty]$. Let $Q \in \mathcal{Q}$, $x \in Q$ and a be large enough as in Theorem 3.1. Then by Theorem 3.1, we find that

$$\begin{aligned}|Q|^{-\alpha/n}\|\mathbf{1}_Q\|^{1-\tau}_{(E^p_r)_t(\mathbb{R}^n)}\left|\psi_{j_Q}*f(x)\right| &\leqslant |Q|^{-\alpha/n}\|\mathbf{1}_Q\|^{1-\tau}_{(E^p_r)_t(\mathbb{R}^n)}\inf_{y\in Q}\left(\psi^*_{j_Q}\right)f(y)\\ &\lesssim \|\mathbf{1}_Q\|^{-\tau}_{(E^p_r)_t(\mathbb{R}^n)}\left\|2^{j_Q\alpha}\left(\psi^*_{j_Q}\right)f\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\ &\lesssim \|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)},\end{aligned}$$

which implies that $\|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)}$.

Then we prove that $\|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$. Assume first that $q \in (0,\infty)$. For any given $P \in \mathcal{Q}$, by the definition of $\|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$, we know that

$$\begin{aligned}&\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j=j_P}^{\infty}\left(2^{j\alpha}\left|\psi_j*f\right|\right)^q\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\ &\lesssim \frac{\|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j=j_P}^{\infty}\left(\sum_{\substack{\ell(\widetilde{Q})=2^{-j}\\ \widetilde{Q}\in\mathcal{Q},\,\widetilde{Q}\subset P}}\left\|\mathbf{1}_{\widetilde{Q}}\right\|^{\tau-1}_{(E^p_r)_t(\mathbb{R}^n)}\mathbf{1}_{\widetilde{Q}}\right)\right]\right\|_{(E^p_r)(P)}.\end{aligned}$$

Similar to estimate I_1 in (i), from Lemma 2.2 and (4.1), we deduce that

$$\begin{aligned}&\frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\left\|\left[\sum_{j=j_P}^{\infty}\left(2^{j\alpha}\left|\psi_j*f\right|\right)^q\right]^{1/q}\mathbf{1}_P\right\|_{(E^p_r)_t(\mathbb{R}^n)}\\ &\lesssim \|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}|P|^{-\frac{\tau q}{p}}|P|^{\frac{q}{p}}\left\{\sum_{j=j_P}^{\infty}2^{-\frac{jn\tau q}{p}}2^{\frac{jnq}{p}}\right\}^{1/q}\\ &\lesssim \|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)},\end{aligned}$$

which implies that $\|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.

The proof of the case $q=\infty$ is similar to that of $q \in (0,\infty)$. Indeed, by repeating the above argument but replaced $\sum_{j=j_P}^{\infty}$ by $\sup_{j\geqslant J_P}$, we conclude that $\|f\|_{(\dot{F}E^{\alpha,\tau,a}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|f\|^{**}_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$, which completes the proof (ii).

The proof of (iii) is similar to that of (ii), the details being omitted. This finishes the proof of Theorem 4.1. $\square$

Let ψ be a Schwartz function satisfying (1.1) and (1.2). For $\alpha \in \mathbb{R}$, $q \in (0,\infty]$, $\lambda \in (0,\infty)$, $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we recall that the generalized g^*_λ-function $G^{\alpha}_{\lambda,q}(f)$

is defined by setting

$$G^{\alpha}_{\lambda,q}(f)(x) := \left\{ \int_0^\infty s^{-\alpha q} \int_{\mathbb{R}^n} |f * \psi_s(y)|^q \left(1 + \frac{|x-y|}{s}\right)^{-\lambda q} dy \frac{ds}{s^{n+1}} \right\}^{1/q}. \tag{4.3}$$

By Theorems 3.1 and 4.1, and an argument similar to that used in the proof of [31, Theorem 2.7], we obtain the following characterization of the slice Trieble–Lizorkin-type space $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ via the generalized g^*_λ-function, which is used in studying the mapping property of Fourier multipliers on $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ in Section 6. We omit the details. To state our result, we define $(E^p_r)^\tau_t(\mathbb{R}^n)$ as the set of all measurable functions f satisfying that

$$\|f\|_{(E^p_r)^\tau_t(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{\|f\|_{(E^p_r)_t(P)}}{\|\mathbf{1}_P\|^\tau_{(E^p_r)_t(\mathbb{R}^n)}}.$$

THEOREM 4.2. *Let $t, r, p \in (0,\infty)$. Let $\alpha \in \mathbb{R}$, $q \in (0,\infty]$ and $\tau \in [0,1)$. Assume that $\lambda \in (n/q,\infty)$. Then $f \in (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $G^{\alpha}_{\lambda,q}(f) \in (E^p_r)^\tau_t(\mathbb{R}^n)$, where $G^{\alpha}_{\lambda,q}(f)$ is as in (4.3). Moreover, there exists a positive constant C such that, for any $f \in (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $C^{-1}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant \|G^{\alpha}_{\lambda,q}(f)\|_{(E^p_r)^\tau_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.*

5. Smooth atomic and molecular characterizations

The purpose of this section is to establish the smooth atomic and molecular characterizations of $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. We first give the boundedness of almost diagonal operators on $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

DEFINITION 5.1. Let $t, r, p \in (0,\infty)$. Let $\alpha\in\mathbb{R}$, $q\in(0,\infty]$, $\tau\in[0,\infty)$ and $\varepsilon\in(0,\infty)$. Let $J := \frac{n}{\min\{1,p\}}$ when $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) := (\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $J := \frac{n}{\min\{1,p,q\}}$ when $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) := (\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. An operator A associated with a matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$, namely, for any sequences $u:=\{u_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{C}$, $Au=\{(Au)_Q\}_{Q\in\mathcal{Q}} := \{\sum_{P\in\mathcal{Q}} a_{QP}u_P\}_{Q\in\mathcal{Q}}$ is said to be ε*-almost diagonal on* $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ if the matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$ satisfies $\sup_{Q,P\in\mathcal{Q}} |a_{QP}|/\omega_{QP}(\varepsilon) < \infty$, where

$$\omega_{QP}(\varepsilon) := \left[\frac{\ell(Q)}{\ell(P)}\right]^\alpha \left[1 + \frac{|x_Q - x_P|}{\max\{\ell(P),\ell(Q)\}}\right]^{-J-\varepsilon} \min\left\{\left[\frac{\ell(Q)}{\ell(P)}\right]^{\frac{n+\varepsilon}{2}}, \left[\frac{\ell(P)}{\ell(Q)}\right]^{\frac{n+\varepsilon}{2}+J-n}\right\}$$

THEOREM 5.1. *Let $\alpha \in \mathbb{R}^n$, $q \in (0,\infty]$ and t, r, $p \in (0,\infty)$. Assume that $\tau \in [0,\infty)$ and $\varepsilon \in (0,\infty)$ satisfying that*

$$0 \leqslant \tau < \left(\frac{\varepsilon}{2n} + \frac{1}{p}\right) p. \tag{5.1}$$

Then every ε-almost diagonal operators on $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ are bounded on $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$

Proof. To prove Theorem 5.1, we borrow some ideas from the proofs of [28, Theorem 4.1] and [33, Theorem 5.2]. Let $u := \{u_Q\}_{Q\in\mathcal{Q}} \in (\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and A be an ε-almost diagonal operator on $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ associated with the matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$ and $\varepsilon\in(0,\infty)$. Without loss of generality, we assume $\alpha=0$. Indeed, once Theorem 5.1 holds true for $\alpha=0$, taking $\widetilde{u}_R := [\ell(R)]^{-\alpha}u_R$ and the ε-almost diagonal operator $\widetilde{A}$ which is associated with the matrix $\{\widetilde{a}_{QP}\}_{Q,R\in\mathcal{Q}}$, where $\widetilde{a}_{QR} := a_{QR}[\ell(R)/\ell(Q)]^{\alpha}$, for any $Q,\ R\in\mathcal{Q}$, we can get

$$\|Au\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} = \left\|\widetilde{A}\widetilde{u}\right\|_{(\dot{a}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|\widetilde{u}\|_{(\dot{a}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sim \|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

This is the desired results.

