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COMPLETE CONSISTENCY AND ASYMPTOTIC NORMALITY FOR THE WEIGHTED ESTIMATOR IN A NONPARAMETRIC REGRESSION MODEL UNDER DEPENDENT ERRORS

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Abstract. In this paper, we investigate the effect of dependent errors in the fixed design nonparametric regression models. Under some mild conditions, we obtain the complete consistency and asymptotic normality for the weighted estimator in the fixed design nonparametric regression models. In addition, a simulation study is undertaken to investigate finite sample behavior of the estimator.

1. Introduction

Nonparametric regression model has been an important object of study in econometrics and statistics for a long time. Because of recent theoretical developments and widespread use of fast and inexpensive computers, nonparametric regression has become a rapidly growing and exciting field of statistics. Researchers have realized that for many real data sets, parametric regression is not sufficiently flexible to adequately fit curves or surfaces. Recent monographs on nonparametric regression (see Muler (1988), Hardle (1990), Fan and Gijbels (1996)) have shown that a variety of interesting examples and applications of nonparametric regression have yielded analyses essentially unobtainable by other techniques.

We consider the following fixed design nonparametric regression model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \quad 1 \leqslant i \leqslant n, \ \ n \geqslant 1, \tag{1.1}$$

where the design points $x_{n1}, \ldots, x_{nn} \in A$, which is a compact set in $\mathbb{R}^d$, g is an unknown real valued function on A, and the $\{\varepsilon_{ni}\}$ are random errors.

To estimate the regression function g, Georgiev (1985) considered the following weighted regression estimator:

$$g_n(x) = \sum_{i=1}^{n} w_{ni}(x) Y_{ni}, \tag{1.2}$$

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where the weight functions $w_{ni}(x)$, $i = 1,\dots,n$, depend on the fixed design points $x_{n1},\dots,x_{nn}$ and on the number of observations.

The above estimator has been studied by many authors and many interesting results have been obtained. For example, Hu et al. (2003) gave the mean consistency, complete consistency, and asymptotic normality of regression models with linear process errors; Liang and Jing (2005) established the consistency, uniform consistency, and asymptotic normality of $g_n(x)$ under negatively associated (NA) samples; Shen (2013) established the Bernstein-type inequality for widely dependent sequence and gave its application to nonparametric regression models; Wang et al. (2014) and Wang and Si (2015) studied the complete consistency of the estimator of nonparametric regression models under widely orthant dependent errors and negatively orthant dependent errors, respectively; Shen et al. (2015) provided the Rosenthal-type inequality for negatively superadditive dependent random variables and gave its application to nonparametric regression models; Shen (2016) established the complete convergence for weighted sums of END random variables and its application to nonparametric regression models; Shen et al. (2016) presented some exponential probability inequalities for WNOD random variables with applications to nonparametric regression models; Wu et al. (2019) studied the asymptotic normality and mean consistency for the weighted estimator in nonparametric regression models. In this article, our aim is to establish the asymptotic properties for the estimator $g_n(x)$ of nonparametric regression models based on (α,β)-mixing errors.

We first introduce the concept of (α,β)-mixing random variables as follows.

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega,\mathcal{F},P)$. Denote $S_n = \sum_{i=1}^{n} X_i, n \geqslant 1$, and $S_0 = 0$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leqslant i \leqslant m)$. Given σ-algebras $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{F}$, let

$$\lambda(\mathcal{A},\mathcal{B}) = \sup_{X\in L_{1/\alpha}(\mathcal{A}),Y\in L_{1/\beta}(\mathcal{B})} \frac{|EXY - EXEY|}{\|X\|_{1/\alpha}\|Y\|_{1/\beta}},$$

where $0 < \alpha,\beta < 1, \alpha+\beta = 1$, and $\|X\|_p = (E|X|^p)^{1/p}$. Define the (α,β)-mixing coefficients by

$$\lambda(n) = \sup_{k\geqslant 1} \lambda(\mathcal{F}_1^k, \mathcal{F}_{k+n}^{\infty}), \quad n \geqslant 0.$$

DEFINITION 1.1. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be (α,β)-mixing if $\lambda(n) \downarrow 0$ as $n \to \infty$.

Since the concept of (α,β)-mixing was introduced by Bradley and Bryc (1985), many limit theorems have been established. Bradley and Bryc (1985) discussed central limit question under absolute regularity for (α,β)-mixing sequences; Shao (1989) established limit theorems of (α,β)-mixing sequences, including strong convergence and complete convergence; Cai (1991) obtained strong consistency and rates for recursive nonparametric conditional probability density estimator under (α,β)-mixing conditions; Lu and Lin (1997) gave the bounds of covariance of (α,β)-mixing sequences; Shen and Zhang (2011) studied some convergence theorems for (α,β)-mixing random variables, and obtained some new strong laws of large numbers for weighted sums of (α,β)-mixing random variables; Gao (2016) investigated the (α,β)-mixing sequences which are stochastically dominated, and presented the strong stability; Yu (2016) showed the Resenthal-type inequality of the (α,β)-mixing sequences, and investigated the strong convergence theorems; Samura et al. (2019) investigated the strong consistency, complete consistency and mean consistency for the estimators of partially linear regression models under (α,β)-mixing errors.

Next, we give the concept of stochastic domination which will be used in this work.

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X, if there exists a positive constant C such that

$$P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Inspired by the literatures above, we want to investigate the complete consistency and asymptotic normality of the estimator $g_n(x)$ in (1.2) for the nonparametric regression model under (α,β)-mixing sequences.

This paper is organized as follows: In Section 2, the main results will be presented and a finite sample performance of the proposed method is tested in simulations. The proofs of the main results are provided in Section 3.

Throughout this paper, C denotes a positive constant not depending on n, which may be different in various places. $a_n = O(b_n)$ represents $a_n \leqslant Cb_n$ for all $n \geqslant 1$. Let $\lfloor x \rfloor$ denote the integer part of x and let $I(A)$ be the indicator function of the set A. Denote $x^+ = xI(X \geqslant 0)$ and $x^- = -xI(x < 0)$. $a \wedge b$ implies $\min(a,b)$.

2. Main results and numerical analysis

2.1. Main results

In order to investigate the asymptotic properties of the estimator $g_n(x)$, we need the following assumptions.

