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INEQUALITIES FOR THE PROBABILITY OF RUIN IN A REINSURANCE RISK MODEL WITH m–DEPENDENCE ASSUMPTIONS

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Abstract. In this article, we investigate a discrete-time risk model. The risk model includes the quota $-(\alpha,\beta)$ reinsurance contract effect on the surplus process. The premium process and claim process are assumed to be m-dependent sequences of identically distributed non-negative random variables. Using Martingale and inductive methods, We obtained upper bounds for the ultimate ruin probability of an insurance company. Finally, we present a numerical example to show the efficiency of the methods.

1. Introduction

The first risk model was developed by Filip Lundberg in 1903 and expanded by Harald Cramers 1930s [12, 17, 20]. They used a Poisson process to model the surplus process of insurance company and estimate its ruin probability. Andersen in 1957 generalized the assumption of Poisson distribution in the number of claims by allowing aribtrary distribution [20]. The model is called a renewal risk model or Sparre Andersen model. The risk models are based on a continuous-time model. However, in reality, claims occur in discrete time. Hence, the discrete-time models often turn out to be more realistic.

Dickson [10] (see, p. 113) showed one of the simplest surplus processes in the discrete-time model, which has the following form:

$$U_n = u + n - \sum_{i=1}^{n} Y_i \quad \text{for } n = 1,2,3,\dots \tag{1}$$

where u is the insurer's initial surplus, the insurer's premium income per unit time is 1 and the claim process is a sequence of independent and identically distributed random $\{Y_i\}_{i>0}$.

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The next, Yang [26] generalized the surplus process (1) by the assumptions whose premium process $\{X_i\}_{i>0}$ and claim process $\{Y_i\}_{i>0}$ are sequences of independent and identically distributed non-negative random variables. Then:

$$U_n = u + \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i \quad \text{for } n = 1,2,3,\dots \tag{2}$$

The studies [1, 2, 5, 8, 9, 16, 21, 22, 23] considered the surplus process (2) with rates of interest. Particularly, Quang [19] introduced the surplus process with homogenous Markov chain claims and homogenous Markov chain premiums.

Recently, Dam and Chung [6, 7] investigated continuously the surplus process (2) under a quota $-(\alpha,\beta)$ reinsurance contract. Then, the surplus process of insurance company as following:

$$U_n = u + \alpha \sum_{i=1}^{n} X_i - \beta \sum_{i=1}^{n} Y_i \quad \text{for } n = 1,2,3,\dots \tag{3}$$

where the premium process and claim process are also assumed to be sequences of identically independent distributed random variables.

The ruin probability calculating problem plays an important role in risk theory. However, the problem is very difficult unless we use simulation program or special conditions for the risk models. The finite time ruin probabilities was approximately by Padé approximants in Xuan [25]. Under the assumption that the claim sizes are integer-valued, Picard and Lefèvre [18], Lefèvre and Loisel [15] showed exact formulas for the probability of ruin within finite time. In some practical problems, we only need a conservative upper bound which is very easy to calculate to approximate for the ruin probability. The Martingale and inductive methods are popular techniques for the idea. See, for example Cai [1, 2, 3], Cai and Dickson [4, 5], Dam and Chung [6, 7], Diasparra and Romera [8, 9], Gajek [11], Hoang [13], Hoang and Bao [14], Lin et al [16], Quang [19], Wei and Hu [22], Yang [26], etc.

The main goal of our paper is to estimate the ruin probability for the surplus process (3) under assumptions where the premium and claim processes are m-dependent sequences of identically distributed non-negative random variables. The m-dependent assumption generalies better than the independent assumption. Hoang [13, 14] considered the risk models with m-dependent assumptions and gave upper bound for the ruin probability by the Martingale method. In this paper, we use Martingale and inductive methods to estimate the ruin probability of an insurance company.