Now we turn to prove Theorem 5.1 for the space $(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)$ in the case $q\in(1,\infty]$ and $p>1$. In this case, $J=n$. Then we decompose $A=A_0+A_1$, where, for any $Q\in\mathcal{Q}$, $(A_0u)_Q := \sum_{\{R:\ \ell(R)\geqslant\ell(Q)\}} a_{QR}u_R$ and $(A_1u)_Q := \sum_{\{R:\ \ell(R)<\ell(Q)\}} a_{QR}u_R$. From Definition 5.1, it follows that, for any $Q\in\mathcal{Q}$,

$$|(A_0u)_Q| \lesssim \sum_{\{R:\ \ell(R)\geqslant\ell(Q)\}} \left[\frac{\ell(Q)}{\ell(R)}\right]^{\frac{n+\varepsilon}{2}} \frac{|u_R|}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}},$$

and hence

$$\begin{aligned}
&\|(A_0u)_Q\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}\\
&\quad\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\\
&\qquad\times\left\{\sum_{j=j_P}^{\infty}\left\|\sum_{\substack{\ell(Q)=2^{-j}\\ Q\subset P}}\ \sum_{\{R:\ \ell(Q)\leqslant\ell(R)\leqslant\ell(P)\}} \frac{[\ell(Q)/\ell(R)]^{\frac{n+\varepsilon}{2}}|u_R||Q|^{-1/2}\mathbf{1}_Q\mathbf{1}_P}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}}\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{1/q}\\
&\qquad+\sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}}\\
&\qquad\times\left\{\sum_{j=j_P}^{\infty}\left\|\sum_{\substack{\ell(Q)=2^{-j}\\ Q\subset P}}\ \sum_{\{R:\ \ell(R)>\ell(P)\}} \frac{[\ell(Q)/\ell(R)]^{\frac{n+\varepsilon}{2}}|u_R||Q|^{-1/2}\mathbf{1}_Q\mathbf{1}_P}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}}\right\|^q_{(E^p_r)_t(\mathbb{R}^n)}\right\}^{1/q}\\
&\quad=: \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

We first estimate I_2. For any $i\in\mathbb{Z}$, $m\in\mathbb{N}$ and $Q\in\mathcal{Q}$, set $U_{0,i}(Q) := \{R\in\mathcal{Q}: \ell(R)=2^{-i}$ and $|x_Q-x_R|<\ell(R)\}$ and $U_{m,i}(Q) := \{R\in\mathcal{Q}:\ \ell(R)=2^{-i}$ and $2^{m-1}\ell(R)\leqslant |x_Q-x_R|<2^m\ell(R)\}$. The geometric property of $\mathbb{R}^n$ implies that the cardinality of $U_{m,i}(Q)$ is at most a multiple of 2^{mn}.

From $\ell(R) > \ell(P)$ and Lemma 2.2, it derives that

$$\|\mathbf{1}_R\|_{(E_r^p)_t(\mathbb{R}^n)} \lesssim 2^{\frac{(j_P-i)n}{p}} \|\mathbf{1}_P\|_{(E_r^p)_t(\mathbb{R}^n)} \quad \text{and} \quad \|\mathbf{1}_P\|_{(E_r^p)_t(\mathbb{R}^n)} \lesssim 2^{\frac{(-j_P+i)n}{p}} \|\mathbf{1}_R\|_{(E_r^p)_t(\mathbb{R}^n)}. \tag{5.2}$$

Notice that $u_R \leqslant \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} |R|^{1/2} \|\mathbf{1}_R\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)} \|\mathbf{1}_R\|^{-1}_{(E_r^p)_t(\mathbb{R}^n)}$. Then, by (5.1) and (5.2) and the fact that the cardinality of $U_{m,i}(Q)$ is at most a multiple of 2^{mn}, we have

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sup_{P\in\mathscr{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} \left\| \sum_{\ell(Q)=2^{-j}} \sum_{i=-\infty}^{j_P-1} \sum_{m=0}^{\infty} \sum_{R\in U_{m,i}(Q)} \left[\frac{\ell(Q)}{\ell(R)}\right]^{\frac{n+\varepsilon}{2}} \right.\right. \\
&\quad \left.\left. \times \frac{|Q|^{-1/2}|R|^{1/2}\|\mathbf{1}_R\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}\|\mathbf{1}_R\|^{-1}_{(E_r^p)_t(\mathbb{R}^n)}\mathbf{1}_Q\mathbf{1}_P}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}} \right\|^q_{(E_r^p)_t(\mathbb{R}^n)} \right\}^{1/q} \\
&\lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sup_{P\in\mathscr{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} \left\| \sum_{i=-\infty}^{j_P-1} \sum_{m=0}^{\infty} 2^{mn} 2^{\frac{(-j+i)(n+\varepsilon)}{2}} \right.\right. \\
&\quad \left.\left. \times 2^{-m(n+\varepsilon)} 2^{\frac{jn}{2}} 2^{-\frac{in}{2}} 2^{\frac{(j_P-i)n\tau}{p}} 2^{\frac{(-j_P+i)n}{p}} \|\mathbf{1}_P\|^{\tau-1}_{(E_r^p)_t(\mathbb{R}^n)} \mathbf{1}_P \right\|^q_{(E_r^p)_t(\mathbb{R}^n)} \right\}^{1/q} \\
&\lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}.
\end{aligned}$$

To deal with I_1, taking v and w the same as in the proof of Lemma 2.6 yields that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \sup_{P\in\mathscr{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}} \\
&\quad \times \left\{ \sum_{j=j_P}^{\infty} \left\| \sum_{\ell(Q)=2^{-j}} \sum_{i=j_P}^{j} 2^{(i-j)(n+\varepsilon)/2} \sum_{\ell(R)=2^{-i}} \frac{|v_R||Q|^{-1/2}\mathbf{1}_Q}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}} \mathbf{1}_P \right\|^q_{(E_r^p)_t(\mathbb{R}^n)} \right\}^{1/q} \\
&\quad + \sup_{P\in\mathscr{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}} \\
&\quad \times \left\{ \sum_{j=j_P}^{\infty} \left\| \sum_{\ell(Q)=2^{-j}} \sum_{i=j_P}^{j} 2^{(i-j)(n+\varepsilon)/2} \sum_{\ell(R)=2^{-i}} \frac{|w_R||Q|^{-1/2}\mathbf{1}_Q}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}} \mathbf{1}_P \right\|^q_{(E_r^p)_t(\mathbb{R}^n)} \right\}^{1/q} \\
&=: \mathrm{J}_1+\mathrm{J}_2.
\end{aligned}$$

Applying [9, Lemma A.2], for any $x\in Q$, we have

$$\sum_{\ell(R)=2^{-i}} \frac{|v_R|\mathbf{1}_Q}{(1+[\ell(R)]^{-1}|x_Q-x_R|)^{n+\varepsilon}} \lesssim \mathcal{M}\left(\sum_{\ell(R)=2^{-i}} |v_R|\mathbf{1}_R\right)(x).$$

By this, Lemma 2.1, $\min\{p,q\}>1$, we conclude that

$$\begin{aligned}
\mathrm{J}_1 &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} \left\| \sum_{i=j_P}^{j} 2^{\frac{(i-j)\varepsilon}{2}} \mathcal{M}\left(\sum_{\ell(R)=2^{-i}} |v_R||R|^{-1/2}\mathbf{1}_R \right) \mathbf{1}_P \right\|^q_{(E^p_r)_t(\mathbb{R}^n)} \right\}^{1/q} \\
&\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \\
&\quad \times \left\{ \sum_{j=j_P}^{\infty} \left[\sum_{i=j_P}^{j} 2^{\frac{(i-j)\varepsilon}{2}} \left\| \mathcal{M}\left(\sum_{\ell(R)=2^{-i}} |v_R||R|^{-1/2}\mathbf{1}_R \right) \mathbf{1}_P \right\|_{(E^p_r)_t(\mathbb{R}^n)} \right]^q \right\}^{1/q}.
\end{aligned}$$

Using Lemma 2.4 and the Hölder inequality, we see

$$\begin{aligned}
\mathrm{J}_1 &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} \sum_{i=j_P}^{j} 2^{\frac{q(i-j)\varepsilon}{4}} \left\| \sum_{\ell(R)=2^{-i}} |u_R||R|^{-1/2}\mathbf{1}_R \right\|^q_{(E^p_r)_t(3P)} \right\}^{1/q} \\
&\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\{ \sum_{i=j_P}^{\infty} \left\| \sum_{\ell(R)=2^{-i}} |u_R||R|^{-1/2}\mathbf{1}_R \right\|^q_{(E^p_r)_t(3P)} \right\}^{1/q} \lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}.
\end{aligned}$$

Next we estimate J_2. Observe that if $R\cap(3P)=\emptyset$, then there exists some $k\in\mathbb{Z}^n$ with $|k|\geqslant 2$ such that $R\subset P+k\ell(P)$ and $[P+k\ell(P)]\cap(3P)=\emptyset$ and $1+[\ell(R)]^{-1}|x_Q-x_R|\sim|k|\ell(P)/\ell(R)$ for any dyadic cube $Q\subset P$. By Lemma 2.2, the above observation $R\subset P+k\ell(P)$, we conclude that

$$\|\mathbf{1}_{P+k\ell(P)}\|_{(E^p_r)_t(\mathbb{R}^n)} = \|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)} \quad \text{and} \quad \|\mathbf{1}_R\|_{(E^p_r)_t(\mathbb{R}^n)} \gtrsim 2^{\frac{(j_P-i)n}{p}} \|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)}. \tag{5.3}$$

