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SPLITTING INEQUALITIES FOR DIFFERENCES OF EXPONENTIALS

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Abstract. The paper is focused on two-sided splitting inequalities for differences of complex exponentials

$$\left|\Delta^k e^{itf(n)}\right|, \quad k \in \mathbb{N}, \quad t \in \mathbb{R},$$

for large $n \in \mathbb{N}$, where $\{f(n)\}_{n=1}^{\infty}$ is real unbounded sequence clustering with appropriate speed. Moreover, it is shown that if $\{e_n\}_{n\in\mathbb{N}}$ is a Riesz basis of a Hilbert space H, then for any $k \geqslant 1$ the system $\left\{\Delta^k e_n\right\}_{n\in\mathbb{N}}$ is complete, minimal but not uniformly minimal in H. Also some properties of systems of functions of real argument t,

$$\left\{\Delta^k e^{itf(n)}\right\}_{n\in\mathbb{N}},$$

where $k \in \mathbb{N} \cup \{0\}$, are discussed.

1. Introduction

In 2005–2010 G. Xu, S. Yung and H. Zwart [22, 24] obtained striking spectral theorem for generators of C_0-semigroups on Hilbert spaces (the XYZ theorem) and gave two conceptually different proofs of it, see also [16, 17] and discussions therein. This theorem provides us with simply formulated three sufficient conditions for an unbounded operator A with point spectrum to generate the Riesz basis of A-invariant subspaces. Riesz bases are main blocks of spectral approach in an infinite-dimensional linear and nonlinear systems theory and frequently appear in problems of mathematical physics as well as in modern signal processing, see, e.g., [1, 3, 4, 5, 8, 9, 12, 17, 22, 23, 24] and the references therein.

The first systematic study of Riesz bases was initiated by R. E. A. C. Paley and N. Wiener in 1934 [21] and the second impulse was given by N. Bari in 1951 [2]. These two works contain a substantial part devoted to such fundamental property of Riesz bases as stability, i.e. that a minor perturbation of Riesz basis turns out again to be a Riesz basis. And everything interesting is encrypted in the phrase "minor perturbation". The celebrated Paley-Wiener Theorem (the first stability theorem for Riesz bases) states that, if the sequence $\{\phi_n\}_{n\in\mathbb{Z}_+}$ is close to some orthonormal basis, then $\{\phi_n\}_{n\in\mathbb{Z}_+}$ forms a Riesz basis, see [21]. In [2] N. Bari proved the second important step, that a

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basis, quadratically close to Riesz basis, is again a Riesz basis. For more concerning stability of bases we refer, e.g., to [1, 2, 3, 4, 6, 9, 10, 11, 12, 14, 15, 23] and the references therein.

In 2015–2017 G. Sklyar and V. Marchenko while verifying the sharpness of conditions of XYZ theorem presented the construction of classes of generators of C_0-groups on special Hilbert and Banach spaces with purely imaginary eigenvalues and corresponding complete minimal families of eigenvectors, which, however, do not form a Schauder basis of the phase space [18, 17]. Essentially using this construction G. Sklyar and V. Marchenko recently proved that the XYZ theorem is sharp, for details see [16]. In [17] it was proved that C_0-groups from constructed classes grow at infinity but not faster than some polynomial, i.e. they are polynomially bounded.

For the study of asymptotic properties of constructed classes of C_0-groups, that is presented in [19], [20], one needs to determine the asymptotics of functions

$$\left|\Delta^k e^{itf(n)}\right| \tag{1}$$

for large n, where $k\in\mathbb{N}$, Δ stands for the standard backward difference operator and

$$\begin{gathered}\{f(n)\}_{n=1}^{\infty}\in\mathscr{S}_k\\ \parallel\\ \left\{\{f(n)\}_{n=1}^{\infty}\subset\mathbb{R}:\ \lim_{n\to\infty}f(n)=+\infty;\left\{n^j\Delta^j f(n)\right\}_{n=1}^{\infty}\in\ell_\infty\ \forall j:\ 1\leqslant j\leqslant k\right\}.\end{gathered}$$

