Journal of Mathematical Inequalities
Volume 18, Number 2 (2024), 719–737 doi:10.7153/jmi-2024-18-39

FURTHER JENSEN—MERCER'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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(*Communicated by M. Krnić*)

Abstract. This article considers the class of convex functions and derives further Jensen-Mercer's-type inequalities. The obtained results improve and generalize some known inequalities. A reverse of Jesnen-Mercer's inequality for scalars and operators is also given. As an application, we provide a new and non-trivial inequality related to the Wigner-Yanase-Dyson function and the logarithmic mean.

1. Motivation and background

Let $\mathcal{L}(\mathcal{H})$ denotes the the C^*-algebra (with the unit $\mathbf{1}_{\mathcal{H}}$) of all bounded linear operators on a Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$. In this paper, the inequality between operators is in the sense of Löewner partial order; that is, $B\leqslant A$ (the same as $A\geqslant B$) signifies that $A-B$ is positive. A positive invertible operator A is symbolized by $A>0$. A linear map $\Phi:\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathcal{H})$ is positive if $\Phi(A)\geqslant 0$, whenever $A\geqslant 0$. It is stated to be unital (or normalized) if $\Phi(\mathbf{1}_{\mathcal{H}})=\mathbf{1}_{\mathcal{H}}$.

A function $f:I\to\mathbb{R}$ on an interval of the real line, for all $a,b\in I$ and $0\leqslant t\leqslant 1$, is called convex if the following inequality holds:

$$f((1-t)a+tb)\leqslant(1-t)f(a)+tf(b).$$

The well-known Jensen's inequality for convex function $f:I\to\mathbb{R}$ says that:

$$f\left(\sum_{j=1}^{k}w_jx_j\right)\leqslant\sum_{j=1}^{k}w_jf(x_j) \tag{1.1}$$

where $x_1,x_2,\ldots,x_k\in I$ and $w_1,w_2,\ldots,w_k$ are positive scalars such that $\sum_{j=1}^{k}w_j=1$. The famous Hermite-Hadamard inequality asserts that if $f:[n,N]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{n+N}{2}\right)\leqslant\frac{1}{N-n}\int_n^N f(t)\,dt\leqslant\frac{f(n)+f(N)}{2}. \tag{1.2}$$

Mathematics subject classification (2020): Primary 26A51, 47A63; Secondary 26D15.
Keywords and phrases: Jensen-Mercer inequality, convex function, continuous functional calculus.

Paper JMI-18-39

The above inequality can be written in the following form

$$f\left(\frac{n+N}{2}\right) \leqslant \int_0^1 f((1-t)n+tN)\,dt \leqslant \frac{f(n)+f(N)}{2},$$

due to

$$\frac{1}{N-n}\int_n^N f(t)\,dt = \int_0^1 f((1-t)n+tN)\,dt = \int_0^1 f((1-t)N+tn)\,dt.$$

The theories of convex functions and inequalities are closely intertwined. In recent years, many researchers have concentrated much on the theory of convexity because of its excellent utility in different areas of pure and applied sciences [6, 13, 14, 15, 16, 17]. A very impressive inequality, which is extensively studied in the literature, is due to Mercer [9]. This superior result reads as follows: If $f:[n,N]\to\mathbb{R}$ is a convex function and $n\leqslant x_1,x_2,\dots,x_k\leqslant N$, then

$$f\left(N+n-\sum_{j=1}^{k} w_j x_j\right) \leqslant f(N)+f(n)-\sum_{j=1}^{k} w_j f(x_j). \tag{1.3}$$

To receive the above inequality, Mercer first proved that

$$f(N+n-x) \leqslant f(N)+f(n)-f(x);\ \ (n\leqslant x\leqslant N). \tag{1.4}$$

Remarkably, the inequality (1.4) is equivalent to the inequality

$$f((1-t)n+tN)+f((1-t)N+tn) \leqslant f(n)+f(N), \quad (0\leqslant t\leqslant 1), \tag{1.5}$$

putting $x:=(1-t)N+tn$ with $0\leqslant t\leqslant 1$. Note that $f:I\to\mathbb{R}$ is called Wright-convex function on $I\subseteq\mathbb{R}$, if we have the inequality (1.5) for any $0\leqslant t\leqslant 1$ and for all $n,N\in I$. We refer the interested reader to [11] for the new findings regarding this class of functions.

Jensen-Mercer's inequality (1.3) has received much concentration in current years, and an impressive variety of improvements and generalizations have been investigated [2, 10, 12]. Let $A_1,A_2,\dots,A_k\in\mathcal{L}(\mathcal{H})$ be self-adjoint operators whose spectra are contained in the interval $[n,N]$. In [8], Matković et al. gave the operator version of Mercer's result as follows: If $f:[n,N]\to\mathbb{R}$ is a convex function, then

$$f\left((N+n)\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j(A_j)\right) \leqslant (f(N)+f(n))\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j(f(A_j)) \tag{1.6}$$

where $\Phi_1,\Phi_2,\dots,\Phi_k$ are positive linear maps such that $\sum_{j=1}^{k}\Phi_j(\mathbf{1}_{\mathcal{H}})=\mathbf{1}_{\mathcal{H}}$.

In the present article, we have established Jensen-Mercer's type inequalities for convex functions. A converse of Jensen-Mercer's inequality for differentiable convex functions is obtained. These results have some connections with known results in the literature.