ASSUMPTION A0.
(i) $\sum_{i=1}^n w_{ni}(x) \to 1$ as $n \to \infty$;
(ii) $\sum_{i=1}^n |w_{ni}(x)| \leqslant C$ for all $n \geqslant 1$, where $C > 0$ is a positive constant;
(iii) $\sum_{i=1}^n |w_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

The Assumption A0 above will be used to prove the consistency of the estimator $g_n(x)$. The following four assumptions will be used to establish the asymptotic normality of the estimator $g_n(x)$.

ASSUMPTION A1.
(i) $g : A \to \mathbb{R}$ is a continuous function on the compact subset A in $\mathbb{R}^d$;
(ii) $\{\varepsilon_i, i \geqslant 1\}$ is a sequence of identically distributed (α,β)-mixing random variables with mixing coefficients $\{\lambda(n), n \geqslant 1\}$ satisfying $\sum_{n=1}^\infty (\lambda(n))^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})} < \infty$, $E\varepsilon_1 = 0$ and $Var(\varepsilon_1) = \sigma^2 \in (0,\infty)$.
(iii) For each $n \geqslant 1$, the joint distribution of $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n\}$ is the same as that of $\{\varepsilon_i, 1 \leqslant i \leqslant n\}$.

Denote

$$w_n(x) = \max\{|w_{ni}(x)| : 1 \leqslant i \leqslant n\} \text{ and } \sigma_n^2(x) = Var(g_n(x)).$$

ASSUMPTION A2.
(i) $\sum_{i=1}^n |w_{ni}(x)| \leqslant C$ *for all* $n \geqslant 1$;
(ii) $w_n(x) = O(\sum_{i=1}^n w_{ni}^2(x))$;
(iii) $\sum_{i=1}^n w_{ni}^2(x) = O(\sigma_n^2(x))$.

ASSUMPTION A3. $E|\varepsilon_1|^r < \infty$ for some $r > 2$.

ASSUMPTION A4. There exist positive integers $p = p(n)$ and $q = q(n)$ such that $p + q \leqslant n$ for sufficiently large n and as $n \to \infty$,

$$nqp^{-1}w_n(x) \to 0,\ pw_n(x) \to 0,\ qp^{-1} \to 0,\ np^{r/2-1}w_n^{r/2} \to 0,$$

$$np^{-1/2}w_n^{1/2}\lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(q) \to 0.$$

REMARK 2.1. (i) Wang et al. (2015) assumed that Assumption A0 is satisfied for the nearest neighbor weights. (ii) It is known that Assumptions A1–A4 are mild regularity conditions and have been used by Yang (2007) and Wang et al. (2017) among others.

Now, we present the main results of this paper. The first one is the complete consistency of the estimator $g_n(x)$.

THEOREM 2.1. *Let $\{\varepsilon_i, i \geqslant 1\}$ be a sequence of (α,β)-mixing random errors stochastically dominated by a random variable X with $\sum_{n=1}^\infty (\lambda(n))^{(1/2\alpha)\wedge(1/2\beta)} < \infty$, where $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. Suppose that Assumption A0 holds, and there exists some $s > 0$ such that $E|X|^{1+1/s} < \infty$, and*

$$\max_{1\leqslant i\leqslant n} |w_{ni}(x)| = O(n^{-s}).$$

Then for all $x \in c(g)$,

$$g_n(x) \to g(x) \ \ completely.$$

Next, we state the result for the asymptotic normality of the estimator $g_n(x)$.

THEOREM 2.2. *Under the Assumptions A1–A4, we have*

$$\sigma_n^{-1}(x)\{g_n(x) - Eg_n(x)\} \xrightarrow{d} N(0,1). \tag{2.2}$$

2.2. Numerical simulation

In this subsection, we carry out a simulation to study the finite sample performance of the estimator of g. We simulate from the following model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \quad 1 \leqslant i \leqslant n, \quad n \geqslant 1,$$

where $g(x) = x^2$. Put $A = [0,1]$ and take $x_{ni} = i/n,\ \ i = 1,2,\ldots,n$. For any $x \in A$, we rewrite

$$|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$$

as follows:

$$|x_{R_1(x),n} - x| \leqslant |x_{R_2(x),n} - x| \leqslant \ldots \leqslant |x_{R_n(x),n} - x|,$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is permutated before $|x_{nj} - x|$ when $x_{ni} < x_{nj}$. Let $1 \leqslant k_n \leqslant n$, the nearest neighbor weight function is defined as follows:

$$\widetilde{w}_{ni}(x) = \begin{cases} 1/k_n, \text{ if } \ |x_{ni} - x| \leqslant |x_{R_{k_n}(x),n} - x|, \\ 0, \quad \text{otherwise}, \end{cases}$$

and the nearest neighbor weight function estimator of g is

$$\widetilde{g}_n(x) = \sum_{i=1}^{n} \widetilde{w}_{ni}(x) Y_{ni}.$$

It is easy to check that Assumption A0 is satisfied for $\widetilde{w}_{ni}(x)$, where $w_{ni}(x)$ is replaced by $\widetilde{w}_{ni}(x)$.

For fixed positive integer m, let $e_i \overset{i.i.d.}{\sim} N(0, \sigma_0^2)$, where $\sigma_0^2 = \frac{1}{m+1}$. Let $\varepsilon_i = \sum_{k=0}^{m} e_{i+k}$ for $i \geqslant 1$. Then $\{\varepsilon_i, i \geqslant 1\}$ is a sequence of m-dependent random variables, and thus a sequence of (α, β)-mixing random variables with $\varepsilon_i \sim N(0,1)$. Take $m = 10$ and $k_n = \lfloor n^{0.6} \rfloor$.

Consistency

We generate the observed data with sample sizes $n = 500$, 1000 and 1500 respectively, from the model above. In Figure 1, we plot $\widetilde{g}_n(x) - g(x)$ with $g(x) = x^2$, $x = 0.25$, 0.5 and 0.75 respectively. The quality of fit for the estimator $\widetilde{g}_n(x)$ increases as increasing of the sample size n.