Clearly $\mathscr{S}_m\subseteq\mathscr{S}_j$ provided that $j\leqslant m$. Note also that for every $k\in\mathbb{N}$ one has $\{\ln n\}_{n=1}^{\infty}\in\mathscr{S}_k$, $\{\ln\ln(n+1)\}_{n=1}^{\infty}\in\mathscr{S}_k$, $\left\{\ln\ln\sqrt{n+1}\right\}_{n=1}^{\infty}\in\mathscr{S}_k$ but $\{\sqrt{n}\}_{n=1}^{\infty}\notin\mathscr{S}_k$.

Consider the following set of sequences:

$$\mathscr{M}=\{\{f(n)\}_{n=1}^{\infty}\subset\mathbb{R}:\exists K>0:\ \forall n\in\mathbb{N}\ n|\Delta f(n)|\geqslant K\}. \tag{2}$$

Obviously $\{\ln(n+1)\}_{n=1}^{\infty}\in\mathscr{S}_k\cap\mathscr{M}$. The main result of the paper is formulated as follows.

THEOREM 1. *Let $k\in\mathbb{N}$ and $\{f(n)\}_{n=1}^{\infty}\in\mathscr{S}_k\cap\mathscr{M}$. Then for any j such that $1\leqslant j\leqslant k$ there exist*

1. *the constant $L>0$;*
2. *the polynomial $\mathscr{P}_j$ with*
$$\deg\mathscr{P}_j=j,$$
with positive coefficients and without a free term;
3. *the polynomial $\mathscr{Q}_j$ with*
$$\deg\mathscr{Q}_j=j,$$
with a positive coefficient in front of the main term and without a free term;

such that for all $t \in \mathbb{R}$ *and for every* $n \geqslant \max\{L|t|+1, j+1\}$ *we have*

$$\frac{\mathscr{Q}_j(|t|)}{n^j} \leqslant \left|\Delta^j e^{itf(n)}\right| \leqslant \frac{\mathscr{P}_j(|t|)}{n^j}. \tag{3}$$

Of course (3) means that for any j such that $1 \leqslant j \leqslant k$ there exists constants $C_j > c_j > 0$ and t_j^* such that for all $t:\ |t| \geqslant t_j^*$ and all $n \geqslant \max\{L|t|+1, j+1\}$ one has

$$c_j \frac{|t|^j}{n^j} \leqslant \left|\Delta^j e^{itf(n)}\right| \leqslant C_j \frac{|t|^j}{n^j}. \tag{4}$$

Thus polynomials $\mathscr{P}_j, \mathscr{Q}_j$ in (3) can be replaced to monomials, but in general only for bigger values of $|t|$ and larger n. It should be emphasized that for the right-hand side of (3) one needs only that

$$\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k.$$

Note that the right-hand side of (3) was used in [17] to prove that constructed there C_0-groups are polynomially bounded.

Also this paper indicates some properties of differences of complex exponentials and gives further extensions of splitting inequalities for differences of complex exponentials of the first order in the case when $\{f(n)\}_{n=1}^{\infty}$ belongs to a much wider class than $\mathscr{S}_1$ – to a class $\mathscr{F}$, see (12), Section 2.

The authors of [17] considered the right shift operator T associated with a given Riesz basis $\{e_n\}_{n=1}^{\infty}$, defined by (16), to construct special classes of Hilbert spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$, where $\{e_n\}_{n=1}^{\infty}$ become complete, minimal but not uniformly minimal system, see Section 2 of [17]. Thus $\{e_n\}_{n=1}^{\infty}$ lose a Schauder basis property in this new space $H_k(\{e_n\})$ for any k. This sequence $\{e_n\}_{n=1}^{\infty}$ in $H_k(\{e_n\})$ serves as a family of eigenvectors for generators of polynomially bounded C_0-groups, acting on $H_k(\{e_n\})$, see [17], [16]. In Section 3 it is shown that systems possessing such properties can be constructed directly in initial Hilbert space H as images of any Riesz basis $\{e_n\}_{n=1}^{\infty}$ under the operator $(I-T)^k = \Delta^k$ for any $k \in \mathbb{N}$. These results also hold for the case of symmetric bases in ℓ_p, $p > 1$, see Proposition 3, and they may be used for the further studies in the theory of C_0-semigroups. Section 4 discusses properties of the functions $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$, where $\{f(n)\}_{n=1}^{\infty} \in \mathscr{F}$, see (12), and Section 5 contains concluding remarks and questions for the further study.