Then, from this, $p>1$ and the Hölder inequality, it is seen that

$$\begin{aligned}
\mathrm{J}_2 \lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} &\left\{ \sum_{j=j_P}^{\infty} 2^{-jq\varepsilon/2}[\ell(P)]^{-q(n+\varepsilon)} \right. \\
&\left. \times \left\| \sum_{\ell(Q)=2^{-j}} \sum_{i=j_P}^{j} 2^{-i(n+\varepsilon)/2} \sum_{\substack{k\in\mathbb{Z}^n\\ |k|\geqslant 2}} |k|^{-n-\varepsilon} \sum_{\substack{\ell(R)=2^{-i}\\ R\subset P+k\ell(P)}} |u_R|\mathbf{1}_Q\mathbf{1}_P \right\|^q_{(E^p_r)_t(\mathbb{R}^n)} \right\}^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{P\in\mathscr{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}} \\
&\quad\times \left\{ \sum_{j=j_P}^{\infty} 2^{-jq\varepsilon/2}[\ell(P)]^{-q(n+\varepsilon)} \left\| \sum_{\ell(Q)=2^{-j}} \sum_{i=j_P}^{j} 2^{-i(n+\varepsilon)/2} \sum_{\substack{k\in\mathbb{Z}^n\\ |k|\geqslant 2}} |k|^{-n-\varepsilon} 2^{n(i-j_P)(1-\frac{1}{p})} \right.\right. \\
&\quad\times \left[\sum_{\substack{\ell(R)=2^{-i}\\ R\subset P+k\ell(P)}} \left(|u_R||R|^{-1/2}\|\mathbf{1}_R\|_{(E_r^p)_t(\mathbb{R}^n)} \right)^p \right]^{\frac{1}{p}} \\
&\quad\times \left.\left. 2^{-in/2} 2^{\frac{-(j_P-i)n}{p}} \|\mathbf{1}_P\|^{-1}_{(E_r^p)_t(\mathbb{R}^n)} \mathbf{1}_Q \mathbf{1}_P \right\|^q_{(E_r^p)_t(\mathbb{R}^n)} \right\}^{1/q}.
\end{aligned}
$$

Furthermore, using Remark 2.3 and (5.3), we get that

$$
\begin{aligned}
\mathrm{J}_2 &\lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \sup_{P\in\mathscr{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)}} \left\{ \sum_{j=j_P}^{\infty} 2^{-jq\varepsilon/2}[\ell(P)]^{-q(n+\varepsilon)} \left\| \sum_{i=j_P}^{j} 2^{-i(n+\varepsilon)/2} \right.\right. \\
&\quad\times 2^{n(i-j_P)(1-\frac{1}{p})} \sum_{\substack{k\in\mathbb{Z}^n\\ |k|\geqslant 2}} |k|^{-n-\varepsilon} \left\|\mathbf{1}_{P+k\ell(P)}\right\|^{\tau}_{(E_r^p)_t(\mathbb{R}^n)} \\
&\quad\times \left.\left. 2^{-in/2} 2^{\frac{-(j_P-i)n}{p}} \|\mathbf{1}_P\|^{-1}_{(E_r^p)_t(\mathbb{R}^n)} \mathbf{1}_P \right\|^q_{(E_r^p)_t(\mathbb{R}^n)} \right\}^{1/q} \\
&\lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \\
&\quad\times \sup_{P\in\mathscr{Q}} \left\{ \sum_{j=j_P}^{\infty} 2^{-jq\varepsilon/2}[\ell(P)]^{-q(n+\varepsilon)} \left[\sum_{i=j_P}^{j} 2^{-i(n+\varepsilon)/2} 2^{-in/2} 2^{n(i-j_P)(1-\frac{1}{p})} 2^{\frac{-(j_P-i)n}{p}} \right]^q \right\}^{1/q} \\
&\lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}.
\end{aligned}
$$

Thus, $\|A_0u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}$. With some estimates similar to I_1, we can also obtain $\|A_1u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}$ and hence $\|Au\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|u\|_{(\dot{b}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.

For the space $(\dot{f}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)$, by Lemma 2.4 and an argument similar to the above, we can also conclude that $\|Au\|_{(\dot{f}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|u\|_{(\dot{f}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.

Now the remain case that $q\in(0,1]$ or $p\in(0,1]$ is a simple consequence of the case $q\in(1,\infty]$ and $p\in(1,\infty)$. In fact, choose an $\eta\in(0,\min\{p,q\})$ and let $\widetilde{A}$ be an operator on $(\dot{a}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)$ associated with the matrix

$$
\{\widetilde{a}_{QP}\}_{Q,P\in\mathscr{Q}} := \{|a_{QP}|^{\eta}[\ell(Q)/\ell(P)]^{n/2-\eta n/2}\}_{Q,P\in\mathscr{Q}}.
$$

Then $\widetilde{A}$ is an $\widetilde{\varepsilon}$ almost diagonal operator on $(\dot{a}E^{0,\tau\eta}_{r/\eta,p/\eta,q/\eta})_t(\mathbb{R}^n)$ with $\widetilde{\varepsilon} := \varepsilon\eta$.

Let $\widetilde{u} = \{[\ell(Q)]^{n/2-\eta n/2}|u_Q|^\eta\}_{Q\in\mathscr{Q}}$. Then $\|\widetilde{u}\|_{(\dot{a}E^{0,\tau\eta}_{r/\eta,p/\eta,q/\eta})_t(\mathbb{R}^n)} = \|u\|_{(\dot{a}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}$. Applying the conclusions for the case $q\in(1,\infty]$ and $p\in(1,\infty)$, we obtain that

$$\|Au\|_{(\dot{a}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \left\|\widetilde{A}\widetilde{u}\right\|_{(\dot{a}E^{0,\tau\eta}_{r/\eta,p/\eta,q/\eta})_t(\mathbb{R}^n)} \lesssim \|\widetilde{u}\|_{(\dot{a}E^{0,\tau\eta}_{r/\eta,p/\eta,q/\eta})_t(\mathbb{R}^n)} \lesssim \|u\|_{(\dot{a}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)},$$

which completes the proof of Theorem 5.1. □

REMARK 5.1. Under the assumption in Theorem 5.1, when t, r, p are as in Remark 2.1(ii), Theorem 5.1 goes back to [28, Theorem 4.1].

Applying Theorem 5.1, we shall establish smooth atomic and molecular characterizations for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

Now we introduce the smooth synthesis molecule for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

DEFINITION 5.2. Let $\alpha\in\mathbb{R}$, $q\in(0,\infty]$, $\tau\in[0,\infty)$ and $t,\ r,\ p\in(0,\infty)$. Let $J := \frac{n}{\min\{1,p\}}$ when $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) := (\dot{b}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $J := \frac{n}{\min\{1,p,q\}}$ when $(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n) := (\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Let $N := \max\{\lfloor J-\alpha-n\rfloor, -1\}$ and $\alpha^* := \alpha - \lfloor\alpha\rfloor$, where $\lfloor\alpha\rfloor$ denotes the maximal integer not more than α.

(i) A function m_Q, with $Q\in\mathscr{Q}$, is called a *smooth synthesis molecule for* $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ *supported near the dyadic cube* Q if there exists $\theta\in(\max\{\alpha^*,(\alpha+n\tau/p)^*\},1]$ and an $M\in(J,\infty)$ such that $\int_{\mathbb{R}^n} x^\gamma m_Q(x)\,dx = 0$ if $|\gamma|\leqslant N$, $|m_Q(x)|\leqslant|Q|^{-1/2}(1+[\ell(Q)]^{-1}|x-x_Q|)^{-\max\{M,M-\alpha\}}$,

$$|\partial^\gamma m_Q(x)| \leqslant |Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}(1+[\ell(Q)]^{-1}|x-x_Q|)^{-M} \tag{5.4}$$

if $|\gamma|\leqslant\lfloor\alpha+n\tau/p\rfloor$, and

$$|\partial^\gamma m_Q(x)-\partial^\gamma m_Q(y)| \leqslant |Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}-\theta/n}|x-y|^\theta \sup_{|z|\leqslant|x-y|}\left(1+\frac{|x-z-x_Q|}{\ell(Q)}\right)^{-M} \tag{5.5}$$

if $|\gamma| = \lfloor\alpha+n\tau/p\rfloor$.

A collection of $\{m_Q\}_{Q\in\mathscr{Q}}$ is called a *family of smooth molecules* for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, if each m_Q is a smooth synthesis for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ supported near Q.