The rest of the paper is organized as follows. In Section 1, we introduce some relative studies in our paper. The risk model in the paper is discussed in Section 2. In Section 3, we give the upper bounds for the ultimate ruin probabilities by the Martingale and inductive methods. Finally, a numerical example is given to illustrate our methods.

2. Model

We will present the notions and risk model of the paper. First of all, we recall the notion of sequence m-dependent random variables.

DEFINITION 1. Let m be a non-negative integer. A sequence of random variables $\{\xi_n\}_{n>0}$ is called m-dependent if the sigma-fields

$$\mathscr{F}_n = \sigma\{\xi_1, \xi_2, \dots, \xi_n\} \quad \text{and} \quad \mathscr{F}^{n+k} = \sigma\{\xi_{n+k}, \xi_{n+k+1}, \dots\}$$

are independent for all $k > m$.

A sequence of independent random variables $\{\xi_n\}_{n>0}$ is 0-dependent.

We now consider the surplus process (3) of a insurer where

- u is the initial capital,
- X_n denotes the premium income in the nth period,
- Y_n denotes the insurer's aggregate claim amount in the nth period,
- α and β $(\alpha, \beta \in [0,1])$ are division ratios to share premiums and claims between the insurer and the reinsurer.

In this study, we present the generalized assumptions of the premium income process and claim size process. Let $X = \{X_n\}_{n>0}$ and $Y = \{Y_n\}_{n>0}$ be sequences of m_1-dependent random variables and m_2-dependent random variables, respectively. Then, $X = \{X_n\}_{n>0}$ and $Y = \{Y_n\}_{n>0}$ are sequences of m-dependent random variables where $m = \max(m_1, m_2)$. Because the article model only consideres the following assumptions.

- $X = \{X_n\}_{n>0}$ and $Y = \{Y_n\}_{n>0}$ are sequences of m-dependent and identically distributed non-negative random variables. Y is independent of X.

We put

$$S_n = \sum_{i=1}^{n} (\beta Y_i - \alpha X_i), \qquad n = 1, 2, \dots$$

and

$$S_n^{(k)} = \sum_{i=1}^{n} \left(\beta Y_{k+(i-1)(m+1)} - \alpha X_{k+(i-1)(m+1)}\right), \qquad n = 1, 2, \dots$$

where $k = 1, 2, \dots, m+1$.

Sequences $\{X_n\}_{n>0}$ and $\{Y_n\}_{n>0}$ are sequences of m-dependent. Hence, subsequences $\left\{S_n^{(k)}\right\}_{n>0}$, $k = 1, 2, \dots, m+1$ are sequences of independent random variables.

The equation (3) can be rewritten as

$$U_n = u - S_n. \tag{4}$$

We say that the reinsurer's ruin occurs at period n if the reinsurer's surplus at period n falls to zero or below. We denote the finite time ruin probability and ultimate ruin probability for model (4) by

$$\psi_n(u, \alpha, \beta) = \mathbb{P}\left(\bigcup_{i=1}^{n} (U_i \leqslant 0)\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} (S_i \geqslant u)\right) \tag{5}$$

and

$$\psi(u,\alpha,\beta) = \mathbb{P}\left(\bigcup_{i=1}^{\infty}(U_i \leqslant 0)\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty}(S_i \geqslant u)\right). \tag{6}$$

Obviously,

$$\lim_{n\to\infty} \psi_n(u,\alpha,\beta) = \psi(u,\alpha,\beta).$$

3. Upper bounds for ultimate ruin probability

We denote

$$\begin{aligned}\psi_n^{(k)}\left(u^{(k)},\alpha,\beta\right) &= \mathbb{P}\left(\bigcup_{i=1}^{n}\left(S_i^{(k)} \geqslant u^{(k)}\right)\right)\\ &= \mathbb{P}\left(\bigcup_{i=1}^{n}\left(\sum_{j=1}^{i}(\beta Y_{k+(j-1)(m+1)} - \alpha X_{k+(j-1)(m+1)}) \geqslant u^{(k)}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\psi^{(k)}\left(u^{(k)},\alpha,\beta\right) &= \mathbb{P}\left(\bigcup_{i=1}^{\infty}\left(S_i^{(k)} \geqslant u^{(k)}\right)\right)\\ &= \mathbb{P}\left(\bigcup_{i=1}^{\infty}\left(\sum_{j=1}^{i}(\beta Y_{k+(j-1)(m+1)} - \alpha X_{k+(j-1)(m+1)}) \geqslant u^{(k)}\right)\right).\end{aligned}$$