2. Splitting inequalities for differences of complex exponentials

2.1. The proof of Theorem 1

Fix any $k \in \mathbb{N}$ and let m be such that $1 \leqslant m \leqslant k$. Then consider the following sets:

$$\begin{aligned}
&\Sigma_1 = \{0,1,2,3,\dots,k-1\}\\
&\Sigma_2 = \{0,1,2,\dots,k-2\}\\
&\dots\\
&\Sigma_{k-1} = \{0,1\},\\
&\Sigma_k = \{0\}.
\end{aligned}$$

Clearly $\Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \dots \supset \Sigma_k$. In the proof of main Theorem 1 we will use the following theorem on splitting inequalities, obtained first in [13], for the proof see also Theorem 3 of [19].

THEOREM 2. *Let $k \in \mathbb{N}$ and $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k$ be given. Then for any m such that $1 \leqslant m \leqslant k$ there exists a polynomial $\mathscr{P}_{m,f}$ with*

$$\deg \mathscr{P}_{m,f} = m,$$

with positive coefficients and without a free term, such that for every $s \in \Sigma_m$, $t \in \mathbb{R}$ and $n > m$ the following inequality holds:

$$\left|\Delta^m e^{(-1)^s it\Delta^s f(n)}\right| \leqslant \frac{\mathscr{P}_{m,f}(|t|)}{n^{m+s}}. \tag{5}$$

The existence of the polynomial $\mathscr{P}_j$ with

$$\deg \mathscr{P}_j = j,$$

with positive coefficients and without a free term, such that for any $n > j$ the right-hand side of (3) holds, is guaranteed by Theorem 2 or by Lemma 10 from [17], for the complete scheme of its proof see Section 2 of [19] or [13].

To prove the left-hand side of (3) one uses the induction over k.

The basis of induction. Consider $k = 1$. Since $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_1$ one has that there exists a constant $L > 0$ such that for all $n \in \mathbb{N}$

$$n\,|\Delta f(n)| \leqslant L.$$

Hence for all $t \in \mathbb{R}$ and every natural n

$$|t\Delta f(n)| \leqslant \frac{L|t|}{n}$$

and thus for all $n \geqslant L|t|$ one has

$$|t\Delta f(n)| \leqslant 1. \tag{6}$$

Note that for all $s \in [0,1]$ one has

$$\sin s \geqslant \frac{s}{\sqrt{2}}. \tag{7}$$

Since $\{f(n)\}_{n=1}^{\infty} \in \mathscr{M}$, combining (6) and (7), one obtains that for arbitrary $t \in \mathbb{R}$ and for all $n \geqslant L|t|$

$$\begin{aligned}\left|\Delta e^{itf(n)}\right|^2 &= \left|e^{itf(n-1)}\left(e^{it\Delta f(n)}-1\right)\right|^2 = \left|e^{it\Delta f(n)}-1\right|^2 \\ &\geqslant \sin^2(t\Delta f(n)) \geqslant \frac{t^2}{2}(\Delta f(n))^2 \geqslant \frac{K^2t^2}{2n^2}.\end{aligned}$$

The step. Assume that lemma holds for some $\ell-1$, where $k-1 \geqslant \ell-1 \geqslant 1$. In order to prove that then it is true also for ℓ: $2 \leqslant \ell \leqslant k$ one will apply the Leibnitz theorem for finite differences:

$$\Delta^{\ell}(u_n v_n) = \sum_{j=0}^{\ell} C_{\ell}^{j} \Delta^{\ell-j} u_{n-j} \Delta^{j} v_n, \quad \ell \in \mathbb{N}. \tag{8}$$