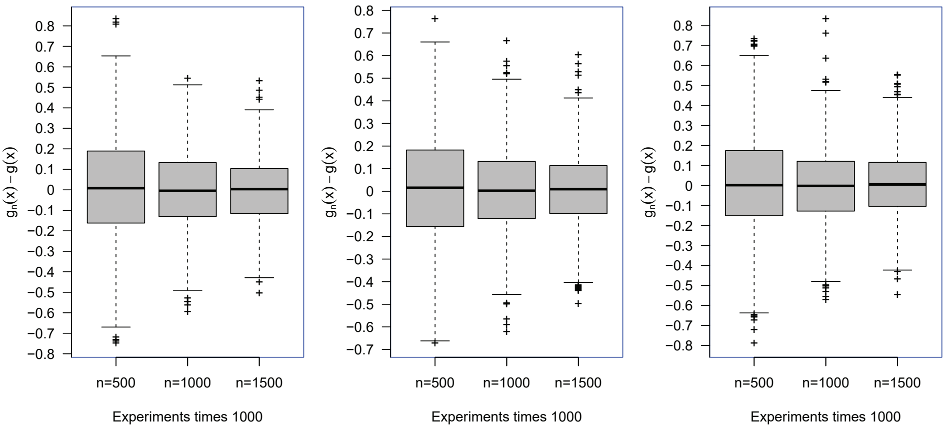

Figure 1: *Boxplots of* $\widetilde{g}_n(x) - g(x)$ *with* $x = 0.25$, 0.5 *and* 0.75, *respectively.*

Asymptotic normality

We examine how good the asymptotic normality of the estimator $\widetilde{g}_n(x)$. In particular, we plot the Q-Q-plots of $\widetilde{g}_n(x)$ at $x = 0.5$, based on 1000 replications with sample sizes $n = 500$, 1000 and 1500, respectively. From Figure 2, it can be seen that the sampling distribution of the estimator fits reasonably normal, and the quality of fit increases as increasing of the sample size n.

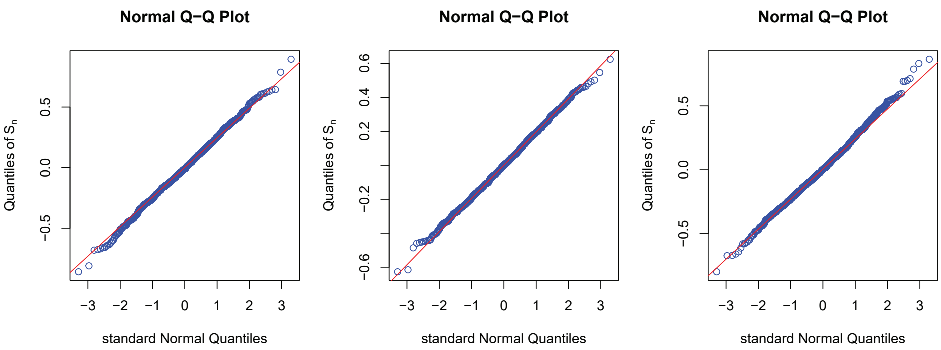

Figure 2: *The Normal Q-Q Plots of* $\widetilde{g}_n(0.5)$ *for* n *=500, 1000 and 1500, respectively.*

3. Proofs of the main results

We first introduce several lemmas which will be used to prove the main results of the paper. The first one is the Rosenthal type inequality for weighted sums of (α,β)-mixing random variables, which can be found in Yu (2016) for instance.

LEMMA 3.1. *Let* $\{X_i, i \geqslant 1\}$ *be a sequence of* (α,β)*-mixing random variables with* $EX_i = 0$, $E|X_i|^p < \infty$ *for some* $p \geqslant 2$ *and* $\sum_{n=1}^{\infty}(\lambda(n))^{(1/2\alpha)\wedge(1/2\beta)} < \infty$*, where* $0 < \alpha, \beta < 1$ *and* $\alpha + \beta = 1$*. Assume that* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is an array of real numbers. Then there exists a positive constant* C *depending only on* α, β *and* $\lambda(\cdot)$ *such that*

$$E\left|\sum_{i=1}^{n} a_{ni}X_i\right|^p \leqslant C\left\{\sum_{i=1}^{n}|a_{ni}|^p E|X_i|^p + \left(\sum_{i=1}^{n} a_{ni}^2 EX_i^2\right)^{p/2}\right\}.$$

The next one is a basic property for stochastic domination. The first inequality is due to Adler and Rosalsky (1987) and the second inequality is due to Adler et al. (1989).

LEMMA 3.2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X*. Then for any* $\alpha > 0$ *and* $b > 0$*, the following two statements hold:*

$$E|X_n|^{\alpha} I(|X_n| \leqslant b) \leqslant C_1[E|X|^{\alpha} I(|X| \leqslant b) + b^{\alpha} P(|X| > b)],$$

$$E|X_n|^{\alpha} I(|X_n| > b) \leqslant C_2 E|X|^{\alpha} I(|X| > b),$$

where C_1 and C_2 are two positive constants.

The following one gives the bound of covariance of (α,β)-mixing sequences, which can be found in Lu and Lin (1997) for instance.

LEMMA 3.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of (α,β)-mixing random variables. Suppose that $X \in L_p(\mathcal{F}_{-\infty}^k)$ and $Y \in L_q(\mathcal{F}_{k+n}^\infty)$, where $p,q \geqslant 1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$|EXY - EXEY| \leqslant 4\lambda^{(1/2\alpha)\wedge(1/2\beta)}(n)\|X\|_p\|Y\|_q.$$

With Lemma 3.3 accounted for, we can get the following inequality about the difference between multiplicative expectations and multiplication of expectations, which will be used to prove the asymptotic normality of the estimator $g_n(x)$ in (1.2).

LEMMA 3.4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of (α,β)-mixing random variables. Suppose that p and q are two positive integers. Let $\eta_l = \sum_{j=(l-1)(p+q)+1}^{(l-1)(p+q)+p} X_j$ for $1 \leqslant l \leqslant k$. Then for any $t \in \mathbb{R}$*

$$\left| E\exp\left(it\sum_{l=1}^k \eta_l\right) - \prod_{l=1}^k E\exp(it\eta_l)\right| \leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{l=1}^k \|\eta_l\|_2.$$

Proof. It is easily checked that

$$\begin{aligned}
&\left| E\exp\left(it\sum_{l=1}^k \eta_l\right) - \prod_{l=1}^k E\exp(it\eta_l)\right| \\
&\leqslant \left| E\exp\left(it\sum_{l=1}^k \eta_l\right) - E\exp\left(it\sum_{l=1}^{k-1} \eta_l\right)E\exp(it\eta_l)\right| \\
&\quad + \left| E\exp\left(it\sum_{l=1}^{k-1} \eta_l\right) - \prod_{l=1}^{k-1} E\exp(it\eta_l)\right| \\
&\doteq J_1 + J_2. \qquad (3.1)
\end{aligned}$$