By convention, $\sum_{i=a}^{b} x_i = 0$ and $\prod_{i=a}^{b} x_i = 1$ if $a > b$.

LEMMA 1. *If* $u = u^{(1)} + u^{(2)} + \ldots + u^{(m+1)}$ *then*

$$\psi(u,\alpha,\beta) \leqslant \sum_{k=1}^{m+1} \psi^{(k)}\left(u^{(k)},\alpha,\beta\right) \tag{7}$$

where $u^{(k)} > 0$ *and* $S_0^{(k)} = 0$.

Proof. We have

$$(S_i \geqslant u) \subset \left(\bigcup_{k=1}^{m+1}\left(S^{(k)}_{\left[\frac{i-k}{m+1}\right]+1} \geqslant u^{(k)}\right)\right)$$

where $\left[\frac{i-k}{m+1}\right]$ is the integer part of $\frac{i-k}{m+1}$. This implies that

$$\begin{aligned}\psi(u,\alpha,\beta) = \mathbb{P}\left(\bigcup_{i=1}^{\infty}(S_i \geqslant u)\right) &\leqslant \mathbb{P}\left(\bigcup_{i=1}^{\infty}\bigcup_{k=1}^{m+1}\left(S^{(k)}_{\left[\frac{i-k}{m+1}\right]+1} \geqslant u^{(k)}\right)\right)\\ &\leqslant \sum_{k=1}^{m+1}\mathbb{P}\left(\bigcup_{i=1}^{\infty}\left(S^{(k)}_{\left[\frac{i-k}{m+1}\right]+1} \geqslant u^{(k)}\right)\right)\\ &= \sum_{k=1}^{m+1}\mathbb{P}\left(\bigcup_{i=1}^{\infty}\left(S^{(k)}_{i} \geqslant u^{(k)}\right)\right)\\ &= \sum_{k=1}^{m+1}\psi^{(k)}\left(u^{(k)},\alpha,\beta\right). \quad \square \end{aligned} \tag{8}$$

The following theorem uses the Martingale method to provide upper bound for ultimate ruin probability.

THEOREM 1. *Let the surplus process* (4). *Assuming that* $R(\alpha,\beta) > 0$ *satisfies*

$$\mathbb{E}\left(e^{R(\alpha,\beta)(\beta Y_1 - \alpha X_1)}\right) = 1 \tag{9}$$

for any (α,β).

Then

$$\psi(u,\alpha,\beta) \leqslant \sum_{i=1}^{m+1} e^{-u^{(k)}R(\alpha,\beta)} \tag{10}$$

for any (α,β) *and* $u^{(k)} > 0$.

Proof. In order to prove (10), we set the stochastic processes $\left\{Z_n^{(k)}\right\}_{n\geqslant 0}$, $k = 1,2,\ldots,m+1$

$$Z_0^{(k)} = e^{-u^{(k)}R(\alpha,\beta)} \text{ and } Z_n^{(k)} = e^{-R(\alpha,\beta)\left(u^{(k)} - S_n^{(k)}\right)}, \quad n = 1,2,\ldots$$

Since, $\left\{X_{k+(n-1)(m+1)}\right\}_{n>0}$ and $\left\{Y_{k+(n-1)(m+1)}\right\}_{n>0}$ are sequences of independent random variables and are mutually independent, $\left\{Z_n^{(k)}\right\}_{n\geqslant 0}$ are Martingale processes.