First note that for each $n \in \mathbb{N}$

$$\Delta^{\ell} e^{itf(n)} = \Delta^{\ell-1}\Delta e^{itf(n)} = \Delta^{\ell-1}\left(e^{itf(n-1)}\left(e^{it\Delta f(n)}-1\right)\right).$$

Applying Leibnitz theorem (8) and reverse triangle inequality to factorization above one observes that

$$\begin{aligned}\left|\Delta^{\ell} e^{itf(n)}\right| &= \left|\sum_{j=0}^{\ell-1} C_{\ell-1}^{j}\Delta^{\ell-1-j}e^{itf(n-1-j)}\Delta^{j}\left(e^{it\Delta f(n)}-1\right)\right| \\ &= \Bigg|\Delta^{\ell-1}e^{itf(n-1)}\cdot\left(e^{it\Delta f(n)}-1\right) \\ &\qquad + \sum_{j=1}^{\ell-1} C_{\ell-1}^{j}\Delta^{\ell-1-j}e^{itf(n-1-j)}\Delta^{j}\left(e^{it\Delta f(n)}-1\right)\Bigg| \\ &\geqslant \left|\Delta^{\ell-1}e^{itf(n-1)}\cdot\left(e^{it\Delta f(n)}-1\right)\right| \\ &\qquad - \left|\sum_{j=1}^{\ell-1} C_{\ell-1}^{j}\Delta^{\ell-1-j}e^{itf(n-1-j)}\Delta^{j}e^{it\Delta f(n)}\right| =: \Phi_n(t)-\Psi_n(t).\end{aligned}$$

The main idea is to minimize the first modulus $\Phi_n(t)$ and maximize the second $\Psi_n(t)$, i.e. to get appropriate estimates from below for $\Phi_n(t)$ and estimates from above for $\Psi_n(t)$.

For $\Phi_n(t)$ one uses inductive assumption and estimation for the case $k=1$ in the fist part of the proof. Thus, by inductive assumption, there exists a polynomial $\mathscr{Q}_{\ell-1}$

of degree $\ell-1$, with a positive coefficient in front of the main term and without a free term and $L>0$ such that for all $t\in\mathbb{R}$ and for every $n\geqslant \max\{L|t|+1,\ell-1\}$ one has

$$\left|\Delta^{\ell-1}e^{itf(n-1)}\right| \geqslant \frac{\mathscr{Q}_{\ell-1}(|t|)}{(n-1)^{\ell-1}} > \frac{\mathscr{Q}_{\ell-1}(|t|)}{n^{\ell-1}}. \tag{9}$$

Hence by (9) one gets for the first modulus $\Phi_n(t)$ that

$$\Phi_n(t) = \left|\Delta^{\ell-1}e^{itf(n-1)}\cdot\left(e^{it\Delta f(n)}-1\right)\right| \geqslant \frac{\mathscr{Q}_{\ell-1}(|t|)}{n^{\ell-1}}\cdot\frac{K|t|}{\sqrt{2}n} = \frac{K|t|\mathscr{Q}_{\ell-1}(|t|)}{\sqrt{2}n^{\ell}} \tag{10}$$

takes place for all $t\in\mathbb{R}$ and every $n\geqslant \max\{L|t|+1,\ell-1\}$.

To estimate the second modulus $\Psi_n(t)$ we essentially use Theorem 2. Application of this theorem yields that there exists a set of polynomials

$$\widetilde{\mathscr{P}}_{m,f}, \quad 1\leqslant m\leqslant \ell-1,$$

satisfying

$$\deg\widetilde{\mathscr{P}}_{m,f} = m,$$

with positive coefficients and without a free term such that for all $t\in\mathbb{R}$ and $n\geqslant\ell$ one has