(ii) A function b_Q, with $Q\in\mathscr{Q}$, is called a *smooth analysis molecule* for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ supported near the dyadic cube Q if there exists a $\rho\in((J-\alpha)^*,1]$ and an $M\in(J,\infty)$ such that $\int_{\mathbb{R}^n} x^\gamma b_Q(x)\,dx = 0$ if $|\gamma|\leqslant\lfloor\alpha+n\tau/p\rfloor$,

$$|b_Q(x)| \leqslant |Q|^{-\frac{1}{2}}(1+[\ell(Q)]^{-1}|x-x_Q|)^{-\max\{M,M+n+\alpha+n\tau/p-J\}},$$
$$|\partial^\gamma b_Q(x)| \leqslant |Q|^{-1/2-|\gamma|/n}(1+[\ell(Q)]^{-1}|x-x_Q|)^{-M} \quad \text{if} \quad |\gamma|\leqslant N, \tag{5.6}$$

and

$$|\partial^\gamma b_Q(x)-\partial^\gamma b_Q(y)|\leqslant |Q|^{-1/2-|\gamma|/n-\rho/n}|x-y|^\rho \sup_{|z|\leqslant|x-y|}\left(1+\frac{|x-z-x_Q|}{\ell(Q)}\right)^{-M} \tag{5.7}$$

if $|\gamma|=N$.

A collection of $\{b_Q\}_{Q\in\mathscr{Q}}$ is called a *family of smooth analysis molecules* for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, if each b_Q is a smooth analysis molecules for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ supported near Q.

We remark that if $\alpha+n\tau/p<0$, then (5.4) and (5.5) are void. If $J+\alpha-n<0$, then (5.6) and (5.7) are void.

To establish the smooth atomic and molecular characterizations for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, we first give some elementary lemmas. The proof of the following estimate is similar to that of [9, Corollary B.3] (see also [28, Lemma 4.1]). We omit the details.

LEMMA 5.1. *Let α, q, p, t, r, J, N, r and ρ be as in Definition 5.2. Assume that*

$$\tau\in\left[0,\frac{1}{p}+\min\left\{\frac{M-J}{2n},\frac{\rho-(J-\alpha)^*}{n}\right\}\right) \quad \text{if } N\geqslant 0\ ,$$

$$\tau\in\left[0,\frac{1}{p}+\min\left\{\frac{M-J}{2n},\frac{\alpha+n-J}{n}\right\}\right) \quad \text{if } N<0 \text{ and } \theta\in(\max\{\alpha^*,(\alpha+n\tau/p)^*\},1].$$

Then there exist positive constants C and $\varepsilon_1\in(2(n\tau/p-n/p),\infty)$ such that, for any family $\{m_Q\}_{Q\in\mathscr{Q}}$ of smooth synthesis molecules for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and family $\{b_Q\}_{Q\in\mathscr{Q}}$ of smooth analysis molecules for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $|\langle m_Q,b_Q\rangle|\leqslant C\omega_{Q,P}(\varepsilon_1)$. Namely, the operators associated with the matrices $\{a_{QP}\}_{Q,P\in\mathscr{Q}}:=\{\langle m_Q,\varphi_P\rangle\}_{Q,P}$ and $\{b_{QP}\}_{Q,P\in\mathscr{Q}}:=\{\langle\varphi_P,b_Q\rangle\}_{Q,P}$ are, respectively, ε_1-almost diagonal operators on $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

As an immediate consequence of Lemma 5.1, we have the following corollary; see [9, Corollaries 5.2 and 5.3] and [28, Corollary 4.1].

COROLLARY 5.1. *Let α, τ, q, t, r, p be as in Lemma 5.1, and ψ satisfy (1.1) and (1.2). Suppose that $\{m_Q\}_{Q\in\mathscr{Q}}$ and $\{b_Q\}_{Q\in\mathscr{Q}}$ are families of smooth synthesis and analysis molecules for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, respectively. Then the operators associated with the matrix $\{a_{QP}\}_{Q,P\in\mathscr{Q}}:=\{\langle m_Q,\psi_P\rangle\}_{Q,P\in\mathscr{Q}}$ and $\{b_{QP}\}_{Q,P\in\mathscr{Q}}:=\{\langle\psi_P,m_Q\rangle\}_{Q,P\in\mathscr{Q}}$ are both ε_1-almost diagonal on $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, where ε_1 is as in Lemma 5.1.*

LEMMA 5.2. *Let α, q, τ, t, r, p be as in Lemma 5.1. $f\in(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and h be a smooth analysis molecule for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ support near some dyadic cube Q. Then $\langle f,h\rangle$ is well defined. Moreover, for Υ and Ψ satisfy (1.1) through (1.3),*

$$\langle f,h\rangle:=\sum_{j\in\mathbb{Z}}\langle\widetilde{\Upsilon}_j*\Psi_j*f,h\rangle=\sum_{P\in\mathscr{Q}}\langle f,\Upsilon\rangle\langle\Upsilon_P,h\rangle \tag{5.8}$$

converge absolutely and its value is independent of the choices of Υ *and* Ψ*, where* $\widetilde{\Upsilon}(\cdot) := \overline{\Upsilon(-\cdot)}$ *and* $\{\widetilde{\Upsilon}_j\}_{j\in\mathbb{Z}}$ *and* $\{\Psi_j\}_{j\in\mathbb{Z}}$ *are as in* (1.4) *with* ψ *replaced, respectively, by* $\widetilde{\Upsilon}$ *and* Ψ*.*

Proof. By similarity, we only prove Lemma 5.2 for the space $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Let h be a smooth analysis molecule for $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ supported near some dyadic cube Q and Υ, Ψ satisfy (1.1) through (1.3). Then the following claim holds true: there exists a matrix $\{a_{\widetilde{Q}P}\}_{\widetilde{Q},P\in\mathscr{Q}}$ such that

$$|\langle f,\Upsilon_P\rangle||\langle\Upsilon_P,h\rangle| \leqslant a_{QP} \quad \text{for any} \quad P\in\mathscr{Q},$$

$a_{\widetilde{Q}P}=0$ for any $\widetilde{Q}\neq Q$, $\widetilde{Q}$, $P\in\mathscr{Q}$, and $\sum_{P\in\mathscr{Q}} a_{QP}<\infty$. Indeed, from Corollary 5.1, there exist positive constants C and ε_1 such that, for any $P\in\mathscr{Q}$, $|\langle\Upsilon_P,h\rangle|\leqslant C\omega_{QP}(\varepsilon_1)$, where $\omega_{QP}(\varepsilon_1)$ is as in Definition 5.1 with ε replaced by ε_1. For any $P\in\mathscr{Q}$, let $a_{QP}:=C|\langle f,\Upsilon_P\rangle|\omega_{QP}(\varepsilon_1)$ and, for any $\widetilde{Q}\neq Q$, $\widetilde{Q}$, $P\in\mathscr{Q}$, let $a_{\widetilde{Q}P}=0$. Then, it is easy to find that $|\langle f,\Upsilon_P\rangle||\langle\Upsilon_P,h\rangle|\leqslant a_{QP}$. Moreover, Theorem 2.1 yields that the sequence $\{|\langle f,\Upsilon_P\rangle|\}_{P\in\mathscr{Q}}$ belongs to $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Observe that the operator associated with the matrix $\{\frac{a_{\widetilde{Q}P}}{|\langle f,\Upsilon_P\rangle|}\}_{\widetilde{Q},P\in\mathscr{Q}}$ is ε_1-almost diagonal on $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. From this, the definition of the $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, and the fact that the sequence $\{|\langle f,\Upsilon_P\rangle|\}_{P\in\mathscr{Q}}$ belongs to $(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, we see

$$\begin{aligned}\left\|\left\{\sum_{P\in\mathscr{Q}} a_{\widetilde{Q}P}\right\}_{\widetilde{Q}\in\mathscr{Q}}\right\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} &= \|\mathbf{1}_Q\|^{1-\tau}_{(E^p_r)_t(\mathbb{R}^n)}|Q|^{-\alpha/n-1/2}\sum_{P\in\mathscr{Q}} a_{QP}\\ &\lesssim \|\{|\langle f,\Upsilon_P\rangle|\}_{P\in\mathscr{Q}}\|_{(\dot{f}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)},\end{aligned}$$

which implies that $\sum_{P\in\mathscr{Q}} a_{QP}<\infty$. This shows the absolute convergence of (5.8) and hence completes the proof of this claim.