Let $\tau_k = \min\left\{i : S_i^{(k)} \geqslant u^{(k)}\right\}$. Then $n \wedge \tau_k = \min(n;\tau_k)$ is a finite stopping time. Thus, by the optional stopping theorem for Martingale $\left\{Z_n^{(k)}\right\}_{n\geqslant 0}$, we have

$$\mathbb{E}\left(Z^{(k)}_{n\wedge\tau_k}\right) = \mathbb{E}\left(Z_0^{(k)}\right) = e^{-u^{(k)}R(\alpha,\beta)}. \tag{11}$$

Equation (11) implies that

$$e^{-u^{(k)}R(\alpha,\beta)} = \mathbb{E}\left(Z^{(k)}_{n\wedge\tau_k}\right) \geqslant \mathbb{E}\left(Z^{(k)}_{n\wedge\tau_k}1_{(\tau_k\leqslant n)}\right) = \mathbb{E}\left(Z^{(k)}_{\tau_k}1_{(\tau_k\leqslant n)}\right). \tag{12}$$

Combining (12) and $Z_{\tau_k}^{(k)} \geqslant 1$ gives us that

$$e^{-u^{(k)}R(\alpha,\beta)} \geqslant \mathbb{E}\left(1_{\tau_k \leqslant n}\right) = \mathbb{P}\left(\tau_k \leqslant n\right) = \psi_n^{(k)}\left(u^{(k)}, \alpha, \beta\right).$$

Letting $n \to \infty$, we have

$$\psi^{(k)}\left(u^{(k)}, \alpha, \beta\right) \leqslant e^{-u^{(k)}R(\alpha,\beta)}. \tag{13}$$

By Inequalities (7) and (13), the ultimate ruin probability of the reinsurance company

$$\psi\left(u, \alpha, \beta\right) \leqslant \sum_{i=1}^{m+1} e^{-u^{(k)}R(\alpha,\beta)}. \quad \square$$

REMARK 1.

1). If $\alpha = \beta$ and $m = 0$ then the theorem 1 deduces the theorem 4.2 in [7].

2). The inequality (10) is the result of theorem 1.3 in [26] where we consider $\alpha = \beta = 1$ and $m = 0$, namely

$$\psi(u, 1, 1) \leqslant e^{-uR(1,1)}. \tag{14}$$

The inequality (14) is known the Lundberg type inequality for ruin probability.

Beside the Martingale method, the inductive method is useful to evaluates the ultimate ruin probability. The inductive method and the following recursive equations are used for the ruin probability in [5, 7, 9, 16, 19]. The following lemma provides recursive equations for $\psi_n^{(k)}\left(u^{(k)}, \alpha, \beta\right)$. We denote distribution functions of X_1 and Y_1 by $H(x)$ and $F(y)$, respectively.

LEMMA 2. *Let the subsequences* $\left\{X_{k+(n-1)(m+1)}\right\}_{n>0}$, $\left\{Y_{k+(n-1)(m+1)}\right\}_{n>0}$ *and for any* (α, β) $(\beta > 0)$, *we have*

$$\begin{aligned}\psi_{n+1}^{(k)}\left(u^{(k)}, \alpha, \beta\right) = &\int_0^{\infty}\int_0^{\frac{1}{\beta}(u^{(k)}+\alpha x)} \psi_n^{(k)}\left(u^{(k)} + \alpha x - \beta y, \alpha, \beta\right) dF(y) dH(x)\\ &+ \int_0^{\infty} \overline{F}\left(\frac{1}{\beta}\left(u^{(k)} + \alpha x\right)\right) dH(x)\end{aligned} \tag{15}$$

and

$$\psi_1^{(k)}\left(u^{(k)}, \alpha, \beta\right) = \int_0^{\infty} \overline{F}\left(\frac{1}{\beta}\left(u^{(k)} + \alpha x\right)\right) dH(x) \tag{16}$$

where $\overline{F}(y) = 1 - F(y)$, $k = 1, 2, \ldots, m+1$.