$$\begin{aligned}
\Psi_n(t) &= \left|\sum_{j=1}^{\ell-1} C_{\ell-1}^{j}\Delta^{\ell-1-j}e^{itf(n-1-j)}\Delta^{j}e^{it\Delta f(n)}\right| \\
&\leqslant \sum_{j=1}^{\ell-1} C_{\ell-1}^{j}\left|\Delta^{\ell-1-j}e^{itf(n-1-j)}\right|\left|\Delta^{j}e^{it\Delta f(n)}\right| \\
&\leqslant \sum_{j=1}^{\ell-1} C_{\ell-1}^{j}\frac{\widetilde{\mathscr{P}}_{\ell-1-j,f}(|t|)}{(n-1-j)^{\ell-1-j}}\cdot\frac{\widetilde{\mathscr{P}}_{j,f}(|t|)}{n^{j+1}} \\
&\leqslant \sum_{j=1}^{\ell-1} C_{\ell-1}^{j}\frac{(2+j)^{\ell-1-j}\widetilde{\mathscr{P}}_{\ell-1-j,f}(|t|)}{n^{\ell-1-j}}\cdot\frac{\widetilde{\mathscr{P}}_{j,f}(|t|)}{n^{j+1}} \\
&=: \frac{1}{n^{\ell}}\mathscr{F}_{\ell-1,f}(|t|),
\end{aligned}$$

where $\mathscr{F}_{\ell-1,f}$ is a polynomial with

$$\deg\mathscr{F}_{\ell-1,f} = \ell-1,$$

with positive coefficients and without a free term.

Combining the last estimate with (10) one concludes that for any $t\in\mathbb{R}$ and every $n\geqslant\max\{L|t|+1,\ell\}$

$$\left|\Delta^{\ell}e^{itf(n)}\right| \geqslant \Phi_n(t)-\Psi_n(t) \geqslant \frac{1}{n^{\ell}}\left(\frac{K|t|\mathscr{Q}_{\ell-1}(|t|)}{\sqrt{2}} - \mathscr{F}_{\ell-1,f}(|t|)\right) =: \frac{\mathscr{Q}_{\ell}(|t|)}{n^{\ell}},$$

where $\mathscr{Q}_{\ell}$ is a polynomial of degree ℓ, with a positive coefficient in front of the main term and without a free term.

2.2. Properties of differences of exponents

Consider $\left\{e^{int}\right\}_{n=-\infty}^{\infty}$ – an orthonormal basis of space $L_2(0,2\pi)$. Computing differences of these functions one gets

$$\left|\Delta e^{int}\right|^2 = \left|e^{int}\left(1-e^{-it}\right)\right|^2 = \left|1-e^{it}\right|^2 = \sin^2 t + (1-\cos t)^2.$$

So one notes that the sequence of functions $\left|\Delta e^{itn}\right|$ does not depend on n and, moreover, $\left|\Delta e^{itn}\right| = 0$ iff $t = 2\pi k$, where $k \in \mathbb{Z}$. Compute the second difference:

$$\left|\Delta^2 e^{int}\right| = \left|e^{int}\left(1-2e^{-it}+e^{-2it}\right)\right| = \left|1-2e^{-it}+e^{-2it}\right|.$$

In general case $k \in \mathbb{N}$ one can show that

$$\left|\Delta^k e^{int}\right| = \left|\sum_{j=0}^{k}(-1)^j C_k^j e^{-jt}\right|,$$

where C_m^k stands for binomial coefficient. Thus, one arrives at the following observation.

REMARK 1. For every $k \in \mathbb{N}$ the sequence of functions

$$\left\{\left|\Delta^k e^{int}\right|\right\}_{n\in\mathbb{Z}}$$

is stationary.

Further it will be shown that the conclusion of remark above is false in the case when instead of sequence of natural numbers one takes $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k$.