Next we prove that $\langle f,h\rangle$ is well defined. We first show that, for any $f\in(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $\sum_{j=0}^{\infty}\widetilde{\Upsilon}_j*\Psi_j*f$ converges in $\mathcal{S}'(\mathbb{R}^n)$. As proved in [27, Lemma 2.2], for any $L\in\mathbb{Z}_+$, $\Upsilon\in\mathcal{S}_\infty(\mathbb{R}^n)$, $\phi\in\mathcal{S}(\mathbb{R}^n)$, $j\in\mathbb{Z}_+$ and $x\in\mathbb{R}^n$,

$$\left|\Upsilon_j*\phi(x)\right| \lesssim \|\phi\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}2^{-jL}\frac{1}{(1+|x|)^{L+n}}, \tag{5.9}$$

where the implicit constant may depend on L. Choosing $a>\frac{n}{\min\{p,q\}}$ and letting

$$L>\max\left\{a, a-\alpha-\frac{\tau n}{p}+\frac{n}{p}\right\},$$

by (5.9), (3.10), Corollary 2.1 and Theorem 3.1, we conclude that, for any $\phi\in\mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}&\sum_{j=0}^{\infty}\left|\langle\widetilde{\Upsilon}_j*\Psi_j*f,\phi\rangle\right| \qquad (5.10)\\ &\quad\lesssim \|\phi\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\sum_{j=0}^{\infty}2^{-jL}\sum_{k\in\mathbb{Z}^n}\int_{Q_{0k}}\frac{|\Psi_j*f(x)|}{(1+|x|)^{n+L}}\,dx\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\phi\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\sum_{j=0}^{\infty}2^{-jL+ja}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-L+a}\inf_{z\in Q_{jk}}\left(\Psi_j^*f\right)_a(z)\\
&\lesssim \|\phi\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-L+a}\sum_{j=0}^{\infty}2^{-jL+ja-j\alpha-\frac{j\tau n}{p}+\frac{jn}{p}}\\
&\sim \|\phi\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{L+1}(\mathbb{R}^n)}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)},
\end{aligned}
$$

which shows that $\sum_{j=0}^{\infty}\widetilde{\Upsilon}_j*\Psi_j*f$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

Since $\Upsilon\in\mathcal{S}_\infty(\mathbb{R}^n)$, for any $x\in\mathbb{R}^n$, $j\in\mathbb{Z}\setminus\mathbb{Z}_+$, $a\in\mathbb{R}_+$, $M_0\in\mathbb{N}$ and multi-indices γ, from [33, p. 141], we know that

$$
\left|\left(\partial^\gamma\widetilde{\Upsilon}_j\right)*\Psi_j*f(x)\right|\lesssim\|\Upsilon\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)}2^{j|\gamma|}\sum_{k\in\mathbb{Z}^n}\frac{(1+2^j|x|)^a}{(1+|k|)^{n+M_0+|\gamma|}}\inf_{z\in Q_{jk}}\left(\Psi_j^*f\right)_a(z). \tag{5.11}
$$

Let $|\gamma|>\alpha+\frac{n\tau}{p}-\frac{n}{p}$. From Corollary 2.1 and Theorem 3.1, we deduce that, for any $\phi\in\mathcal{S}(\mathbb{R}^n)$,

$$
\begin{aligned}
&\sum_{j=-\infty}^{-1}\left|\left\langle\left(\partial^\gamma\widetilde{\Upsilon}_j\right)*\Psi_j*f,\phi\right\rangle\right| \qquad (5.12)\\
&\quad\lesssim\|\Upsilon\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)}\sum_{j=-\infty}^{-1}\sum_{k\in\mathbb{Z}^n}\frac{2^{j|\gamma|}\|(\Psi_j^*f)_a\|_{(E_r^p)_t(Q_{jk})}}{\|\mathbf{1}_{Q_{jk}}\|_{(E_r^p)_t(\mathbb{R}^n)}}\frac{1}{(1+|k|)^{n+M_0+|\gamma|}}\\
&\qquad\times\int_{\mathbb{R}^n}(1+2^j|x|)^a|\phi(x)|\,dx\\
&\quad\lesssim\|\phi\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-M_0-|\gamma|}\\
&\qquad\times\sum_{j=-\infty}^{-1}2^{-j[-|\gamma|+\alpha+\frac{n\tau}{p}-\frac{n}{p}]}\\
&\quad\sim\|\phi\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)}\|\Upsilon\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)}\|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.
\end{aligned}
$$

Together with (5.10) and (5.12), using the proof of [9, pp. 153–154], we know that there exists a sequence $\{P_N\}_{N\in\mathbb{N}}$ of polynomials, with degree less than $\Gamma:=\lfloor\alpha+\frac{n\tau}{p}-\frac{n}{p}\rfloor$, and $g\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$
g=\lim_{N\to\infty}\left(\sum_{j=-N}^{\infty}\widetilde{\Upsilon}_j*\Psi_j*f+P_N\right)
$$

is in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representive of the equivalence class $f+\mathcal{P}(\mathbb{R}^n)$.

To prove that (5.8) is independent of the choices of Υ and Ψ, Let Υ^0, Ψ^0, $\{P_N^0\}_{N\in\mathbb{N}}$ and g^0 are another choice as in the previous paragraph, namely,

$$
g^0=\lim_{N\to\infty}\left(\sum_{j=-N}^{\infty}\widetilde{\Upsilon}_j^0*\Psi_j^0*f+P_N^0\right)
$$

in $\mathcal{S}'(\mathbb{R}^n)$. Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\widehat{\eta}(\xi)=1$ when $|\xi|\leqslant 2$ and $\widehat{\eta}(\xi)=0$ when $|\xi|>4$. As an argument similar to that used in the proof of [28, Lemma 4.2] (see also, [33, p. 142]), we know that, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left|\langle \partial^\gamma (g-g^0),\phi\rangle\right| \lesssim \lim_{N\to\infty} \sum_{j=-N}^{-N+2} \left\{ \left| \langle ((\partial^\gamma \widetilde{\Upsilon}_j) * \Psi_j * f)^\sim * \phi, \eta_{-N}\rangle \right| \right.$$
$$\left. + \left| \langle ((\partial^\gamma \widetilde{\Upsilon}_j^0) * \Psi_j^0 * f)^\sim * \phi, \eta_{-N}\rangle \right| \right\},$$

where, for any $x\in\mathbb{R}^n$, $f^\sim(x) := f(-x)$. Thus, for multi-indices γ with $|\gamma| > \lfloor \alpha + n\tau/p - n/p \rfloor$, similar to the estimate (5.12), using (5.11), we conclude that, for any $y\in\mathbb{R}^n$ and $j\in\mathbb{Z}\setminus\mathbb{Z}_+$,

$$\begin{aligned}
&\left| ((\partial^\gamma \widetilde{\Upsilon}_j^0) * \Psi_j^0 * f)^\sim * \phi(y) \right| \\
&\lesssim \|\Upsilon^0\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)} 2^{j|\gamma|}(1+2^j|y|)^a \int_{\mathbb{R}^n} (1+2^j|z|)^a \phi(z)\,dz \sum_{k\in\mathbb{Z}^n} \inf_{x\in Q_{jk}} \left(\Psi_j^* f\right)_a(x) \\
&\lesssim \|\phi\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)} \|\Upsilon\|_{\mathcal{S}_{M_0+1}(\mathbb{R}^n)} \|f\|_{(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} 2^{j[|\gamma|-\alpha-\frac{n\tau}{p}+\frac{n}{p}]}(1+2^j|y|)^a,
\end{aligned}$$

and the same estimate holds true also for $|((\partial^\gamma \widetilde{\Upsilon}_j) * \Psi_j * f)^\sim * \phi(y)|$. Therefore, we know that

$$\begin{aligned}
|\langle \partial^\gamma (g-g^0),\phi\rangle| &\lesssim \lim_{N\to\infty} \sum_{j=-N}^{-N+2} 2^{j[|\gamma|-\alpha+\frac{n\tau}{p}-\frac{n}{p}]} \int_{\mathbb{R}^n} |\eta_{-N}|(1+2^j|y|)^a\,dy \\
&\lesssim \lim_{N\to\infty} \sum_{j=-N}^{-N+2} 2^{j[|\gamma|-\alpha+\frac{n\tau}{p}-\frac{n}{p}]} = 0,
\end{aligned}$$

if $|\gamma| > \alpha + \frac{n\tau}{p} - \frac{n}{p}$. Therefore, the degree of $g-g_0$ is not more than $\lfloor \alpha + \frac{n\tau}{p} - \frac{n}{p} \rfloor$. Notice that, if h is a smooth analysis molecule, then $\int_{\mathbb{R}^n} x^\gamma h(x)\,dx = 0$ for any $\lfloor\gamma\rfloor \leqslant \lfloor \alpha + \frac{n\tau}{p} - \frac{n}{p} \rfloor$. Then by the argument used in [9, p. 155], we complete the proof of Lemma 5.2. $\square$

Using Lemmas 5.1 and 5.2, by the method pioneered by Frazier and Jawerth (see [9, Theorems 3.5 and 3.7], we obtain the following Theorem 5.2 and we omit the details here.