Proof.

$$
\begin{aligned}
\psi_{n+1}^{(k)}\left(u^{(k)},\alpha,\beta\right) =& \mathbb{P}\left(\bigcup_{i=1}^{n+1}(S_i^{(k)} \geqslant u^{(k)})\right)\\
=& \int_0^\infty \int_0^\infty \mathbb{P}\left(\bigcup_{i=1}^{n+1}(S_i^{(k)} \geqslant u^{(k)}) \mid X_k = x, Y_k = y\right) dF(y)dH(x)\\
=& \int_0^\infty \int_0^{\frac{1}{\beta}(u^{(k)}+\alpha x)} \mathbb{P}\left(\bigcup_{i=1}^{n+1}(S_i^{(k)} \geqslant u^{(k)}) \mid X_k = x, Y_k = y\right) dF(y)dH(x)\\
&+ \int_0^\infty \int_{\frac{1}{\beta}(u^{(k)}+\alpha x)}^\infty \mathbb{P}\left(\bigcup_{i=1}^{n+1}(S_i^{(k)} \geqslant u^{(k)}) \mid X_k = x, Y_k = y\right) dF(y)dH(x).
\end{aligned}
\tag{17}
$$

If $y \geqslant \frac{1}{\beta}\left(u^{(k)}+\alpha x\right)$, then

$$
\mathbb{P}\left(S_1^{(k)} \geqslant u^{(k)} | X_k = x, Y_k = y\right) = 1,
$$

which implies that for $y \geqslant \frac{1}{\beta}\left(u^{(k)}+\alpha x\right)$,

$$
\mathbb{P}\left(\bigcup_{i=1}^{n+1}\left(S_i^{(k)} \geqslant u^{(k)}\right) | X_k = x, Y_k = y\right) = 1.
$$

While if $0 \leqslant y < \frac{1}{\beta}\left(u^{(k)}+\alpha x\right)$, then $\mathbb{P}\left(S_1^{(k)} \geqslant u^{(k)} | X_k = x, Y_k = y\right) = 0$, which implies that for $0 \leqslant y < \frac{1}{\beta}\left(u^{(k)}+\alpha x\right)$,

$$
\begin{aligned}
&\mathbb{P}\left(\bigcup_{i=1}^{n+1}\left(S_i^{(k)} \geqslant u^{(k)}\right) | X_k = x, Y_k = y\right) = \mathbb{P}\left(\bigcup_{i=2}^{n+1}\left(S_i^{(k)} \geqslant u^{(k)}\right) | X_k = x, Y_k = y\right)\\
&= \mathbb{P}\left(\bigcup_{i=2}^{n+1}\left(\sum_{j=1}^{i}\left(\beta Y_{k+(j-1)(m+1)} - \alpha X_{k+(j-1)(m+1)}\right) \geqslant u^{(k)}\right) | X_k = x, Y_k = y\right)\\
&= \mathbb{P}\left(\bigcup_{i=2}^{n+1}\left(\sum_{j=2}^{i}\left(\beta Y_{k+(j-1)(m+1)} - \alpha X_{k+(j-1)(m+1)}\right) \geqslant u^{(k)} + \alpha x - \beta y\right)\right)\\
&= \psi_n^{(k)}\left(u^{(k)} + \alpha x - \beta y, \alpha, \beta\right).
\end{aligned}
$$

Therefore, (17) implies that

$$\begin{aligned}\psi_{n+1}^{(k)}\left(u^{(k)},\alpha,\beta\right) &= \int_0^{\infty}\int_0^{\frac{1}{\beta}(u^{(k)}+\alpha x)} \psi_n^{(k)}\left(u^{(k)}+\alpha x-\beta y,\alpha,\beta\right)dF(y)dH(x) \\ &\quad + \int_0^{\infty}\int_{\frac{1}{\beta}(u^{(k)}+\alpha x)}^{\infty} dF(y)dH(x) \\ &= \int_0^{\infty}\int_0^{\frac{1}{\beta}(u^{(k)}+\alpha x)} \psi_n^{(k)}\left(u^{(k)}+\alpha x-\beta y,\alpha,\beta\right)dF(y)dH(x) \\ &\quad + \int_0^{\infty}\overline{F}\left(\frac{1}{\beta}(u^{(k)}+\alpha x)\right)dH(x). \end{aligned} \tag{18}$$

Formulas (18) is called the recursive equation for $\psi_n^k(u^{(k)},\alpha,\beta)$.