Noting that $e^{ix} = \cos x + i\sin x$, we have

$$\begin{aligned}
J_1 \leqslant & \left| Cov\left(\cos\left(t\sum_{l=1}^{k-1}\eta_l\right), \cos(t\eta_k)\right)\right| + \left| Cov\left(\sin\left(t\sum_{l=1}^{k-1}\eta_l\right), \sin(t\eta_k)\right)\right| \\
& + \left| Cov\left(\sin\left(t\sum_{l=1}^{k-1}\eta_l\right), \cos(t\eta_k)\right)\right| + \left| Cov\left(\cos\left(t\sum_{l=1}^{k-1}\eta_l\right), \sin(t\eta_k)\right)\right| \\
\doteq & J_{11} + J_{12} + J_{13} + J_{14}. \qquad (3.2)
\end{aligned}$$

It follows from Lemma 3.3 and $|\sin x| \leqslant |x|$ that

$$\begin{aligned}
J_{14} &\leqslant C\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(n)\left\|\cos\left(t\sum_{l=1}^{k-1}\eta_l\right)\right\|_2 \|\sin(t\eta_k)\|_2 \\
&\leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2 \qquad (3.3)
\end{aligned}$$

and

$$J_{12} \leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2. \tag{3.4}$$

Noting that $\cos(2x) = 1 - 2\sin^2 x$, and hence applying Lemma 3.3 and invoking again the inequality $\sin^2 x \leqslant |\sin x| \leqslant |x|$, we find that

$$J_{13} = 2\left|Cov\left(\sin(t\sum_{l=1}^{k-1}\eta_l), \sin^2\left(\frac{t\eta_k}{2}\right)\right)\right| \leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(n)\|\eta_k\|_2, \tag{3.5}$$

and

$$J_{11} \leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2. \tag{3.6}$$

By (3.1)–(3.6), we have

$$\left|E\exp\left(it\sum_{l=1}^{k}\eta_l\right) - \prod_{l=1}^{k}E\exp(it\eta_l)\right| \leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2 + J_2.$$

Proceeding in this manner, we obtain

$$\begin{aligned}
&\left|E\exp\left(it\sum_{l=1}^{k}\eta_l\right) - \prod_{l=1}^{k}E\exp(it\eta_l)\right| \\
&\leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2 + C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_{k-1}\|_2 \\
&\quad + \left|E\exp\left(it\sum_{l=1}^{k-2}\eta_l\right) - \prod_{l=1}^{k-2}E\exp(it\eta_l)\right| \\
&\leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{l=1}^{k}\|\eta_k\|_2.
\end{aligned}$$

This completes the proof of the lemma. □

Now, we turn to prove the main results of the paper.

Proof of Theorem 2.1. For any $a > 0$ and $x \in c(g)$, we obtain from (1.1) and (1.2) that

$$\begin{aligned}
|Eg_n(x) - g(x)| \leqslant &\sum_{i=1}^{n}|w_{ni}(x)|.|g(x_{ni}) - g(x)|I(\|x_{ni} - x\| \leqslant a) \\
&+ \sum_{i=1}^{n}|w_{ni}(x)|.|g(x_{ni}) - g(x)|I(\|x_{ni} - x\| > a) \\
&+ |g(x)| \cdot \left|\sum_{i=1}^{n}w_{ni}(x) - 1\right|.
\end{aligned} \tag{3.7}$$

It follows from $x \in c(g)$ that for all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all x' satisfying $\|x' - x\| < \delta$, $|g(x') - g(x)| < \varepsilon$. Setting $0 < a < \delta$ in (3.7), we have by Assumption A0 and the arbitrariness of $\varepsilon > 0$ that for all $x \in c(g)$,

$$\lim_{n \to \infty} E g_n(x) = g(x). \tag{3.8}$$

Hence, to prove (2.1), it suffices to show that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} w_{ni}(x)\varepsilon_{ni}\right| > \varepsilon\right) < \infty. \tag{3.9}$$

Since $w_{ni}(x) = w_{ni}^{+}(x) - w_{ni}^{-}(x)$, we may assume without loss of generality that $w_{ni}(x) \geqslant 0$ and $\max_{1 \leqslant i \leqslant n} w_{ni}(x) \leqslant n^{-s}$. For any fixed $n \geqslant 1$, we denote

$$X_{ni} = w_{ni}(x)\varepsilon_{ni} I(|w_{ni}(x)\varepsilon_{ni}| \leqslant 1), \quad i = 1, 2, \ldots, n. \tag{3.10}$$

It is easy to check that

$$\left(\left|\sum_{i=1}^{n} w_{ni}(x)\varepsilon_{ni}\right| > \varepsilon\right) \subset \left(\max_{1 \leqslant l \leqslant n} |w_{ni}(x)\varepsilon_{ni}| > 1\right) \bigcup \left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right), \tag{3.11}$$

which implies

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} w_{ni}(x)\varepsilon_{ni}\right| > \varepsilon\right) &\leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|w_{ni}(x)\varepsilon_{ni}| > 1) + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon\right) \\ &\doteq I_1 + I_2. \end{aligned} \tag{3.12}$$

Hence to prove (3.9), we only need to show $I_1 < \infty$ and $I_2 < \infty$. By Assumption A0 and $E|X|^{1+1/s} < \infty$, we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|w_{ni}(x)\varepsilon_{ni}| > 1) &\leqslant C \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|w_{ni}(x)X| > 1) \\ &\leqslant C \sum_{n=1}^{\infty}\sum_{i=1}^{n} w_{ni}(x) E|X| I(|w_{ni}(x)X| > 1) \\ &\leqslant C \sum_{n=1}^{\infty} E|X| I(|X| > n^{s}) \\ &\leqslant C \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} E|X| I(k^{s} < |X| \leqslant (k+1)^{s}) \\ &\leqslant C \sum_{k=1}^{\infty}\sum_{n=1}^{k} E|X| I(k^{s} < |X| \leqslant (k+1)^{s}) \\ &\leqslant C \sum_{k=1}^{\infty} k E|X| I(k^{s} < |X| \leqslant (k+1)^{s}) \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}E|X|^{1+1/s}I(k^s<|X|\leqslant(k+1)^s)\\
&\leqslant CE|X|^{1+1/s}<\infty, \qquad (3.13)
\end{aligned}
$$

which implies that $I_1<\infty$. Next, we will prove that $I_2<\infty$. Firstly, we shall show that

$$\left|\sum_{i=1}^{n}EX_{ni}\right|\to 0, \quad \text{as} \quad n\to\infty. \qquad (3.14)$$