THEOREM 5.2. *Let $t, r, p \in (0,\infty)$. Let $\alpha\in\mathbb{R}$, $q\in(0,\infty]$ and let τ and ε_1 be as in Lemma 5.1.*

(i) If $\{m_Q\}_{Q\in\mathcal{Q}}$ is a family of synthesis molecules for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, then there exists a positive constant C such that, for any $u := \{u_Q\}_{Q\in\mathcal{Q}} \in (\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$,

$$\left\| \sum_{Q\in\mathcal{Q}} u_Q m_Q \right\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C\|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

(ii) If $\{b_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth analysis molecules for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, then there exists a positive constant C such that, for any $f\in(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$,

$$\|\{\langle f,b_Q\rangle\}_{Q\in\mathcal{Q}}\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant C\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

DEFINITION 5.3. Let α, q, τ, t, p, r and J be as in Definition 5.2. A function a_Q, with $Q\in\mathcal{Q}$, is called a smooth atom for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ supported near a dyadic cube Q if there exist $\widetilde{K}\in\mathbb{N}$ and $\widetilde{N}\in\mathbb{N}$ with $\widetilde{K}\geqslant\max\{\lfloor\alpha+n\tau/p\rfloor+1,0\}$ and $\widetilde{N}\geqslant\max\{\lfloor J-n-\alpha\rfloor,-1\}$ such that $\operatorname{supp}(a_Q)\subset 3Q$, $\int_{\mathbb{R}^n}x^\gamma a_Q(x)\,dx=0$ if $|\gamma|\leqslant\widetilde{N}$, and $|\partial^\gamma a_Q(x)|\leqslant|Q|^{-1/2-|\gamma|/n}$ for any $x\in\mathbb{R}^n$ if $|\gamma|\leqslant\widetilde{K}$.

A collection $\{a_Q\}_Q$ is called a *family of smooth atoms* for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, if each a_Q is a smooth atom for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ supported near Q.

Using Theorem 5.2 and repeating the argument as in [9, pp. 60–61] yield the following result; we omit the details.

THEOREM 5.3. *Let t, r, $p\in(0,\infty)$. Let $\alpha\in\mathbb{R}$, $q\in(0,\infty]$ and let τ and ε_1 be as in Lemma 5.1. Then for any $f\in(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, there exist smooth atoms $\{a_Q\}_{Q\in\mathcal{Q}}$ for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, and coefficients $u:=\{u_Q\}_{Q\in\mathcal{Q}}\in(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ such that $f=\sum_{Q\in\mathcal{Q}}u_Qa_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and*

$$\|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant C\|f\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)},$$

where C is a positive constant independent of f, u and t.

Conversely, there exists a positive constant C such that, for any family $\{a_Q\}_{Q\in\mathcal{Q}}$ of smooth atoms for $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $u:=\{u_Q\}_{Q\in\mathcal{Q}}\in(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$,

$$\left\|\sum_{Q\in\mathcal{Q}}u_Qa_Q\right\|_{(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}\leqslant C\|u\|_{(\dot{a}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)}.$$

6. Boundedness of Fourier multipliers on $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$

In this section, we first study the mapping property on $(\dot{A}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ for a class of Fourier multipliers, which was originally introduced by Cho and Kim [7] and Cho [6].

For $\ell\in\mathbb{N}$ and $\beta\in\mathbb{R}$, assume that $m\in C^\ell(\mathbb{R}^n\setminus\{\vec{0}_n\})$ satisfies that, for any $\sigma\in\mathbb{Z}^n_+$ and $|\sigma|\leqslant\ell$,

$$\sup_{R\in(0,\infty)}\left[R^{-n+2\beta+2|\sigma|}\int_{R\leqslant|\xi|\leqslant 2R}\left|\partial^\sigma_\xi m(\xi)\right|^2\,d\xi\right]\leqslant A_\sigma<\infty. \tag{6.1}$$

The Fourier multiplier T_m is defined by setting, for any $f\in\mathcal{S}_\infty(\mathbb{R}^n)$, $(\widehat{T_mf}):=m\widehat{f}$. Let K be the distribution whose Fourier transform is m. Recall that it was proved in [31, Lemma 3.1] that $K\in\mathcal{S}'_\infty(\mathbb{R}^n)$.

When $\beta=0$, the condition (6.1) is just the classical Hörmander condition (see, for example, [18, p. 263]). A typical example satisfying (6.1) with $\beta=0$ is the kernel of the Riesz transform R_j given by $\widehat{R_j f}(\xi) := -i\frac{\xi_j}{|\xi|}\widehat{(f)}(\xi)$ for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$ and $j \in \{1,\dots,n\}$. When $\beta \neq 0$, a typical example satisfying (6.1) for any $\ell \in \mathbb{N}$ is given by $m(\xi) := |\xi|^{-\beta}$ for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$; another example is the symbol of the differential operator ∂^σ of order $\beta := \sigma_1 + \cdots + \sigma_n$ with $\sigma := (\sigma_1,\dots,\sigma_n) \in \mathbb{Z}_+^n$.

In a suitable way, T_m can be defined on the whole spaces $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ and $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$. Let Υ and Ψ be Schwartz functions satisfy (1.1) through (1.3). For any $f \in (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ or $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, we define $T_m f$ by setting, for any $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\langle T_m f, \phi\rangle := \sum_{i\in\mathbb{Z}} f * \Upsilon_i * \Psi_i * K(\vec{0}_n) \tag{6.2}$$

as long as the right-hand side converges. In this sense, we say $T_m f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. The following result shows that $T_m f$ in (6.2) is well defined.

LEMMA 6.1. *Let $\ell \in (n/2,\infty)$, $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$, t, r, $p \in (0,\infty)$ and $q \in (0,\infty]$. Then $T_m f$ in (6.2) is independent of the choice of the pair (Υ,Ψ) of Schwartz functions satisfying (1.1) through (1.3). Moreover, $T_m f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.*

This lemma was proved in [14, Lemma 10.18] when taking $\mathcal{L} := (E^p_r)_t(\mathbb{R}^n)$, $\alpha_1 = \alpha_2 = \alpha$, $\alpha_3 = 0$ and $a \in (\frac{n}{\min\{p,q\}},\infty)$ for $(\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$ or $a \in (\frac{n}{p},\infty)$ for $(\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$.

The following lemma was proved in [31, Lemma 3.5].

LEMMA 6.2. *Let $\beta \in \mathbb{R}$, $\lambda \in (0,\infty)$, $\gamma \in [2,\infty)$, $\ell \in \mathbb{N}$, and ψ and Ψ be Schwartz functions satisfying (1.1) and (1.2). Assume that m satisfies (6.1) and $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $T_m f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.*

(i) If $\ell > \lambda + n/2$ and $\Upsilon = \Psi * \psi$, then for any x, $y \in \mathbb{R}^n$ and $s \in (0,\infty)$,

$$|(T_m f * \Upsilon_s)(y)| \leqslant C s^\beta \left(1 + \frac{|x-y|}{s}\right)^\lambda (\psi_s^* f)_\lambda(x).$$

(ii) If $\ell > \lambda + n(1/2 - 1/\gamma)$, then for any x, $y \in \mathbb{R}^n$ and $s \in (0,\infty)$ satisfying that $|x-y| < s$, $|(T_m f * \psi_s)(y)| \leqslant C s^\beta G^0_{\lambda,\gamma}(f)(x)$, where $G^0_{\lambda,\gamma}(f)(x)$ is as in (4.3).

THEOREM 6.1. *Let t, r, $p \in (0,\infty)$. Let α, $\gamma \in \mathbb{R}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$. Suppose that m satisfies (6.1) with $\ell \in \mathbb{N}$.*

(i) If $\ell > \frac{n}{\min\{p,q\}} + \frac{n}{2}$, then there exists a positive constant C such that, for any $f \in (\dot{F}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $\|T_m f\|_{(\dot{F}E^{\alpha+\gamma,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{F}E^{\gamma,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.

(ii) If $\ell > \frac{n}{p} + \frac{n}{2}$, then there exists a positive constant C such that, for any $f \in (\dot{B}E^{\alpha,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $\|T_m f\|_{(\dot{B}E^{\alpha+\gamma,\tau}_{r,p,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{B}E^{\gamma,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.

REMARK 6.1. Liang, Yang, Yuan, Sawano and Ullrich [13] obtained Theorem 6.1. However, we reobtain Theorem 6.1 by a different method.

Proof of Theorem 6.1. We only give the proof of (i) by similarity. Let ψ and Ψ be Schwartz functions satisfying (1.1) and (1.2). Then $\Upsilon := \psi * \Psi$ also satisfies (1.1) and (1.2). Since $\ell > \frac{n}{\min\{p,q\}} + \frac{n}{2}$, we can choose $a > \frac{n}{\min\{p,q\}} + \frac{n}{2}$ such that $\ell > a + \frac{n}{2}$. Thus, by Lemma 6.2(i), we conclude that, for any $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, $2^{j\alpha}(\Psi_j^*(T_m f)_a)(x) \lesssim (\psi_j^* f)_a(x)$, which together with Theorem 3.1 and Corollary 2.2, implies that $\|T_m f\|_{(\dot{F}E^{\alpha+\gamma,\tau}_{r,p,q})_t(\mathbb{R}^n)} \lesssim \|f\|_{(\dot{F}E^{\gamma,\tau}_{r,p,q})_t(\mathbb{R}^n)}$ and hence completes the proof of Theorem 6.1. $\square$

REMARK 6.2. Let t, r, p be as in Remark 2.1(ii). Then, Theorem 6.1 coincides with [31, Theorem 1.5].