Similarly, Equation (16) holds.

This ends the proof of Lemma 2. □

A distribution F, concentrated on $(0,\infty)$, is said to be new worse than used in convex (NWUC) ordering if for all $x \geqslant 0, y \geqslant 0$,

$$\int_{x+y}^{\infty}\overline{F}(z)dz \geqslant \overline{F}(y)\int_{x}^{\infty}\overline{F}(z)dz.$$

THEOREM 2. *Let the surplus process (4). Assuming that $R(\alpha,\beta)>0$ given in (9). Then, for any (α,β) $(\beta>0)$ and $u^{(k)}>0$*

$$\psi(u,\alpha,\beta) \leqslant \gamma\sum_{i=1}^{m+1} e^{-u^{(k)}R(\alpha,\beta)} \tag{19}$$

where

$$\gamma^{-1} = \inf_{z\geqslant 0}\frac{\int_z^{\infty} e^{\beta R(\alpha,\beta)y}dF(y)}{e^{\beta R(\alpha,\beta)z}\overline{F}(z)}.$$

In particular, if F is new worse than used in convex ordering (NWUC), then

$$\psi(u,\alpha,\beta) \leqslant \frac{1}{\mathbb{E}\left(e^{\beta R(\alpha,\beta)Y_1}\right)}\sum_{i=1}^{m+1} e^{-u^{(k)}R(\alpha,\beta)}. \tag{20}$$

Proof. We have

$$\begin{aligned}\overline{F}(z) &= \left(\frac{\int_z^{\infty} e^{\beta R(\alpha,\beta)y} dF(y)}{e^{\beta R(\alpha,\beta)z}\overline{F}(z)}\right)^{-1} e^{-\beta R(\alpha,\beta)z}\int_z^{\infty} e^{\beta R(\alpha,\beta)y} dF(y)\\ &\leqslant \gamma e^{-\beta R(\alpha,\beta)z}\int_z^{\infty} e^{\beta R(\alpha,\beta)y} dF(y) \qquad (21)\\ &\leqslant \gamma e^{-\beta R(\alpha,\beta)z}\mathbb{E}\left(e^{\beta R(\alpha,\beta)Y_1}\right). \qquad (22)\end{aligned}$$

Replacing $z=\frac{1}{\beta}(u^{(k)}+\alpha x)$ in (22) and using (16), we show that

$$\begin{aligned}\psi_1^{(k)}\left(u^{(k)},\alpha,\beta\right) &\leqslant \gamma e^{-u^{(k)}R(\alpha,\beta)}\mathbb{E}\left(e^{\beta R(\alpha,\beta)Y_1}\right)\int_0^{\infty} e^{-\alpha R(\alpha,\beta)x} dH(x)\\ &= \gamma e^{-u^{(k)}R(\alpha,\beta)}\mathbb{E}\left(e^{R(\alpha,\beta)(\beta Y_1-\alpha X_1)}\right) = \gamma e^{-u^{(k)}R(\alpha,\beta)}. \qquad (23)\end{aligned}$$

Under an inductive hypothesis, we assume that

$$\psi_n^{(k)}\left(u^{(k)},\alpha,\beta\right) \leqslant \gamma e^{-u^{(k)}R(\alpha,\beta)}. \qquad (24)$$

We prove (24) holds for $n+1$.