Actually by the condition $E\varepsilon_{ni}=0$, Lemma 3.2 and $E|X|^{1+1/s}<\infty$, we can see that

$$
\begin{aligned}
\left|\sum_{i=1}^{n}EX_{ni}\right| &= \left|\sum_{i=1}^{n}Ew_{ni}(x)\varepsilon_{ni}I(|w_{ni}(x)\varepsilon_{ni}|\leqslant 1)\right|\\
&\leqslant C\sum_{i=1}^{n}E|w_{ni}(x)\varepsilon_{ni}|^{1+1/s}I(|w_{ni}(x)\varepsilon_{ni}|>1)\\
&\leqslant C\sum_{i=1}^{n}w_{ni}^{1+1/s}(x)E|X|^{1+1/s}I(|w_{ni}(x)X|>1)\\
&\leqslant C\left(\max_{1\leqslant i\leqslant n}w_{ni}(x)\right)^{1/s}\sum_{i=1}^{n}w_{ni}(x)E|X|^{1+1/s}I(|X|>n^s)\\
&\leqslant C(n^{-s})^{1/s}E|X|^{1+1/s}I(|X|>n^s)\\
&= Cn^{-1}E|X|^{1+1/s}I(|X|>n^s)\to 0, \quad \text{as} \quad n\to\infty, \qquad (3.15)
\end{aligned}
$$

which implies (3.14). Hence, to prove $I_2<\infty$, we only need to show that for any $\varepsilon>0$,

$$I_2^*=\sum_{i=1}^{\infty}P\left(\left|\sum_{i=1}^{n}(X_{ni}-EX_{ni})\right|>\frac{\varepsilon}{2}\right)<\infty. \qquad (3.16)$$

By Markov's inequality, Lemma 3.1, C_r-inequality and Jensen's inequality, we have for $p\geqslant 2$ that

$$
\begin{aligned}
I_2^* &\leqslant C\sum_{n=1}^{\infty}E\left(\left|\sum_{i=1}^{n}(X_{ni}-EX_{ni})\right|^p\right)\\
&\leqslant C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}E|X_{ni}|^2\right)^{p/2}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}E|X_{ni}|^p\\
&\doteq I_{21}+I_{22}. \qquad (3.17)
\end{aligned}
$$

Take

$$p>\max\left\{2,\frac{2}{s},1+\frac{1}{s}\right\},$$

which implies that $-sp/2<-1$ and $-s(p-1)<-1$.

For I_{21}, by C_r-inequality and Lemma 3.2, we have

$$I_{21} \leqslant C\sum_{i=1}^{\infty}\left[\sum_{i=1}^{n}P(|w_{ni}(x)X|>1)+\sum_{i=1}^{n}E|w_{ni}(x)X|^2I(|w_{ni}(x)X|\leqslant 1)\right]^{p/2}. \quad (3.18)$$

If $s>1$, then we have by Markov's inequality and $E|X|^{1+1/s}<\infty$ that

$$\begin{aligned} I_{21} &\leqslant C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}w_{ni}^{1+1/s}(x)E|X|^{1+1/s}\right)^{p/2} \\ &\leqslant C\sum_{n=1}^{\infty}\left[\left(\max_{1\leqslant i\leqslant n}w_{ni}(x)\right)^{1/s}\sum_{i=1}^{n}w_{ni}(x)\right]^{p/2} \\ &\leqslant C\sum_{n=1}^{\infty}n^{-p/2}<\infty. \end{aligned} \quad (3.19)$$

If $0<s\leqslant 1$, then we have by Markov's inequality and $E|X|^{1+1/s}<\infty$ again that

$$\begin{aligned} I_{21} &\leqslant C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}w_{ni}^{2}(x)E|X|^{2}\right)^{p/2} \\ &\leqslant C\sum_{n=1}^{\infty}\left[\left(\max_{1\leqslant i\leqslant n}w_{ni}(x)\right)\sum_{i=1}^{n}w_{ni}(x)\right]^{p/2} \\ &\leqslant C\sum_{n=1}^{\infty}n^{-sp/2}<\infty. \end{aligned} \quad (3.20)$$

From (3.18)–(3.20), we have proved that $I_{21}<\infty$.

For I_{22}, by C_r-inequality and Lemma 3.2, we can see that

$$\begin{aligned} I_{22} &\leqslant C\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|w_{ni}(x)X|>1)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}E|w_{ni}(x)X|^{p}I(|w_{ni}(x)X|\leqslant 1) \\ &\doteq I_3+I_4. \end{aligned} \quad (3.21)$$

$I_3<\infty$ has been proved by (3.13). In the following, we will show that $I_4<\infty$. Denote

$$I_{nj}=\left\{i:[n(j+1)]^{-s}<w_{ni}(x)\leqslant (nj)^{-s}\right\}, n\geqslant 1, j\geqslant 1. \quad (3.22)$$

It can be easily seen that $I_{nk}\bigcap I_{nj}=\emptyset$ for $k\neq j$ and $\bigcup_{j=1}^{\infty}I_{nj}=\{1,2,\ldots,n\}$ for all $n\geqslant 1$. Hence, we have

$$\begin{aligned} I_4 &\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}E|w_{ni}(x)X|^{p}I(|w_{ni}(x)X|\leqslant 1) \\ &\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-sp}E|X|^{p}I(|X|\leqslant [n(j+1)]^{s}) \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-sp}\sum_{k=0}^{n(j+1)}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&=C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-sp}\sum_{k=0}^{2n}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\quad+\leqslant C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-sp}\sum_{k=2n+1}^{n(j+1)}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\doteq I_{41}+I_{42}.
\end{aligned}
\tag{3.23}
$$