THEOREM 6.2. *Let $t, r, p \in (0,\infty)$. Let $\alpha, \beta \in \mathbb{R}$ with $\beta > \alpha$ and $\gamma, q \in (0,\infty]$. Let $p_0 \in (0,\infty)$ be such that $\alpha - n/p_0 = \beta - n/p$ and m satisfy (6.1) with $\ell \in \mathbb{N}$ and $\ell > n/2$. Let $\tau^* = \frac{\tau p_0}{p}$, $r^* = rp_0/p$, $p^* = p_0$ and $s = p_0/p$.*

(i) *If $\tau \in [0,1) \cup [1,\infty)$, then there exists a positive constant C such that, for any $f \in (\dot{F}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $\|T_m f\|_{(\dot{F}E^{\alpha,\tau^*}_{r^*,p^*,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)}$.*

(ii) *If $p_0 > p$, then there exists a positive constant C such that, for any $\tau \in [0,\infty)$ and $f \in (\dot{F}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)$, $\|T_m f\|_{(\dot{F}E^{\alpha,\tau^*}_{r^*,p^*,q})_t(\mathbb{R}^n)} \leqslant C\|f\|_{(\dot{F}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)}$.*

Proof. To show (i), we consider two cases for τ.

Case I: $\tau \in [0,1)$. In this case, assume that $f \in (\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)$ and $\gamma \in [2,\infty]$. By the assumption that $\ell > n/2$, we know that there exists $\lambda > n/\gamma$ such that $\ell > \lambda + n/2 - n/\gamma$. Then by Lemma 6.2(ii), we conclude that, for any $x, y \in \mathbb{R}^n$ and $s \in (0,\infty)$ satisfying that $|x-y| < s$,

$$|U(y,s)| \lesssim s^{\beta} G^0_{\lambda,\gamma}(f)(x), \tag{6.3}$$

where $U(y,s) := (T_m f * \Psi_s)(y)$, for any $y \in \mathbb{R}^n$ and $s \in (0,\infty)$, and Ψ is as in the proof of Theorem 6.1.

If $\|f\|_{(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)} = 0$, from Theorem 4.2, we deduce that $\|G^0_{\lambda,\gamma}(f)\|_{(E^p_r)^\tau_t(\mathbb{R}^n)} = 0$, and hence $G^0_{\lambda,\gamma}(f)$ for almost every $x \in \mathbb{R}^n$, which, together with (6.3), implies that $U(y,s) = 0$ for any $y \in \mathbb{R}^n$ and $s \in (0,\infty)$. We then conclude that $\|T_m f\|_{(\dot{F}E^{\alpha,\tau^*}_{r^*,p^*,q})_t(\mathbb{R}^n)}$.

If $\|f\|_{(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)} > 0$, from Theorem 4.2, we deduce that $\|G^0_{\lambda,\gamma}(f)\|_{(E^p_r)^\tau_t(\mathbb{R}^n)} > 0$. Let P be a dyadic cube and $s \in (0,\ell(P))$. Then, it holds that $\{y : \operatorname{dist}(y,P) < s\} \subset 3P$. By (6.3) and the fact that $\|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)} \gtrsim |P|^{\frac{1}{p}}$, we see that $|U(y,s)| \lesssim s^{\beta-\frac{n}{p}}\|G^0_{\lambda,\gamma}(f)\|_{(E^p_r)_t(3P)}$. By this, (6.3), $\beta > \alpha$ and an argument similar to that used in the proof of [31, (3.34)], we conclude that, for any $x \in P$,

$$\left\{\int_0^{\ell(P)} s^{-\alpha q}\int_{|x-y|<s} |U(y,s)|^q\,dy\,\frac{ds}{s^{n+1}}\right\}^{1/q} \tag{6.4}$$

$$\lesssim \left[G^0_{\lambda,\gamma}(f)(x)\right]^{\frac{p}{p_0}} \left\|G^0_{\lambda,\gamma}(f)\right\|^{(1-\frac{p}{p_0})}_{(E^p_r)^\tau_t(\mathbb{R}^n)} \|\mathbf{1}_P\|^{\tau(1-\frac{p}{p_0})}_{(E^p_r)_t(\mathbb{R}^n)}.$$

Then by Theorem 3.1, (6.4), $\lambda > n/\gamma$ and Theorem 4.2, we obtain that

$$\begin{aligned}
&\|T_m f\|_{(\dot{F}E^{\alpha,\tau^*}_{r^*,p^*,q})_t(\mathbb{R}^n)} \\
&= \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau^*}_{(E^{p^*}_{r^*})_t(\mathbb{R}^n)}} \left\| \left\{ \int_0^{\ell(P)} s^{-\alpha q} \int_{|\cdot-y|<s} |U(y,s)|^q\,dy\,\frac{ds}{s^{n+1}} \right\}^{1/q} \right\|_{(E^{p^*}_{r^*})_t(P)} \\
&\lesssim \left[\sup_{P\in\mathcal{Q}} \frac{\|\mathbf{1}_P\|^{\tau(1-\frac{p}{p_0})}_{(E^p_r)_t(\mathbb{R}^n)}}{\|\mathbf{1}_P\|^{\tau^*}_{(E^{p^*}_{r^*})_t(\mathbb{R}^n)}} \left\| \left[G^0_{\lambda,\gamma}(f)\right]^{\frac{p}{p_0}} \right\|_{(E^{p^*}_{r^*})_t(P)} \right] \left\|G^0_{\lambda,\gamma}(f)\right\|^{1-\frac{p}{p_0}}_{(E^p_r)^\tau_t(\mathbb{R}^n)} \\
&= \left[\sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|^{\tau}_{(E^p_r)_t(\mathbb{R}^n)}} \left\|G^0_{\lambda,\gamma}(f)\right\|_{(E^p_r)_t(P)} \right]^{\frac{p}{p_0}} \left\|G^0_{\lambda,\gamma}(f)\right\|^{1-\frac{p}{p_0}}_{(E^p_r)^\tau_t(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)}.
\end{aligned}$$

When $f \in (\dot{F}E^{0,\tau}_{r,p,q})_t(\mathbb{R}^n)$ with $r \in (0,2)$, the desired result is a direct consequence of the case $\gamma \in [2,\infty]$, together with the embedding $(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n) \subset (\dot{F}E^{0,\tau}_{r,p,2})_t(\mathbb{R}^n)$ (see Proposition 2.1).

Case II: $\tau \in [1,\infty)$. In this case, from $\beta > \alpha$, it follows that $p_0 > p$. By this we see

$$\tau^* \geqslant \frac{p_0}{p} > 1.$$

By the assumption that $\ell > n/2$, we know that there exists $\lambda > 0$ such that $\ell > \lambda + n/2$. Then letting Υ be as in Lemma 6.2, from Theorem 4.1(ii), Lemma 6.2(i), $\alpha - n/p_0 = \beta - n/p$ and the fact that $\|\mathbf{1}_P\|_{(E^p_r)_t(\mathbb{R}^n)} \gtrsim |P|^{\frac{1}{p}}$, it follows that

$$\begin{aligned}
\|T_m f\|_{(\dot{F}E^{\alpha,\tau^*}_{r^*,p^*,q})_t(\mathbb{R}^n)} &\sim \sup_{Q\in\mathcal{Q}} \inf_{x\in Q} |Q|^{-\frac{\alpha}{n}} \|\mathbf{1}_Q\|^{1-\tau^*}_{(E^{p^*}_{r^*})_t(\mathbb{R}^n)} \left(\Upsilon^*_{j_Q}(T_m f)\right)_\lambda (x) \\
&\lesssim \sup_{Q\in\mathcal{Q}} \inf_{x\in Q} |Q|^{\frac{\beta}{n}-\frac{\alpha}{n}} \|\mathbf{1}_Q\|^{\frac{p}{p_0}-\tau}_{(E^p_r)_t(\mathbb{R}^n)} \left(\psi^*_{j_Q} f\right)_\lambda (x) \\
&\lesssim \sup_{Q\in\mathcal{Q}} \|\mathbf{1}_Q\|^{-\tau}_{(E^p_r)_t(\mathbb{R}^n)} \left\| \left(\psi^*_{j_Q} f\right)_\lambda \right\|_{(E^p_r)_t(\mathbb{R}^n)} \lesssim \|f\|_{(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of *Case II* and hence (i).