Indeed, for $0 \leqslant \beta y < u^{(k)}+\alpha x$, replacing $u^{(k)}$ by $u^{(k)}+\alpha x-\beta y$ in (24), we have

$$\psi_n^{(k)}\left(u^{(k)}+\alpha x-\beta y,\alpha,\beta\right) \leqslant \gamma e^{(-u^{(k)}+\alpha x-\beta y)R(\alpha,\beta)}. \qquad (25)$$

From (15), (25) and z is replaced by $\frac{1}{\beta}(u^{(k)}+\alpha x)$ in (21), we obtain

$$\begin{aligned}\psi_{n+1}^{(k)}\left(u^{(k)},\alpha,\beta\right) &\leqslant \int_0^{\infty}\int_0^{\frac{1}{\beta}(u^{(k)}+\alpha x)} \gamma e^{-R(\alpha,\beta)(u^{(k)}+\alpha x-\beta y)} dF(y)dH(x)\\ &\quad + \int_0^{\infty}\int_{\frac{1}{\beta}(u^{(k)}+\alpha x)}^{\infty} \gamma e^{-R(\alpha,\beta)(u^{(k)}+\alpha x-\beta y)} dF(y)dH(x)\\ &= \int_0^{\infty}\int_0^{\infty} \gamma e^{-R(\alpha,\beta)(u^{(k)}+\alpha x-\beta y)} dF(y)dH(x)\\ &= \gamma e^{-u^{(k)}R(\alpha,\beta)}\mathbb{E}\left(e^{R(\alpha,\beta)(\beta Y_1-\alpha X_1)}\right)\\ &= \gamma e^{-u^{(k)}R(\alpha,\beta)}. \qquad (26)\end{aligned}$$

Then, $\psi_n^{(k)}\left(u^{(k)},\alpha,\beta\right) \leqslant \gamma e^{-u^{(k)}R(\alpha,\beta)}$ holds for all $n=1,2,\ldots$ Therefore

$$\psi^{(k)}\left(u^{(k)},\alpha,\beta\right) \leqslant \gamma e^{-u^{(k)}R(\alpha,\beta)}. \qquad (27)$$

From (7) and (27), we have

$$\psi(u,\alpha,\beta) \leqslant \gamma \sum_{i=1}^{m+1} e^{-u^{(k)}R(\alpha,\beta)}.$$

Following Willmot and Lin [24] (see, p. 96–97), we get

$$\gamma^{-1} = \mathbb{E}\left(e^{\beta R(\alpha,\beta)Y_1}\right).$$

Finally, replacing this equality in (19), we obtain (20). □

REMARK 2. Obviously,

$$\gamma^{-1} = \inf_{z\geqslant 0} \frac{\int_z^\infty e^{\beta R(\alpha,\beta)y} dF(y)}{e^{\beta R(\alpha,\beta)z}\overline{F}(z)} \geqslant \inf_{z\geqslant 0} \frac{\int_z^\infty e^{\beta R(\alpha,\beta)z} dF(y)}{e^{\beta R(\alpha,\beta)z}\overline{F}(z)} = 1.$$

Since, $0 < \gamma \leqslant 1$, the upper bound derived by the inductive method is tighter than that derived by the Martingale method.

In Theorems 1 and 2, we assumed that there exists $R(\alpha,\beta) > 0$ satisfying (9). The following proposition can be seen as the definition of the adjustment coefficient.

PROPOSITION 1. *Suppose* $\mathbb{E}\left(e^{r(\beta Y_1 - \alpha X_1)}\right) < \infty$ $(r \geqslant 0)$, $\alpha\mathbb{E}(X_1) > \beta\mathbb{E}(Y_1)$ *and* $\mathbb{P}(\beta Y_1 - \alpha X_1 > 0) > 0$ *for any* (α,β). *Then, there exists a unique positive number,* $R(\alpha,\beta)$, *such that*

$$\mathbb{E}\left(e^{R(\alpha,\beta)(\beta Y_1 - \alpha X_1)}\right) = 1. \tag{28}$$