2. Jensen-Mercer's type inequalities

2.1. Scalar case

We begin this section by generalizing Jensen-Mercer inequality (1.3).

THEOREM 2.1. *Let $f:[n,N]\to\mathbb{R}$ be a convex function, let $n\leqslant x_j,y_j\leqslant N$ $(j=1,2,\ldots,k)$, and let w_j be positive scalars such that $\sum_{j=1}^k w_j=1$. Then*

$$f\left(N+n-\sum_{j=1}^k w_jy_j\right)$$
$$\leqslant f(N)+f(n)-\sum_{j=1}^k w_jf(x_j)+\frac{\sum_{j=1}^k w_jy_j-\sum_{j=1}^k w_jx_j}{N-n}\left(f(n)-f(N)\right).$$

Proof. If $f:[n,N]\to\mathbb{R}$ is a convex function, then

$$f(t)\leqslant\frac{N-t}{N-n}f(n)+\frac{t-n}{N-n}f(N);\ \ (n\leqslant t\leqslant N). \tag{2.1}$$

We substitute t by $N+n-t$, in (2.1), we reach

$$f(N+n-t)\leqslant\frac{t-n}{N-n}f(n)+\frac{N-t}{N-n}f(N). \tag{2.2}$$

Choosing $t=x_j$ $(j=1,2,\ldots,k)$, in (2.1), to obtain

$$f(x_j)\leqslant\frac{N-x_j}{N-n}f(n)+\frac{x_j-n}{N-n}f(N). \tag{2.3}$$

Multiplying (2.3) by $w_j\geqslant 0$ $(j=1,2,\ldots,k)$ and then summing over j from 1 to k, we infer

$$\sum_{j=1}^k w_jf(x_j)\leqslant\frac{N-\sum_{j=1}^k w_jx_j}{N-n}f(n)+\frac{\sum_{j=1}^k w_jx_j-n}{N-n}f(N). \tag{2.4}$$

Choosing $t=\sum_{j=1}^k w_jy_j$ $(j=1,2,\ldots,k)$, in (2.2), we obtain

$$f\left(N+n-\sum_{j=1}^k w_jy_j\right)\leqslant\frac{\sum_{j=1}^k w_jy_j-n}{N-n}f(n)+\frac{N-\sum_{j=1}^k w_jy_j}{N-n}f(N). \tag{2.5}$$

Adding two inequalities (2.4) and (2.5) together, we receive

$$f\left(N+n-\sum_{j=1}^k w_jy_j\right)$$
$$\leqslant f(N)+f(n)-\sum_{j=1}^k w_if(x_j)+\frac{\sum_{j=1}^k w_jy_j-\sum_{j=1}^k w_jx_j}{N-n}\left(f(n)-f(N)\right)$$

as expected. □

REMARK 2.1. Let the assumptions of Theorem 2.1 hold. If

$$\sum_{j=1}^{k} w_j x_j = \sum_{j=1}^{k} w_j y_j,$$

then

$$f\left(N+n-\sum_{j=1}^{k} w_j x_j\right) \leqslant f(N)+f(n)-\sum_{j=1}^{k} w_j f(x_j).$$

Thus, Theorem 2.1 is a generalization of Jensen-Mercer's inequality (1.3).

The following result presents a counterpart of Mercer's result (1.4).

THEOREM 2.2. *Let $f:[n,N]\to\mathbb{R}$ be a differentiable convex function and let $n\leqslant t\leqslant N$. Then*

$$f(N)+f(n)-f(t) \leqslant f(N+n-t)+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right).$$

Proof. We understand that any differentiable convex function f satisfies the inequality

$$f(a)+f'(a)(b-a) \leqslant f(b) \leqslant f(a)+f'(b)(b-a) \tag{2.6}$$

for any a,b in the domain of f. If we replace a by t and b by N, in the second inequality of (2.6), we get

$$f(N) \leqslant f(t)+f'(N)(N-t). \tag{2.7}$$

If we replace a by t and b by n, in the second inequality of (2.6), we have

$$f(n) \leqslant f(t)+f'(n)(n-t). \tag{2.8}$$

If we replace a by $N+n-t$ and b by N, in (2.7), we get

$$f(N) \leqslant f(N+n-t)+f'(N)(N-(N+n-t)). \tag{2.9}$$

If we replace a by $N+n-t$ and b by n, in (2.7), we obtain

$$f(n) \leqslant f(N+n-t)+f'(n)(n-(N+n-t)). \tag{2.10}$$

Summing inequalities (2.7), (2.8), (2.9), and (2.10), we reach

$$f(N)+f(n)-f(t) \leqslant f(N+n-t)+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right)$$

as desired. □

As a consequence of Theorem 2.2, we give a complementary inequality of (1.3).