COROLLARY 1. *Let* $k \in \mathbb{N}$, $-\infty < a < b < \infty$ *and* $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k$. *Consider any* j *such that* $1 \leqslant j \leqslant k$ *and the sequence* $\left\{\alpha_j(n)\right\}_{n=1}^{\infty} \subset \mathbb{C}$ *such that*

$$\lim_{n\to\infty} \frac{\left|\alpha_j(n)\right|}{n^j} = 0.$$

Then for the sequence of functions $\left\{\alpha_j(n)\Delta^j e^{itf(n)}\right\}_{n=1}^{\infty}$ *the following holds:*

1. *For any* $t \in \mathbb{R}$ *one has*
$$\lim_{n\to\infty}\left|\alpha_j(n)\Delta^j e^{itf(n)}\right| = 0$$
(pointwise convergence to 0*).*

2. $\lim\limits_{n\to\infty}\left\|\alpha_j(n)\Delta^j e^{itf(n)}\right\|_{C[a,b]} = 0$ *(uniform convergence to* 0 *on any* $[a,b]$*).*

3. *For all* $p \geqslant 1$ *one has* $\lim\limits_{n\to\infty}\left\|\alpha_j(n)\Delta^j e^{itf(n)}\right\|_{L_p[a,b]} = 0.$

4. Let $p \geqslant 1$. For any $t \in \mathbb{R}$ one has

$$\left\{\beta_j(n)\Delta^j e^{itf(n)}\right\}_{n=1}^{\infty} \in \ell_p$$

provided that $\left\{\frac{\beta_j(n)}{n^j}\right\}_{n=1}^{\infty} \in \ell_p$.

Proof. To prove the first part it is enough to recall that condition $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k$ is sufficient for the right-hand side of (3), as it was noted after Theorem 1, for details see Section 2 of [19].

To prove the second part consider j such that $1 \leqslant j \leqslant k$. Then, by Theorem 1 or by Lemma 4 of [19], there exists the polynomial $\mathscr{P}_j$ with

$$\deg \mathscr{P}_j = j,$$

with positive coefficients and without a free term, such that for $n \geqslant j+1$ one has

$$\left|\Delta^j e^{itf(n)}\right| \leqslant \frac{\mathscr{P}_j(|t|)}{n^j}. \tag{11}$$

Hence for $n \geqslant j+1$ one obtains that

$$\left\|\alpha_j(n)\Delta^j e^{itf(n)}\right\|_{C[a,b]} = \max_{t\in[a,b]}\left|\alpha_j(n)\Delta^j e^{itf(n)}\right| \leqslant \frac{|\alpha_j(n)|}{n^j}\max_{t\in[a,b]}\mathscr{P}_j(|t|) \to 0$$

as $n \to \infty$.

The third part follows trivially from the second.

To prove the fourth note that $\forall t \in \mathbb{R}$

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\beta_j(n)\Delta^j e^{itf(n)}\right|^p &= \sum_{n=1}^{j+1}\left|\beta_j(n)\Delta^j e^{itf(n)}\right|^p + \sum_{n=j+1}^{\infty}\left|\beta_j(n)\Delta^j e^{itf(n)}\right|^p \\ &\leqslant \sum_{n=1}^{j+1}\left|\beta_j(n)\Delta^j e^{itf(n)}\right|^p + \left|\mathscr{P}_j(|t|)\right|^p\sum_{n=j+1}^{\infty}\left|\frac{\beta_j(n)}{n^j}\right|^p < \infty,\end{aligned}$$

since $\left\{\frac{\beta_j(n)}{n^j}\right\}_{n=1}^{\infty} \in \ell_p$, where $\mathscr{P}_j$ is the polynomial from (11). $\square$

Combining Corollary 1 with Theorem 1 one arrives at the following.