Proof. The equation (28) follows by considering the properties of the function

$$g(r) = \mathbb{E}\left(e^{r(\beta Y_1 - \alpha X_1)}\right) - 1, \quad r \geqslant 0.$$

The function $g(r)$ has $g(0) = 0$, $g'(0) < 0$, $g''(r) > 0$ và $\lim_{r\to+\infty} g(r) = +\infty$. Thus, $g(r)$ must intersect the x-axis at a positive real number and $g(x)$ is a strictly convex on an interval $[0,+\infty)$. This shows that $R(\alpha,\beta)$ is a unique positive root of the equation (28). □

4. Numerical example

In this section, we give an example for Theorem 1 and Theorem 2. Suppose that $\{X_n\}_{n>0}$ and $\{Y_n\}_{n>0}$ are sequences of 2-dependent random variables and $u^{(1)} = u^{(2)} = u^{(3)} = \frac{u}{3}$. Let X_1 has a Poisson distribution with parameter $\lambda = 1.1$ and Y_1 has a gamma density with

$$f(y) = \frac{\lambda_1^{\alpha_1} y^{\alpha_1 - 1} e^{-\lambda_1 y}}{\Gamma(\alpha_1)}, \quad y \geqslant 0 \tag{29}$$

where $\alpha_1 = \frac{1}{2}$ and $\lambda_1 = \frac{1}{2}$. Furthermore, the gamma distribution is NWUC. We will show the values of upper bounds (10), (14) and (20) with cases (α,β). The values in the columns of 'Martingale', 'Induction' and 'Lundberg' mean that the upper bounds are calculated by (10), (20) and (14), respectively.

Case 1: Let $\alpha = \beta = 1$, i.e. the risk model does not consider a reinsurance contract. We obtain $R(1,1) = 0.147187$ by the Matlab program. From, we have Table 1 with a range of values of u.

u	Martingale	Induction	Lundberg
50	0.2580752	0.2167872	0.0006366
55	0.2019337	0.1696274	0.0003050
60	0.1580051	0.1327267	0.0001461
65	0.1236328	0.1038535	0.0000700
70	0.0967378	0.0812612	0.0000335
75	0.0756935	0.0635837	0.0000161

Table 1: *Upper bounds by different methods with* $\alpha = \beta = 1$

Case 2: Let $\alpha = 0.75$ and $\beta = 0.5$. We have Table 2 where $R(0.75, 0.5) = 0.7612898$.

u	Martingale	Induction	Lundberg
50	0.0000093	0.0000045	0.0006366
55	0.0000026	0.0000013	0.0003050
60	0.0000007	0.0000003	0.0001461
65	0.0000002	0.0000001	0.0000700
70	0.0000001	0.0000000	0.0000335
75	0.0000000	0.0000000	0.0000161

Table 2: *Upper bounds by different methods with* $\alpha = 0.75$ *and* $\beta = 0.5$

Case 3: The first two cases $\alpha \geqslant \beta$. In this case, we consider $\alpha < \beta$, namely $\alpha = 0.52$ and $\beta = 0.55$. We obtain $R(0.52, 0.55) = 0.6099072$.

u	Martingale	Induction	Lundberg
50	0.0001155	0.0000663	0.0006366
55	0.0000418	0.0000240	0.0003050
60	0.0000151	0.0000087	0.0001461
65	0.0000055	0.0000032	0.0000700
70	0.0000020	0.0000011	0.0000335
75	0.0000007	0.0000004	0.0000161

Table 3: *Upper bounds by different methods with* $\alpha = 0.52$ *and* $\beta = 0.55$

Table 1, Table 2 and Table 3 show that the upper bound derived by the inductive method is tighter than that derived by the Martingale method. This suits the results of Theorem 1 and Theorem 2. The upper bounds (10), (20) with $\alpha \geqslant \beta$ are tighter than the one with $\alpha < \beta$.

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