COROLLARY 2.1. *Let $f:[n,N]\to\mathbb{R}$ be a differentiable convex function, let $n\leqslant x_j\leqslant N$ $(j=1,2,\ldots,k)$, and let w_j be positive scalars such that $\sum_{j=1}^{k} w_j=1$. Then*

$$f(N)+f(n)-\sum_{j=1}^{k} w_j f(x_j)\leqslant f\left(N+n-\sum_{j=1}^{k} w_j x_j\right)+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right).$$

Proof. If we replace t by $\sum_{j=1}^{k} w_j x_j$, in Theorem 2.2, we get

$$\begin{aligned}
&f(N)+f(n)-\sum_{j=1}^{k} w_j f(x_j)\\
&\leqslant f(N)+f(n)-f\left(\sum_{j=1}^{k} w_j x_j\right)\\
&\leqslant f\left(N+n-\sum_{j=1}^{k} w_j x_j\right)+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right),
\end{aligned}$$

where we have used (1.1) to obtain the first inequality. □

It has been shown in [7, Eq. (2.1)] and [7, Eq. (2.2)] that if $f:[n,N]\to\mathbb{R}$ is convex, then for any $n\leqslant x,y\leqslant N$,

$$f\left(N+n-\frac{x+y}{2}\right)\leqslant f(N)+f(n)-\int_0^1 f((1-t)x+ty)\,dt,$$

and

$$\frac{1}{y-x}\int_x^y f(N+n-t)\,dt\leqslant f(N)+f(n)-\frac{f(x)+f(y)}{2}. \tag{2.11}$$

The following result provides counterparts of the above inequalities.

THEOREM 2.3. *Let $f:[n,N]\to\mathbb{R}$ be a differentiable convex function and let $n\leqslant x,y\leqslant N$. Then*

$$f(N)+f(n)-\int_0^1 f((1-t)x+ty)\,dt\leqslant f\left(N+n-\frac{x+y}{2}\right)+\frac{N-n}{2}\left(f'(N)-f'(n)\right),$$

and

$$f(N)+f(n)-\frac{f(x)+f(y)}{2}\leqslant\frac{1}{y-x}\int_x^y f(N+n-t)\,dt+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right).$$

Proof. We first prove the first inequality. If we replace t by $\frac{x+y}{2}$, in Theorem 2.2, we get

$$\begin{aligned} & f(N)+f(n)-\int_0^1 f((1-t)x+ty)\,dt \\ & \leqslant f(N)+f(n)-f\left(\frac{x+y}{2}\right) \\ & \leqslant f\left(N+n-\frac{x+y}{2}\right)+\frac{N-n}{2}\left(f'(N)-f'(n)\right), \end{aligned} \tag{2.12}$$

where the first inequality follows from the first inequality in Hermite-Hadamard inequality.

To prove the second inequality, if we take integral over $x \leqslant t \leqslant y$, in Theorem 2.2, we obtain

$$\begin{aligned} & f(N)+f(n)-\frac{f(x)+f(y)}{2} \\ & \leqslant f(N)+f(n)-\frac{1}{y-x}\int_x^y f(t)\,dt \\ & \leqslant \frac{1}{y-x}\int_x^y f(N+n-t)\,dt+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right), \end{aligned} \tag{2.13}$$

where the first inequality follows from the second inequality in Hermite-Hadamard inequality. $\square$

REMARK 2.2. From (1.3) and (2.12), we get

$$\begin{aligned} & f(N)+f(n)-\int_0^1 f((1-t)x+ty)\,dt \\ & \leqslant f(N)+f(n)-f\left(\frac{x+y}{2}\right) \\ & \leqslant f\left(N+n-\frac{x+y}{2}\right)+\frac{N-n}{2}\left(f'(N)-f'(n)\right) \\ & \leqslant f(N)+f(n)-\frac{f(x)+f(y)}{2}+\frac{N-n}{2}\left(f'(N)-f'(n)\right) \end{aligned}$$

i.e.,

$$\begin{aligned} & \frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(t)\,dt \\ & =\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt \end{aligned}$$

$$\leqslant \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$$
$$\leqslant f\left(N+n-\frac{x+y}{2}\right) - f(N) - f(n) + \frac{f(x)+f(y)}{2} + \frac{N-n}{2}\left(f'(N)-f'(n)\right)$$
$$\leqslant \frac{N-n}{2}\left(f'(N)-f'(n)\right).$$

From (2.11) and (2.13), we also have

$$f(N)+f(n)-\frac{f(x)+f(y)}{2}$$
$$\leqslant f(N)+f(n)-\frac{1}{y-x}\int_x^y f(t)\,dt$$
$$\leqslant \frac{1}{y-x}\int_x^y f(N+n-t)\,dt + \frac{N-n}{2}\left(f'(N)-f'(n)\right)$$
$$\leqslant f(N)+f(n)-\frac{f(x)+f(y)}{2} + \frac{N-n}{2}\left(f'(N)-f'(n)\right)$$

i.e.,

$$\frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)\,dt$$
$$\leqslant \frac{1}{y-x}\int_x^y f(N+n-t)\,dt - f(N) - f(n) + \frac{f(x)+f(y)}{2} + \frac{N-n}{2}\left(f'(N)-f'(n)\right)$$
$$\leqslant \frac{N-n}{2}\left(f'(N)-f'(n)\right).$$

It is known that $Hz_t(a,b) \leqslant A(a,b)$ for any $a,b>0$, where

$$A(a,b) := \frac{a+b}{2}, \quad Hz_t(a,b) := \frac{a^{1-t}b^t + a^t b^{1-t}}{2}; \ (0 \leqslant t \leqslant 1),$$

and $Hz_t(a,b)$ is called the Heinz mean. We give a reverse of the above inequality.