It is easily seen that for all $m\geqslant 1$,

$$
\begin{aligned}
C&\geqslant\sum_{i=1}^{n}w_{ni}(x)=\sum_{j=1}^{\infty}\sum_{i\in I_j}w_{ni}(x)\geqslant\sum_{j=1}^{\infty}(\sharp I_{nj})[n(j+1)]^{-s}\\
&\geqslant\sum_{j=m}^{\infty}(\sharp I_{nj})[n(j+1)]^{-s}\geqslant\sum_{j=m}^{\infty}(\sharp I_{nj})[n(j+1)]^{-s}\left[\frac{n(m+1)}{n(j+1)}\right]^{s(p-1)}\\
&=\sum_{j=m}^{\infty}(\sharp I_{nj})[n(j+1)]^{-sp}[n(m+1)]^{s(p-1)},
\end{aligned}
\tag{3.24}
$$

which implies that for all $m\geqslant 1$,

$$
\sum_{j=m}^{\infty}(\sharp I_{nj})(nj)^{-sp}\leqslant Cn^{-s(p-1)}\cdot m^{-s(p-1)}.
\tag{3.25}
$$

Therefore, we have by $E|X|^{1+1/s}<\infty$ again that

$$
\begin{aligned}
I_{41}&=C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-sp}\sum_{k=0}^{2n}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\leqslant C\sum_{n=1}^{\infty}n^{-s(p-1)}\sum_{k=0}^{2n}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\leqslant C\sum_{k=0}^{2}\sum_{n=1}^{\infty}n^{-s(p-1)}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\quad+C\sum_{k=2}^{\infty}\sum_{n=[k/2]}^{\infty}n^{-s(p-1)}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\leqslant C+C\sum_{k=2}^{\infty}k^{1-s(p-1)}E|X|^{p}I(k\leqslant|X|^{1/s}<k+1)\\
&\leqslant C+C\sum_{k=2}^{\infty}E|X|^{p+1/s-(p-1)}I(k\leqslant|X|^{1/s}<k+1)\\
&\leqslant C+CE|X|^{1+1/s}<\infty,
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
I_{42} &= C\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-sp}\sum_{k=2n+1}^{n(j+1)} E|X|^p I(k \leqslant |X|^{1/s} < k+1)\\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=2n+1}^{\infty}\sum_{j\geqslant k/n-1}(\sharp I_{nj})(nj)^{-sp}E|X|^p I(k \leqslant |X|^{1/s} < k+1)\\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=2n+1}^{\infty} n^{-s(p-1)}\left(\frac{k}{n}\right)^{-s(p-1)} E|X|^p I(k \leqslant |X|^{1/s} < k+1)\\
&\leqslant C\sum_{k=2}^{\infty}\sum_{n=1}^{\lfloor k/2\rfloor} k^{-s(p-1)}E|X|^p I(k \leqslant |X|^{1/s} < k+1)\\
&\leqslant C\sum_{k=2}^{\infty} k^{1-s(p-1)}E|X|^p I(k \leqslant |X|^{1/s} < k+1)\\
&\leqslant C\sum_{k=2}^{\infty} E|X|^{p+1/s-(p-1)} I(k \leqslant |X|^{1/s} < k+1)\\
&\leqslant CE|X|^{1+1/s} < \infty. \qquad (3.27)
\end{aligned}
$$

Thus, the inequality (3.16) follows from (3.17)–(3.21), (3.23), (3.26) and (3.27). This completes the proof of Theorem 2.1. $\square$

To prove Theorem 2.2, we need the following notations and lemmas. For simplicity, we omit everywhere the argument x and set $S_n = \sigma_n^{-1}(g_n - Eg_n)$, $Z_{ni} = \sigma_n^{-1} w_{ni}\varepsilon_{ni}$, $i = 1,\ldots,n$. So that $S_n = \sum_{i=1}^n Z_{ni}$. Following the familiar procedure, partition the set $\Delta_n = \{1,\ldots,n\}$ into $2k+1$ subsets Δ'_{nm}, Δ''_{nm}, $m = 1,\ldots,k$, and Δ'''_{nk}, where

$$
\begin{aligned}
\Delta'_{nm} &= \{(m-1)(p+q)+1,\ldots,(m-1)(p+q)+p, m = 1,\ldots,k\},\\
\Delta''_{nm} &= \{(m-1)(p+q)+p+1,\ldots,m(p+q), m = 1,\ldots,k\},\\
\Delta'''_{nk} &= \{k(p+q)+1,\ldots,n\}.
\end{aligned}
$$

Let $k = \lfloor n/(p+q)\rfloor$. Then S_n can be split as $S_n = S'_n + S''_n + S'''_n$, where

$$
S'_n = \sum_{m=1}^{k} y_{nm}, \quad S''_n = \sum_{m=1}^{k} y'_{nm}, \quad S'''_n = y'_{nk+1},
$$

$$
y_{nm} = \sum_{i=k_m}^{k_m+p-1} Z_{ni}, \;\; y'_{nm} = \sum_{i=l_m}^{l_m+q-1} Z_{ni}, \;\; y'_{nk+1} = \sum_{i=k(p+q)+1}^{n} Z_{ni},
$$

$$
k_n = (m-1)(p+1)+1, \;\; l_m = (m-1)(p+q)+p+1, \;\; m = 1,2,\ldots,k.
$$

Thus, to prove Theorem 2.2, it suffices to show that

$$
E(S''_n)^2 \to 0, \quad E(S'''_n)^2 \to 0, \quad S'_n \to N(0,1). \qquad (3.28)
$$

LEMMA 3.5. *Suppose that Assumptions A1–A4 are satisfied. Then*

$$E(S_n'')^2 \to 0, \quad E(S_n''')^2 \to 0.$$

Proof. It is easy to check that

$$\begin{aligned}
E(S_n'')^2 &= E\left(\sum_{m=1}^{k} y'_{nm}\right)^2 \\
&= \sum_{m=1}^{k} Var(y'_{nm}) + 2\sum_{1\leqslant i<j\leqslant k} Cov(y'_{ni}, y'_{nj}) \\
&\doteq A_{n1} + A_{n2}.
\end{aligned}$$

From the definition of Z_{ni}, we have by Lemma 3.3 that

$$EZ_{ni} = 0, \;\; Var(Z_{ni}) \leqslant \sigma_n^{-2} w_{ni}^2 \sigma^2,$$

$$|Cov(Z_{ni}, Z_{nj})| \leqslant \sigma_n^{-2} |w_{ni} w_{nj}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i).$$