COROLLARY 2. *Let $k \in \mathbb{N}$ and $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k$. Consider any j such that $1 \leqslant j \leqslant k$. Then for any $t \in \mathbb{R}$*

$$\left\{\Delta^j e^{itf(n)}\right\}_{n=1}^{\infty} \begin{array}{l} \in \ell_p, \ \textit{provided that } jp > 1, \\ \notin \ell_p, \ \textit{provided that } jp \leqslant 1. \end{array}$$

2.3. Further extensions of splitting inequalities, obtained in Theorem 1

In this subsection one examines what kind of splitting inequalities could be obtained if one weakens the condition $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_1$ and considers differences of the first order. Consider instead the following more general condition:

$$\mathscr{F} = \left\{ \{f(n)\}_{n=1}^{\infty} \subset \mathbb{R} : \lim_{n\to\infty} f(n) = +\infty,\ \lim_{n\to\infty} |\Delta f(n)| = 0 \right\}. \tag{12}$$

Note that for every $k \in \mathbb{N}$ one obviously has that

$$\mathscr{S}_k \subset \mathscr{S}_{k-1} \subset \ldots \subset \mathscr{S}_2 \subset \mathscr{S}_1 \subset \mathscr{F}.$$

E.g., clearly $\{\sqrt{n}\}_{n=1}^{\infty} \in \mathscr{F} \setminus \mathscr{S}_1$.

PROPOSITION 1. *Let $\{f(n)\}_{n=1}^{\infty} \in \mathscr{F}$. Then the following holds.*

1. *If g, h be any real functions such that for all $s \in \mathbb{R}$*

$$\sin^2 s \leqslant g(s) \qquad (1-\cos s)^2 \leqslant h(s),$$

then for all $t \in \mathbb{R}$

$$\left|\Delta e^{itf(n)}\right|^2 \leqslant g\left(t\Delta f(n)\right) + h\left(t\Delta f(n)\right).$$

2. *If v is any real nonnegative function such that for all $s \in [0,1]$ one has*

$$\sin s \geqslant v(s), \tag{13}$$

then for all $t \in \mathbb{R}$ and for any n such that $|t\Delta f(n)| \leqslant 1$ one has

$$\left|\Delta e^{itf(n)}\right| \geqslant v\left(t\Delta f(n)\right).$$

Proof. Using properties of functions g, h it is easy to see that for all $t \in \mathbb{R}$

$$\begin{aligned}\left|\Delta e^{itf(n)}\right|^2 &= \left|e^{itf(n-1)}\left(e^{it\Delta f(n)} - 1\right)\right|^2 = \left|e^{it\Delta f(n)} - 1\right|^2 \\ &= \sin^2\left(t\Delta f(n)\right) + \left(1-\cos\left(t\Delta f(n)\right)\right)^2 \leqslant g\left(t\Delta f(n)\right) + h\left(t\Delta f(n)\right),\end{aligned}$$

so the first part is proved.

To prove the second part one notes that, since $\{f(n)\}_{n=1}^{\infty} \in \mathscr{F}$, there exists the sequence $\{\gamma(n)\}_{n=1}^{\infty} \in \mathbb{R}$ such that $\gamma(n) \to \infty$ as $n \to \infty$ and the constant $L > 0$ such that for all $n \in \mathbb{N}$

$$\gamma(n)\,|\Delta f(n)| \leqslant L.$$

Hence for all $t \in \mathbb{R}$ and every $n \in \mathbb{N}$

$$|t\Delta f(n)| \leqslant \frac{L|t|}{\gamma(n)}$$

and thus for all n such that $\frac{L|t|}{\gamma(n)} \leqslant 1$ one has

$$|t\Delta f(n)| \leqslant 1. \tag{14}$$

Combining (14) and (13) one obtains that for arbitrary $t \in \mathbb{R}$ and for all n such that $\frac{L|t|}{\gamma(n)} \leqslant 1$

$$\begin{aligned} \left|\Delta e^{itf(n)}\right|^2 &= \left|e^{itf(n-1)}\left(e^{it\Delta f(n)}-1\right)\right|^2 = \left|e^{it\Delta f(n)}-1\right|^2 \\ &\geqslant \sin^2(t\Delta f(n)) \geqslant (v(t\Delta f(n)))^2. \quad \square \end{aligned}$$

Proposition 1 implies the following.