COROLLARY 2.2. *Let* $a,b>0$. *Then*

$$A(a,b) \leqslant Hz_t(a,b) + \frac{1}{4}(b-a)(\log b - \log a).$$

Proof. Taking $n=0$ and $N=1$ in Theorem 2.2, we have for a differentiable convex function f:

$$f(1)+f(0) \leqslant f(t)+f(1-t)+\frac{1}{2}(f'(1)-f'(0)). \tag{2.14}$$

Taking a convex function $f(t) := x^t$, $(x>0,\ 0 \leqslant t \leqslant 1)$, (since $f''(t) = x^t(\log x)^2 \geqslant 0$), we have for $x>0$

$$\frac{x+1}{2} \leqslant \frac{x^t + x^{1-t}}{2} + \frac{1}{4}(x-1)\log x.$$

Putting $x := \frac{b}{a} > 0$ and multiplying $a>0$ to both sides, we obtain the desired result. □

REMARK 2.3. Notice that for $0 \leqslant t \leqslant 1$, the function $g(t) = f((1-t)a+tb)$ is convex, whenever f is convex. So, from (2.14), we have

$$f(b)+f(a) \leqslant f((1-t)a+tb)+f(ta+(1-t)b)+\frac{f'(b)-f'(a)}{2},$$

which is a converse of the following inequality for a convex function f:

$$f((1-t)a+tb)+f(ta+(1-t)b) \leqslant f(a)+f(b).$$

In particular,

$$0 \leqslant \frac{f(1)+f(0)}{2} - f\left(\frac{1}{2}\right) \leqslant \frac{f'(1)-f'(0)}{4}.$$

The above inequality implies the second inequality in the following

$$0 \leqslant \frac{f(1)+f(0)}{2} - \int_0^1 f(t)\,dt \leqslant \frac{f'(1)-f'(0)}{4}.$$

The second inequality is a special case of [5, Theorem 3.2].

It is also known [3, Theorem 2.3] that $L(a,b) \leqslant W_t(a,b) \leqslant K(h)^{t(1-t)}L(a,b)$ for $a,b>0$ and $0 \leqslant t \leqslant 1$, where the logarithmic mean and the Wigner-Yanase-Dyson function are defined by

$$L(a,b) := \frac{b-a}{\log b - \log a}, \quad W_t(a,b) := \frac{t(1-t)(b-a)^2}{(b^t-a^t)(b^{1-t}-a^{1-t})}$$

and $K(h) := \frac{(h+1)^2}{4h}$ is the Kantorovich constant with $h := \frac{b}{a}$.

Applying the same method above corollary, we give a new reverse of the inequality $L(a,b) \leqslant W_t(a,b)$.

THEOREM 2.4. *Let $a,b>0$. Then*

$$W_t(a,b) \leqslant \sqrt{\frac{I(a,b)}{G(a,b)}}L(a,b),$$

where the geometric mean $G(a,b) := \sqrt{ab}$ and the identric mean $I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ are defined for $a,b>0$.

Proof. It is known that $G(u,1) \leqslant L(u,1)$ for $u > 0$. From this, for $u > 0$ we have $\left(\frac{u-1}{\log u}\right)^2 \geqslant u$ which is equivalent to the inequality $(u-1)^2 - u(\log u)^2 \geqslant 0$ for $u > 0$. Take $f(t) := \log\left(\frac{x^t-1}{t}\right)$ for $1 \neq x > 0$ and $0 \leqslant t \leqslant 1$. Then we have

$$\frac{df(t)}{dt} = \frac{x^t \log x^t + x^t - 1}{t(x^t-1)}, \quad \frac{d^2 f(t)}{dt^2} = \frac{(x^t-1)^2 - x^t(\log x^t)^2}{t^2(x^t-1)^2} \geqslant 0.$$

Therefore $f(t)$ is convex in $t \in [0,1]$. Thus the inequality

$$f(1) + f(0) \leqslant f(t) + f(1-t) + \frac{1}{2}(f'(1) - f'(0))$$

gives

$$\log(x-1) + \log(\log x) \leqslant \log\left(\frac{x^t-1}{t}\right) + \log\left(\frac{x^{1-t}-1}{1-t}\right) + \frac{1}{2}\left(-1 + \frac{x}{x-1}\log x - \frac{1}{2}\log x\right)$$

which implies

$$(x-1)\log x \leqslant \frac{(x^t-1)(x^{1-t}-1)}{t(1-t)}\left(\frac{x^{\frac{x}{x-1}}}{e\sqrt{x}}\right)^{1/2}.$$

Multiplying $\frac{1}{(x-1)^2} > 0$ to both sides and taking an inverse of the inequality, we get

$$\frac{x-1}{\log x} \geqslant \frac{t(1-t)(x-1)^2}{(x^t-1)(x^{1-t}-1)}\left(\frac{x^{\frac{x}{x-1}}}{e\sqrt{x}}\right)^{-1/2},$$

which is equivalent to

$$W_t(x,1) \leqslant \sqrt{\frac{I(x,1)}{G(x,1)}} L(x,1).$$

Putting $x := \frac{b}{a} > 0$ with $a \neq b$ and multiplying $a > 0$ to both sides, and taking accounts for $L(a,a) := \lim_{b\to a} L(a,b) = a$, $I(a,a) := \lim_{b\to a} I(a,b) = a$ and $W_t(a,a) := \lim_{b\to a} W_t(a,b) = a$, we obtain the desired result. $\square$

REMARK 2.4. It may be of interest to compare the ordering between $K(h)^{t(1-t)}$ and $\sqrt{\frac{I(h,1)}{G(h,1)}}$ for $h > 0$ and $t \in [0,1]$. However, there is no ordering between them.