Noting that

$$\begin{aligned}
Var(y'_{nm}) &= Var\left(\sum_{i=l_m}^{l_m+q-1} Z_{ni}\right) \\
&= \sum_{i=l_m}^{l_m+q-1} Var(Z_{ni}) + 2\sum_{l_m\leqslant i<j\leqslant l_m+q-1} Cov(Z_{ni}, Z_{nj}),
\end{aligned}$$

we have by Assumptions A1–A4 that for

$$\begin{aligned}
A_{n1} &= \sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1} Var(Z_{ni}) + 2\sum_{m=1}^{k}\sum_{l_m\leqslant i<j\leqslant l_m+q-1} Cov(Z_{ni}, Z_{nj}) \\
&\leqslant C\sigma_n^{-2}\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1} w_{ni}^2 + C\sigma_n^{-2}\sum_{m=1}^{k}\sum_{l_m\leqslant i<j\leqslant l_m+q-1} |w_{ni}w_{nj}|\lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i) \\
&\leqslant C\sigma_n^{-2}\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1} w_{ni}^2 + C\sigma_n^{-2}w_n^2\sum_{m=1}^{k}\sum_{l_m\leqslant i<j\leqslant l_m+q-1} \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i) \\
&\leqslant C\sigma_n^{-2}\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1} w_{ni}^2 + C\sigma_n^{-2}w_n^2\sum_{m=1}^{k}\sum_{i=1}^{q}\sum_{l=1}^{q-1} \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(l) \\
&\leqslant C\sigma_n^{-2}qkw_n^2 + C\sigma_n^{-2}qkw_n^2 \\
&\leqslant Cqkw_n \leqslant C(1+qp^{-1})^{-1}nqp^{-1}w_n \to 0, \;\; \text{as } n\to\infty. \qquad (3.29)
\end{aligned}$$

For A_{n2}, notice that

$$
\begin{aligned}
Cov(y'_{ni}, y'_{nj}) &= Cov\left(\sum_{k=l_i}^{l_i+q-1} Z_{nk}, \sum_{l=l_j}^{l_j+q-1} Z_{nl}\right) \\
&= \sum_{k=l_i}^{l_i+q-1} \sum_{l=l_j}^{l_j+q-1} Cov(Z_{nk}, Z_{nl}).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\left|\sum_{1\leqslant i<j\leqslant k} Cov(y'_{ni}, y'_{nj})\right| &\leqslant \sum_{1\leqslant i<j\leqslant k} |Cov(y'_{ni}, y'_{nj})| \\
&\leqslant \sum_{1\leqslant i<j\leqslant k} \sum_{l=l_i}^{l_i+q-1} \sum_{r=l_j}^{l_j+q-1} |Cov(Z_{nl}, Z_{nr})| \\
&\leqslant \sum_{i=1}^{n-p} \sum_{j=i+p}^{n} |Cov(Z_{ni}, Z_{nj})| \\
&\leqslant C\sigma_n^{-2} \sum_{i=1}^{n-p} \sum_{j=i+p}^{n} |w_{ni} w_{nj}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i) \\
&\leqslant C\sigma_n^{-2} w_n \sum_{i=1}^{n-p} \sum_{j=i+p}^{n} |w_{nj}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i) \\
&\leqslant C\sigma_n^{-2} w_n \sum_{i=1}^{n} \sum_{l=p}^{n} |w_{ni}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(l) \\
&\leqslant C\left(\sum_{i=1}^{n} |w_{ni}|\right) \sigma_n^{-2} w_n \sum_{l=p}^{\infty} \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(l) \\
&\leqslant C \sum_{l=p}^{\infty} \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(l) \to 0, \text{ as } p\to\infty.
\end{aligned} \tag{3.30}
$$

Combining (3.29) and (3.30), we have $E(S''_n)^2 \to 0$. Next we prove $E(S'''_n)^2 \to 0$. In fact, we have

$$
\begin{aligned}
E(S'''_n)^2 &= Var\left(\sum_{i=k(p+q)+1}^{n} Z_{ni}\right) \\
&\leqslant \sum_{i=k(p+q)+1}^{n} Var(Z_{ni}) + 2 \sum_{k(p+q)+1\leqslant i<j\leqslant n} |Cov(Z_{ni}, Z_{nj})| \\
&\leqslant C(n-k(p+q))\sigma_n^{-2} w_n^2 \\
&\leqslant C(1+qp^{-1}) p w_n \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.31}
$$

This completes the proof of the lemma. □

LEMMA 3.6. *Suppose that Assumptions A1–A4 are satisfied. Let*

$$s_n^2 = \sum_{m=1}^{k} Var(y_{nm}).$$

Then

$$E(S_n')^2 \to 1, \quad s_n^2 \to 1.$$

Proof. Let $\Gamma_n = \sum_{1\leqslant i<j\leqslant k} Cov(y_{ni}, y_{nj})$. Then $s_n^2 = E(S_n')^2 - 2\Gamma_n$. Noting that

$$\begin{aligned} E|S_n(S_n''+S_n''')| &\leqslant E^{1/2}S_n^2 E^{1/2}(S_n''+S_n''')^2 \\ &= E^{1/2}(S_n''+S_n''')^2 \\ &\leqslant E^{1/2}(S_n'')^2 + E^{1/2}(S_n''')^2 \\ &\to 0, \text{ as } n\to\infty, \end{aligned}$$

we have

$$\begin{aligned} E(S_n')^2 &= E\left[S_n - (S_n''+S_n''')\right]^2 \\ &= 1 + E(S_n''+S_n''')^2 - 2E[S_n(S_n''+S_n''')] \to 1, \text{ as } n\to\infty. \end{aligned}$$

This will also imply that $s_n \to 1$, provided that we show $\Gamma_n \to 0$ as $n\to\infty$. Indeed, it follows from Assumption A2 that

$$\begin{aligned} |\Gamma_n| &\leqslant \sum_{1\leqslant i<j\leqslant k} \sum_{\mu=k_i}^{k_i+q-1} \sum_{v=k_j}^{k_j+q-1} |Cov(Z_{n\mu}, Z_{nv})| \\ &\leqslant \sum_{i=1}^{n-q} \sum_{j=i+q}^{n} |Cov(Z_{ni}, Z_{nj})| \\ &\leqslant C\sigma_n^{-2} \sum_{i=1}^{n-q} \sum_{j=i+q}^{n} |w_{ni}w_{nj}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i) \\ &\leqslant C\sigma_n^{-2} w_n \sum_{i=1}^{n-q} \sum_{j=i+q}^{n} |w_{n\mu}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(j-i) \\ &\leqslant C\sigma_n^{-2} w_n \sum_{i=1}^{n} \sum_{l=q}^{n} |w_{ni}| \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(l) \\ &\leqslant C\left(\sum_{i=1}^{n} |w_{ni}|\right) \sigma_n^{-2} w_n \sum_{l=q}^{\infty} \lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(l) \to 0, \text{ as } q\to\infty. \end{aligned} \tag{3.32}$$