(ii) When $\tau \in [0,1)$, the conclusion is a consequence of (i). To complete the proof of (ii), it suffices to prove the result for the case that $\tau \in [1,\infty)$. Since $p_0 > p$, we deduce that

$$\tau^* \geqslant \frac{p_0}{p} > 1,$$

which, together with Theorem 4.1(ii) and an argument similar to that used in the proof of *Case II* in (i), implies that $\|T_m f\|_{(\dot{F}E^{\alpha,\tau^*}_{r^*,p^*,q})_t(\mathbb{R}^n)} \lesssim \|f\|_{(\dot{F}E^{0,\tau}_{r,p,\gamma})_t(\mathbb{R}^n)}$. This finishes the proof of (ii) and hence Theorem 6.2. □

REMARK 6.3. Let t, r, p be as in Remark 2.1(ii). Then, Theorem 6.2 coincides with [31, Theorem 1.7].

Competing interests. The authors declare that they have no competing interests.

Funding. The research was supported by Natural Science Foundation of China (Grant No. 12061069).

Acknowledgements. All authors would like to express their thanks to the referees for valuable advice regarding previous version of this paper.

REFERENCES

[1] Z. V. DE P. ABLÉ AND J. FEUTO, *Atomic decomposition of Hardy-amalgam spaces*, (English summary), J. Math. Anal. Appl. **455** (2017), no. 2, 1899–1936.
[2] P. AUSCHER AND M. MOURGOGLOU, *Representation and uniqueness for boundary value elliptic problems via first order systems*, Rev. Mat. Iberoam. **359** (2019), 241–315.
[3] P. AUSCHER AND C. PRISUELOS-ARRIBAS, *Tent space boundedness via extrapolation*, (English summary), Math. Z. **286** (2017), no. 3–4, 1575–1604.
[4] H.-Q. BUI, M. PALUSZYŃSKI AND M. TAIBLESON, *A maximal function characterization of weighted Besov–Lipschitz and Triebel–Lizorkin spaces*, (English summary), Studia Math. **119** (1996), no. 3, 219–246.
[5] H.-Q. BUI, M. PALUSZYŃSKI AND M. TAIBLESON, *Characterization of the Besov–Lipschitz and Triebel–Lizorkin spaces. The case* $q < 1$, Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), J. Fourier Anal. Appl. **3** (1997), Special Issue, 837–846.
[6] Y.-K. CHO, *Continuous characterization of the Triebel–Lizorkin spaces and Fourier multipliers*, (English summary), Bull. Korean Math. Soc. **47** (2010), no. 4, 839–857.
[7] Y.-K. CHO AND D. KIM, *A Fourier multiplier theorem on the Besov-Lipschitz spaces*, Korean J. Math. **16** (2008), 85–90.
[8] M. FRAZIER AND B. JAWERTH, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799.
[9] M. FRAZIER AND B. JAWERTH, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34–171.
[10] P. LI, J. XIAO AND Q. YANG, *Global mild solutions to modified Navier-Stokes equations with small initial data in critical Besov-Q spaces*, Electron. J. Differential Equations (2014), no. **185**, 37 pp.
[11] P. LI AND Z. ZHAI, *Generalized Navier-Stokes equations with initial data in local Q-type spaces*, J. Math. Anal. Appl. **369** (2010), no. 2, 595–609.
[12] P. LI AND Z. ZHAI, *Well-posedness and regularity of generalized Navier–Stokes equations in some critical Q-spaces*, J. Funct. Anal. **259** (2010), no. 10, 2457–2519.
[13] Y. LIANG, Y. SAWANO, T. ULLRICH, D. YANG AND W. YUAN, *New characterizations of Besov–Triebel–Lizorkin-Hausdorff spaces including coorbits and wavelets*, J. Fourier Anal. Apll. **18** (2012), 1067–1111.
[14] Y. LIANG, D. YANG, W. YUAN, Y. SAWANO AND T. ULLRICH, *A new framework for generalized Besov-type and Triebel–Lizorkin-type spaces*, Dissertationes Math. **489** (2013), 114 pp.
[15] L. LIU, *Hardy spaces via distribution spaces*, Front. Math. China **2** (2007), 599–611.
[16] Y. LU, J. ZHOU AND S. WANG, *Necessary and sufcient conditions for boundedness of commutators associated with Calderón–Zygmund operators on slice spaces*, Ann. Funct. Anal. **13** (2022), no. 61, 19 pp.
[17] V. S. RYCHKOV, *On a theorem of Bui, Paluszyński, and Taibleson*, (Russian), Tr. Mat. Inst. Steklova 227 (1999), Issled. po Teor. Differ. Funkts. Mnogikh Perem. i ee Prilozh. 18, 286–298; translation in Proc. Steklov Inst. Math. (1999), no. **4** (227), 280–292.
[18] E. M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[19] H. TRIEBEL, *Theory of function spaces. II*, Monographs in Mathematics, **84**, Birkhäuser Verlag, Basel, (1992), viii+370 pp.

[20] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*, Second edition, Johann Ambrosius Barth, Heidelberg, (1995), 532 pp.

[21] H. TRIEBEL, *Theory of function spaces. III*, Monographs in Mathematics, **100**, Birkhäuser Verlag, Basel, (2006), xii+426 pp.

[22] H. TRIEBEL, *Theory of function spaces*, Reprint of 1983 edition [MR0730762]. Also published in 1983 by Birkhüser Verlag [MR0781540]. Modern Birkhüser Classics, Birkhäuser/Springer Basel AG, Basel, (2010), 285 pp.

[23] T. ULLRICH, *Continuous characterization of Besov–Lizorkin–Triebel space and new interpretations as coorbits*, J. Funct. Space Appl. Article ID 163213 (2012).

[24] N. WIENER, *On the representation of functions by trigonometrical integrals*, Math. Z. **24** (1926), no. 1, 575–616.

[25] J. XU, *Variable Besov and Triebel–Lizorkin spaces*, Ann. Acad. Sci. Fenn. Math. **33** (2008), 511–522.

[26] J. XU, *The relation between variable Bessel potential spaces and Triebel–Lizorkin spaces*, Integral Transforms Spec. Funct. **19** (2008), 599–605.

[27] D. YANG AND W. YUAN, *A new class of function spaces connecting Triebel–Lizorkin spaces and Q spaces*, J. Funct. Anal. **255** (2008), 2760–2809.

[28] D. YANG AND W. YUAN, *New Besov-type spaces and Triebel–Lizorkin-type spaces including Q spaces*, Math. Z. **265** (2010), 451–480.

[29] D. YANG AND W. YUAN, *Characterizations of Besov-type and Triebel–Lizorkin-type spaces via maximal functions and local means*, Nonlinear Anal. **73** (2010), no. 12, 3805–3820.

[30] D. YANG AND W. YUAN, *Relations among Besov-type spaces, Triebel–Lizorkin-type spaces and generalized Carleson measure spaces*, Appl. Anal. **92** (2013), no. 3, 549–561.

[31] D. YANG, W. YUAN AND C. ZHUO, *Fourier multipliers on Triebel–Lizorkin-type spaces*, J. Funct. Spaces Appl. Article ID 431016 (2012).

[32] D. YANG, W. YUAN AND C. ZHUO, *Complex interpolation on Besov-type and Triebel–Lizorkin-type spaces*, Anal. Appl. (Singap.) **11** (2013), no. 5, 1350021, 45 pp.

[33] D. YANG, W. YUAN AND C. ZHUO, *Musielak–Orlicz Besov-type and Triebel–Lizorkin-type spaces*, Rev. Mat. Complut. **27** (2014), 93–157.

[34] D. YANG, C. ZHUO AND W. YUAN, *Besov-type spaces with variable smoothness and integrability*, J. Funct. Anal. **269** (2015), no. 6, 1840–1898.

[35] D. YANG, C. ZHUO AND W. YUAN, *Triebel–Lizorkin type spaces with variable exponents*, Banach J. Math. Anal. **9** (2015), no. 4, 146–202.

[36] W. YUAN, W. SICKEL AND D. YANG, *Morrey and Campanato meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics, **2005**, Springer-Verlag, Berlin, (2010), xii+281 pp.

[37] W. YUAN, W. SICKEL AND D. YANG, *On the coincidence of certain approaches to smoothness spaces related to Morrey spaces*, (English summary), Math. Nachr. **286** (2013), no. 14–15, 1571–1584.

[38] Y. ZHANG, D. YANG, W. YUAN AND S. WANG, *Real-variable characterizations of Orlicz-slice Hardy spaces*, Anal. Appl. **17** (2019), 597–664.

(Received May 30, 2022)

Yuan Lu
College of Mathematics and System Science
Xinjiang University
Urumqi, 830046, P.R. China
e-mail: Luyuan_y@163.com

Jiang Zhou
College of Mathematics and System Science
Xinjiang University
Urumqi, 830046, P.R. China
e-mail: zhoujiang@xju.edu.cn