Since $I(h,1) \leqslant A(h,1)$, we have $\dfrac{h^{\frac{h}{h-1}}}{e\sqrt{h}} \leqslant \dfrac{h+1}{2\sqrt{h}}$ which implies $\left(\dfrac{I(h,1)}{G(h,1)}\right)^2 \leqslant K(h) \Longleftrightarrow \sqrt{\dfrac{I(h,1)}{G(h,1)}} \leqslant K(h)^{1/4}$.

On the other hand, we have the numerical computation such as

$$\sqrt{\frac{I(h,1)}{G(h,1)}} - K(h)^{t(1-t)} \simeq 0.121015$$

when $h := 10$ and $t := 0.1$.

THEOREM 2.5. *Let $f : [n,N] \to \mathbb{R}$ be a differentiable convex function, let $n \leqslant x_j, y_j \leqslant N$ $(j = 1,2,\ldots,k)$, and let w_j be positive scalars such that $\sum_{j=1}^{k} w_j = 1$. Then*

$$\begin{aligned} & f(N) + f(n) - \sum_{j=1}^{k} w_j f(x_j) \\ & \leqslant f\left(N + n - \sum_{j=1}^{k} w_i y_j\right) \\ & \quad + \left(\frac{N-n}{2}\right)\left(f'(N) - f'(n)\right) + \left(\frac{\sum_{j=1}^{k} w_j y_j - \sum_{j=1}^{k} w_j x_j}{2}\right)\left(f'(N) + f'(n)\right). \end{aligned}$$

Proof. By adding two inequalities (2.7) and (2.8) together, we have

$$f(N) + f(n) \leqslant 2f(t) + f'(N)(N-t) + f'(n)(n-t). \tag{2.15}$$

If we employ (2.15) for the selection $t = x_j$ $(j = 1,2,\ldots,k)$ and then multiplying by w_j $(j = 1,2,\ldots,k)$ and summing over j from 1 to k, we get

$$\begin{aligned} & f(N) + f(n) \\ & \leqslant 2\sum_{j=1}^{k} w_j f(x_j) + f'(N)\left(N - \sum_{j=1}^{k} w_j x_j\right) + f'(n)\left(n - \sum_{i=1}^{k} w_j x_j\right). \end{aligned} \tag{2.16}$$

Replacing t by $N + n - \sum_{j=1}^{k} t_j y_j$, in (2.15), implies

$$\begin{aligned} & f(N) + f(n) \\ & \leqslant 2f\left(N + n - \sum_{j=1}^{k} w_j y_j\right) + f'(N)\left(\sum_{j=1}^{k} w_j y_j - n\right) + f'(n)\left(\sum_{j=1}^{k} w_j y_j - N\right). \end{aligned} \tag{2.17}$$

Merging two inequalities (2.16) and (2.17) gives

$$\begin{aligned}
&f(N)+f(n)-\sum_{j=1}^{k} w_j f(x_j)\\
&\leqslant f\left(N+n-\sum_{j=1}^{k} w_j y_j\right)\\
&\quad+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right)+\left(\frac{\sum_{j=1}^{k} w_j y_j-\sum_{i=1}^{k} w_j x_j}{2}\right)\left(f'(N)+f'(n)\right)
\end{aligned}$$

as expected. □

THEOREM 2.6. *Let $f : I \to \mathbb{R}$ be a differentiable convex function on I^o (interior of I) and let $f' \in \mathcal{S}[n,N]$ (the space of Riemann integrable function on $[n,N]$), where $n,N \in I$ with $n<N$, then for each $x \in [n,N]$*

$$f\left(\frac{n+N}{2}\right) \leqslant \delta_{n,N}(x)+\frac{1}{N-n}\int_n^N f(t)\,dt \leqslant \frac{f(n)+f(N)}{2}$$

where

$$\begin{aligned}
\delta_{n,N}(x)=\frac{1}{2(N-n)}&\left((N-x)^2\left(\int_0^1 \nu f'(n+\nu(N-x))\,d\nu-\int_0^1 \nu f'(N+\nu(x-N))\,d\nu\right)\right.\\
&\left.+(n-x)^2\left(\int_0^1 \nu f'(n+\nu(x-n))\,d\nu-\int_0^1 \nu f'(N+\nu(n-x))\,d\nu\right)\right).
\end{aligned}$$

In particular,

$$\begin{aligned}
&f\left(\frac{1}{2}\right)\\
&\leqslant \frac{1}{2}\left((1-x)^2\left(\int_0^1 \nu f'(\nu(1-x))\,d\nu-\int_0^1 \nu f'(1-\nu(1-x))\,d\nu\right)\right.\\
&\quad\left.+x^2\left(\int_0^1 \nu f'(\nu x)\,d\nu-\int_0^1 \nu f'(1-\nu x)\,d\nu\right)\right)+\int_0^1 f(t)\,dt\\
&\leqslant \frac{f(0)+f(1)}{2}.
\end{aligned}$$

Proof. From [1], we know that if $f : I \to \mathbb{R}$ is a differentiable function on I^o and if $f' \in \mathcal{S}[n,N]$, then for each $x \in [n,N]$

$$f(x) - \frac{1}{N-n}\int_n^N f(t)\,dt = \frac{(x-n)^2}{N-n}\int_0^1 v f'((1-v)n + vx)\,dv - \frac{(x-N)^2}{N-n}\int_0^1 v f'((1-v)N + vx)\,dv. \tag{2.18}$$