The proof is completed. □

LEMMA 3.7. *Suppose that Assumptions A1–A4 are satisfied. Then*

$$S_n' \xrightarrow{d} N(0,1).$$

Proof. In order to establish asymptotic normality, we assume that $\{\eta_{nm}, m = 1,2,\dots,k\}$ are independent random variables, and the distribution of η_{nm} is the same as that of y_{nm} for each $m = 1,2,\dots,k$. Thus, $E\eta_{nm} = 0$ and $Var(\eta_{nm}) = Var(y_{nm})$. Let $T_{nm} = \eta_{nm}/s_n$. Then $\{T_{nm}, m = 1,2,\dots,k\}$ are independent random variables with $ET_{nm} = 0$ and $\sum_{m=1}^{k} Var(T_{nm}) = \frac{1}{s_n^2}\sum_{n=1}^{k} Var(\eta_{nm}) = 1$. Let $\phi_X(t)$ be the characteristic function of X. It is easy to check that

$$
\begin{aligned}
&\left|\phi_{\sum_{m=1}^{k} y_{nm}}(t) - e^{-\frac{t^2}{2}}\right| \\
\leqslant &\left|E\exp\left(it\sum_{m=1}^{k} y_{nm}\right) - \prod_{m=1}^{k} E\exp(ity_{nm})\right| + \left|\prod_{m=1}^{k} E\exp(ity_{nm}) - e^{-\frac{t^2}{2}}\right| \\
\leqslant &\left|E\exp\left(it\sum_{m=1}^{k} y_{nm}\right) - \prod_{m=1}^{k} E\exp(ity_{nm})\right| + \left|\prod_{m=1}^{k} E\exp(it\eta_{nm}) - e^{-\frac{t^2}{2}}\right| \\
\doteq\ & I_3 + I_4.
\end{aligned}
$$

Since $\{\varepsilon_{ni}\}$ are random errors with zero mean and finite variance σ^2, we have by Lemma 3.4, Lemma 3.1 and Assumptions A2–A4 that

$$
\begin{aligned}
I_3 &= \left|E\exp\left(it\sum_{m=1}^{k} y_{nm}\right) - \prod_{m=1}^{k} E\exp(ity_{nm})\right| \\
&\leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{m=1}^{k}\|y_{nm}\|_2 \\
&\leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{m=1}^{k}\left(E\left|\sum_{i=k_m}^{k_m+p-1} Z_{ni}\right|^2\right)^{\frac{1}{2}} \\
&\leqslant C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{m=1}^{k}\left(\sum_{i=k_m}^{k_m+p-1} \sigma_n^{-2} w_{ni}^2 E\varepsilon_{ni}^2\right)^{\frac{1}{2}} \\
&\leqslant Ck|t|(pw_n)^{\frac{1}{2}}\lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(q) \\
&\leqslant C|t|np^{-\frac{1}{2}}w_n^{1/2}\lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(q) \to 0,\ as\ n\to\infty.
\end{aligned} \tag{3.33}
$$

So it suffices to show that $\sum_{m=1}^{k}\eta_{nm} \xrightarrow{d} N(0,1)$ which on account of $s_n^2 \to 1$, will follow from the convergence $\sum_{m=1}^{k} T_{nm} \xrightarrow{d} N(0,1)$. By the Lyapunov's condition, it suffices to show that for for some $r > 2$,

$$\frac{1}{s_n^r}\sum_{m=1}^{k} E|\eta_{nm}|^r \to 0,\ \ as\ \ n\to\infty. \tag{3.34}$$

Using Lemma 3.1 and Assumptions A2–A4, we have

$$\begin{aligned}
\sum_{m=1}^{k} E|\eta_{nm}|^r &= \sum_{m=1}^{k} E|y_{nm}|^r \\
&= \sum_{m=1}^{k} E\left|\sum_{i=k_m}^{k_m+p-1} \sigma_n^{-1} w_{ni}\varepsilon_{ni}\right|^r \\
&= \sigma_n^{-r} \sum_{m=1}^{k} E\left|\sum_{i=k_m}^{k_m+p-1} w_{ni}\varepsilon_{ni}\right|^r \\
&\leqslant Ck\sigma_n^{-r}\left\{\sum_{i=k_m}^{k_m+p-1} E|w_{ni}\varepsilon_{ni}|^r + \left(\sum_{i=k_m}^{k_m+p-1} E(w_{ni}\varepsilon_{ni})^2\right)^{r/2}\right\} \\
&\leqslant Ck\sigma_n^{-r}\left\{\sum_{i=k_m}^{k_m+p-1} w_n^r E|\varepsilon_i|^r + \left(\sum_{i=k_m}^{k_m+p-1} w_n^2 E\varepsilon_1^2\right)^{r/2}\right\} \\
&\leqslant Ck\sigma_n^{-r}\left\{pw_n^r E|\varepsilon_1|^r + \left(pw_n^2 E\varepsilon_1^2\right)^{r/2}\right\} \\
&\leqslant Ck\sigma_n^{-r}\left\{pw_n^r E|\varepsilon_1|^r + p^{r/2} w_n^r E|\varepsilon_1|^r\right\} \\
&\leqslant Ck\sigma_n^{-r} w_n^r p^{r/2} \\
&\leqslant Cnp^{r/2-1} w_n^{r/2} \to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

which together with $s_n^2 \to 1$ yields (3.34). This completes the proof of the lemma. □

Based on the Lemmas 3.5–3.7, we could provide the proof of Theorem 2.2 as follows.

Proof of Theorem 2.2. By Lemmas 3.5, 3.6 and 3.7, we have that

$$\sigma_n^{-1}(x)\{g_n(x) - Eg_n(x)\} = \sum_{n=1}^{n} Z_{ni} \xrightarrow{d} N(0,1).$$

This completes the proof of the theorem. □

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