If we replace x by $N+n-x$ in the above equality, we get

$$f(N+n-x) - \frac{1}{N-n}\int_n^N f(t)\,dt = \frac{(N-x)^2}{N-n}\int_0^1 v f'(n + v(N-x))\,dv - \frac{(n-x)^2}{N-n}\int_0^1 v f'(N + v(n-x))\,dv. \tag{2.19}$$

Now, summing two inequalities (2.18) and (2.19), we have

$$\begin{aligned} f(N+n-x) &+ f(x) - \frac{2}{N-n}\int_n^N f(t)\,dt \\ &= \frac{(N-x)^2}{N-n}\left(\int_0^1 v f'(n+v(N-x))\,dv - \int_0^1 v f'(N+v(x-N))\,dv\right) \\ &\quad + \frac{(n-x)^2}{N-n}\left(\int_0^1 v f'(n+v(x-n))\,dv - \int_0^1 v f'(N+v(n-x))\,dv\right). \end{aligned} \tag{2.20}$$

From this and the convexity of the function f, we reach the desired inequality, since

$$\begin{aligned} f\left(\frac{n+N}{2}\right) &\leqslant \frac{f(N+n-x)+f(x)}{2} \\ &= \delta_{n,N}(x) + \frac{1}{N-n}\int_n^N f(t)\,dt \\ &\leqslant \frac{f(n)+f(N)}{2}, \end{aligned}$$

where the second inequality observes from Jensen-Mercer's inequality □

REMARK 2.5. From the convexity of f, f' is monotone increasing. The condition $n + v(N-x) \leqslant N + v(x-N)$ for $0 \leqslant v \leqslant 1$ and $n \leqslant x \leqslant N$ implies $x \geqslant \dfrac{N+n}{2}$.

The condition $n\nu(x-n) \leqslant N+\nu(n-x)$ for $0 \leqslant \nu \leqslant 1$ and $n \leqslant x \leqslant N$ also implies $x \leqslant \frac{N+n}{2}$. Thus we could show $\delta_{n,N}\left(\frac{N+n}{2}\right) \leqslant 0$. Therefore the inequality

$$f\left(\frac{N+n}{2}\right) \leqslant \delta_{n,N}\left(\frac{N+n}{2}\right) + \frac{1}{N-n}\int_n^N f(t)dt$$

gives a refinement of the first inequality in the Hermite-Hadamard inequality (1.2).

REMARK 2.6. It has been shown in [4, Eq. (25)] that

$$\frac{f(N)+f(n)}{2} - \frac{1}{N-n}\int_n^N f(t)\,dt = \frac{1}{N-n}\int_n^N \left(t-\frac{N+n}{2}\right) f'(t)\,dt. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$\begin{aligned}
&f(N)+f(n)-f(x)-f(N+n-x)\\
&= \frac{2}{N-n}\int_n^N \left(t-\frac{N+n}{2}\right) f'(t)\,dt\\
&\quad - \frac{(N-x)^2}{N-n}\left(\int_0^1 \nu f'(n+\nu(N-x))\,d\nu - \int_0^1 \nu f'(N+\nu(x-N))\,d\nu\right)\\
&\quad - \frac{(n-x)^2}{N-n}\left(\int_0^1 \nu f'(n+\nu(x-n))\,d\nu - \int_0^1 \nu f'(N+\nu(n-x))\,d\nu\right).
\end{aligned}$$

The two sides of the overhead equality are positive whenever f is convex.

2.2. Operator version

This subsection presents the non-commutative version of the previously obtained results.

THEOREM 2.7. *Let $f:[n,N]\to\mathbb{R}$ be a convex function, let $A_j, B_j \in \mathcal{L}(\mathcal{H})$ $(j=1,2,\ldots,k)$ be self-adjoint operators whose spectra are contained on the interval $[n,N]$, and let $\Phi_j : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ $(j=1,2,\ldots,k)$ be positive linear maps such that $\sum_{j=1}^k \Phi_j(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{H}}$. Then*

$$\begin{aligned}
f\left((N+n)\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^k \Phi_j(A_j)\right) &\leqslant (f(N)+f(n))\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^k \Phi_j(f(B_j))\\
&\quad + (f(n)-f(N))\frac{\sum_{j=1}^k \Phi_j(A_j) - \sum_{j=1}^k \Phi_j(B_j)}{N-n}.
\end{aligned}$$

Proof. Utilizing continuous functional calculus, we conclude from the inequality (2.1) that

$$f(B_j) \leqslant \frac{N\mathbf{1}_{\mathcal{H}} - B_j}{N-n} f(n) + \frac{B_j - n\mathbf{1}_{\mathcal{H}}}{N-n} f(N).$$

Applying positive linear maps Φ_j and summing, it observes that

$$\sum_{j=1}^{k} \Phi_i(f(B_j)) \leqslant \frac{N\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k}\Phi_j(B_j)}{N-n} f(n) + \frac{\sum_{j=1}^{k}\Phi_j(B_j) - n\mathbf{1}_{\mathcal{H}}}{N-n} f(N). \tag{2.22}$$

Again, by using continuous functional calculus, we infer from the inequality (2.1) that

$$f\left((N+n)\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k}\Phi_j(A_j)\right) \leqslant \frac{\sum_{j=1}^{k}\Phi_j(A_j) - n\mathbf{1}_{\mathcal{H}}}{N-n} f(n) + \frac{N\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k}\Phi_j(A_j)}{N-n} f(N). \tag{2.23}$$

Utilizing inequalities (2.22) and (2.23), we obtain

$$\begin{aligned} &f\left((N+n)\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k}\Phi_i(A_j)\right) + \sum_{j=1}^{k}\Phi_i(f(B_j)) \\ &\leqslant (f(N)+f(n))\mathbf{1}_{\mathcal{H}} + (f(n)-f(N))\frac{\sum_{j=1}^{k}\Phi_j(A_j) - \sum_{j=1}^{k}\Phi_j(B_j)}{N-n}, \end{aligned}$$

as wished. □

REMARK 2.7. Let the assumptions of Theorem 2.7 hold. If

$$\sum_{j=1}^{k}\Phi_j(A_j) = \sum_{j=1}^{k}\Phi_j(B_j),$$

then

$$f\left((N+n)\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k}\Phi_j(A_j)\right) \leqslant (f(N)+f(n))\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k}\Phi_j(f(A_j)).$$

Hence, Theorem 2.7 is a generalization of Jensen-Mercer's inequality (1.6).

REMARK 2.8. For example, in the cases $n=0$, $N=1$, $k=1$ and $\Phi_1(X) := X$ for any $X \in \mathcal{L}(\mathcal{H})$ in Theorem 2.7, we have for convex f,

$$f(\mathbf{1}_{\mathcal{H}} - A) + f(B) - (f(0)+f(1))\mathbf{1}_{\mathcal{H}} \leqslant (f(0)-f(1))(A-B).$$

Replacing A and B by $h(A)$ and $h(B)$ respectively under the assumption $A, B \geqslant 0$ and h is an operator monotone function with $h(1)=1$, we have

$$f(\mathbf{1}_{\mathcal{H}} - h(A)) + f(h(B)) - (f(0)+f(1))\mathbf{1}_{\mathcal{H}} \leqslant (f(0)-f(1))(h(A)-h(B)).$$

Thus, if (i) f is increasing and $A \geqslant B \geqslant 0$, or (ii) f is decreasing and $0 \leqslant A \leqslant B$, then we have

$$f(\mathbf{1}_{\mathcal{H}} - h(A)) + f(h(B)) \leqslant (f(0) + f(1))\mathbf{1}_{\mathcal{H}}.$$

For the case (i), take a convex and increasing function $f(t) := e^t$ in $[0,1]$ and $A \geqslant B$, then we have

$$e^{\mathbf{1}_{\mathcal{H}} - h(A)} + e^{h(B)} \leqslant (1+e)\mathbf{1}_{\mathcal{H}}.$$

The sufficient condition to hold the above inequality is $B \leqslant \mathbf{1}_{\mathcal{H}} \leqslant A$ trivially. However, an application of Theorem 2.7 says that the above inequality holds for any $A \geqslant B$ and an operator monotone function h with $h(1) = 1$. For case (ii), take a convex and decreasing function $f(t) := e^{-t}$ in [0,1] and $A \leqslant B$, then we have

$$e^{h(A) - \mathbf{1}_{\mathcal{H}}} + e^{-h(B)} \leqslant (1+e^{-1})\mathbf{1}_{\mathcal{H}}.$$

We also find the sufficient condition to hold the above inequality is $A \leqslant \mathbf{1}_{\mathcal{H}} \leqslant B$ trivially. However, an application of Theorem 2.7 says that the above inequality holds for any $A \leqslant B$ and an operator monotone function h with $h(1) = 1$.

THEOREM 2.8. *Let $f : [n,N] \to \mathbb{R}$ be a differentiable convex function, let $A_j, B_j \in \mathcal{L}(\mathcal{H})$ $(j = 1,2,\ldots,k)$ be self-adjoint operators whose spectra are contained on the interval $[n,N]$, and let $\Phi_j : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ $(j = 1,2,\ldots,k)$ be positive linear maps such that $\sum_{j=1}^{k} \Phi_j(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{H}}$. Then,*

$$\begin{aligned}
&(f(N) + f(n))\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k} \Phi_j f(B_j) \\
&\leqslant f\left((N+n)\mathbf{1}_{\mathcal{H}} - \sum_{j=1}^{k} \Phi_j(A_j)\right) \\
&\quad + \left(\frac{N-n}{2}\right)\left(f'(N) - f'(n)\right)\mathbf{1}_{\mathcal{H}} \\
&\quad + \left(\frac{f'(N) + f'(n)}{2}\right)\left(\sum_{j=1}^{k} \Phi_j(A_j) - \sum_{j=1}^{k} \Phi_j(B_j)\right).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\left(f(N) + f(n) - \left(\frac{N-n}{2}\right)\left(f'(N) - f'(n)\right)\right)\mathbf{1}_{\mathcal{H}} \\
&\leqslant f(B) + f((N+n)1_H - A) + \left(\frac{f'(N) + f'(n)}{2}\right)(A - B).
\end{aligned}$$

Proof. By applying continuous functional calculus, we infer from the inequality (2.15) that

$$
\begin{aligned}
(f(N)+f(n))\mathbf{1}_{\mathcal{H}} \leqslant 2f\left((N+n)\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j(A_j)\right)& \\
+f'(N)\left(\sum_{j=1}^{k}\Phi_j(A_j)-n\mathbf{1}_{\mathcal{H}}\right)& \qquad (2.24)\\
+f'(n)\left(\sum_{j=1}^{k}\Phi_j(A_j)-N\mathbf{1}_{\mathcal{H}}\right)&.
\end{aligned}
$$

Again, by applying continuous functional calculus, we infer from the inequality (2.15) that

$$
\begin{aligned}
(f(N)+f(n))\mathbf{1}_{\mathcal{H}} \leqslant 2\sum_{j=1}^{k}\Phi_j f(B_j)+f'(N)\left(N\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j(B_j)\right)& \\
+f'(n)\left(n\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j(B_j)\right)&. \qquad (2.25)
\end{aligned}
$$

Incorporating two inequalities (2.24) and (2.25) implies the desired inequality. □

REMARK 2.9. Let the assumptions of Theorem 2.8 hold. If

$$\sum_{j=1}^{k}\Phi_j(A_j)=\sum_{j=1}^{k}\Phi_j(B_j),$$

then

$$
\begin{aligned}
&(f(N)+f(n))\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j f(A_j)\\
&\leqslant f\left((N+n)\mathbf{1}_{\mathcal{H}}-\sum_{j=1}^{k}\Phi_j(A_j)\right)+\left(\frac{N-n}{2}\right)\left(f'(N)-f'(n)\right)\mathbf{1}_{\mathcal{H}}.
\end{aligned}
$$

Indeed, Theorem 2.7 is an extension of Jensen-Mercer's inequality (1.6).

From Theorem 2.4, we have the following inequality.

COROLLARY 2.3. *Let $A,B\in\mathcal{L}(\mathcal{H})$ be positive operators with $\alpha A\leqslant B\leqslant\beta A$ for $0<\alpha\leqslant\beta$ and let $0\leqslant t\leqslant 1$. Then we have*

$$W_t(A,B)\leqslant R_{I/G}\cdot L(A,B),$$

where

$$R_{I/G}:=\max_{\alpha\leqslant x\leqslant\beta}\sqrt{\frac{I(x,1)}{G(x,1)}}=\max\left\{\sqrt{\frac{I(\alpha,1)}{G(\alpha,1)}},\sqrt{\frac{I(\beta,1)}{G(\beta,1)}}\right\}$$

and the operator logarithmic mean and the operator Wigner–Yanase–Dyson function [3] *are defined by:*

$$L(A,B) := \int_0^1 A\sharp_p B dp,$$

$$W_t(A,B) := \frac{t(1-t)}{2}(A-B)\left(A\nabla B - Hz_t(A,B)\right)^{-1}(A-B),\ (A \neq B)$$

with $W_t(A,A) := A$.

Proof. Applying $x := A^{-1/2}BA^{-1/2}$ to the inequality:

$$W_t(x,1) \leqslant \sqrt{\frac{I(x,1)}{G(x,1)}} L(x,1)$$

given in Theorem 2.4 with $L(x,1) = \int_0^1 x^p dp$, and multiplying $A^{1/2}$ to both sides, we obtain the desired inequality. It is sufficient to prove that $\frac{I(x,1)}{G(x,1)} = \frac{x^{\frac{x}{x-1}}}{e\sqrt{x}} = \frac{1}{e}x^{\frac{x+1}{2(x-1)}}$ is monotone decreasing in $x \in (0,1)$, monotone increasing $x \in (1,\infty)$ and $\lim_{x\to 1} x^{\frac{x+1}{2(x-1)}} = e$. Since $\lim_{x\to 1}\frac{(x+1)\log x}{2(x-1)} = \lim_{x\to 1}\frac{\log x + 1 + 1/x}{2} = 1$, we have $\lim_{x\to 1} x^{\frac{x+1}{2(x-1)}} = e$ which means $\lim_{x\to 1}\frac{I(x,1)}{G(x,1)} = 1$. To show the monotonicity, we set the function $f(x) := x^{\frac{x+1}{2(x-1)}}$ for $x > 0$. Then we have

$$f'(x) = x^{\frac{x+1}{2(x-1)}}\left\{\frac{(x+1)(x-1) - 2x\log x}{2x(x-1)^2}\right\}.$$

In general, we have the inequality $\frac{x-1}{\log x} \geqslant \frac{2x}{x+1}$ for $x > 0$. Therefore we have $f'(x) \leqslant 0$ for $0 < x < 1$, and $f'(x) \geqslant 0$ for $x > 1$. Thus we have

$$R_{I/G} = \max\left\{\sqrt{\frac{I(\alpha,1)}{G(\alpha,1)}}, \sqrt{\frac{I(\beta,1)}{G(\beta,1)}}\right\}. \quad \square$$

As we stated in Remark 2.4, there is no superiority or inferiority for Corollary 2.3 and [3, Corollary 4.2].

Declarations

Availability of data and materials. Not applicable.

Competing interests. The authors declare no competing interests.

Funding. This research is supported by a grant (JSPS KAKENHI, Grant Number: 21K03341) awarded to the author, S. Furuichi.

Authors' contributions. Authors declare that they have contributed equally to this paper. All authors have read and approved this version.

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(Received April 16, 2023)

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