REMARK 2. Note that under the condition $\{f(n)\}_{n=1}^{\infty} \in \mathscr{F}$ the sequence of functions $\left\{\Delta e^{itf(n)}\right\}_{n=1}^{\infty}$ is pointwise convergent to 0, i.e. for all $t \in \mathbb{R}$

$$\lim_{n\to\infty}\left|\Delta e^{itf(n)}\right| = 0.$$

To see this it is enough to choose in Proposition 1

$$g(s) = s^2, \qquad h(s) = 0.55s^2.$$

Using inequalities involving linear functions one deduces from Proposition 1 the following.

COROLLARY 3. *Let $\{f(n)\}_{n=1}^{\infty} \in \mathscr{F}$. Then for all $t \in \mathbb{R}$*

$$\left|\Delta e^{itf(n)}\right| \leqslant 1.5\sqrt{|t\Delta f(n)|}. \tag{15}$$

Moreover, for all $t \in \mathbb{R}$ and for any n such that $|t\Delta f(n)| \leqslant 1$ one has

$$\left|\Delta e^{itf(n)}\right| \geqslant 0.8\,|t\Delta f(n)|.$$

Proof. To prove (15) one can take

$$g(s) = 0.75|s|, \qquad h(s) = 1.5|s|$$

and apply Proposition 1.

Clearly for all $s \in [0,1]$ one has

$$\sin s \geqslant 0.8|s|.$$

Therefore taking $v(s) = 0.8|s|$ and applying Proposition 1 one observes that

$$\left|\Delta e^{itf(n)}\right| \geqslant 0.8\,|t\Delta f(n)|$$

for all $t \in \mathbb{R}$ and for any n such that $|t\Delta f(n)| \leqslant 1$. $\square$

3. Sequences of the form $\{(I-T)^k e_n\}_{n\in\mathbb{N}}$, $k\in\mathbb{N}$

Let H be a separable Hilbert space with norm $\|\cdot\|$, scalar product $\langle\cdot,\cdot\rangle$ and a Riesz basis $\{e_n\}_{n\in\mathbb{N}}$. E.g. it might be a Riesz basis of complex exponentials in $H = L_2(-a,a)$, $a>0$. Consider $T: H\to H$ – the right shift operator associated with Riesz basis $\{e_n\}_{n\in\mathbb{N}}$, i.e.

$$Te_n = e_{n+1}, \quad n\in\mathbb{N}. \tag{16}$$

Then consider the sequence of the form

$$\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}}, \tag{17}$$

where $k\in\mathbb{N}$. It is easy to see that the sequence (17) does not form a Riesz basis of H, since $0\in\sigma\left((I-T)^k\right)$ for any k, which means that $(I-T)^k$ is not an isomorphism of H. Recall that the sequence $\{e_n\}_{n\in\mathbb{N}}\subset H$ is called minimal in space H with distance ρ provided that for any n

$$\rho\left(e_n, \overline{Lin}\{e_j\}_{j\neq n}\right) > 0,$$

i.e. $e_n\notin\overline{Lin}\{e_j\}_{j\neq n}$, and uniformly minimal if, additionally,

$$\inf_n \rho\left(e_n, \overline{Lin}\{e_j\}_{j\neq n}\right) > 0.$$

It is well known that $\{e_n\}_{n\in\mathbb{N}}\subset H$ is minimal if and only if $\{e_n\}_{n\in\mathbb{N}}$ has biorthogonal sequence [2]. Next proposition describes some other properties of the sequence (17) in H.

PROPOSITION 2. *Let $k\in\mathbb{N}$. Then the following is true.*

1. $\overline{Lin}\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}} = H$;
2. $\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}}$ *has a unique biorthogonal sequence*

$$\left\{\psi_n = (I-T^*)^{-k} e_n^*\right\}_{n\in\mathbb{N}}, \tag{18}$$

   in H, where $\langle e_n, e_m^*\rangle = \delta_n^m$;
3. $\{\psi_n\}_{n\in\mathbb{N}}$ *is uniformly minimal in H while the sequence (17) is minimal but not uniformly minimal in H;*
4. $\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}}$ *does not form a Schauder basis of H;*
5. $\{\psi_n\}_{n\in\mathbb{N}}$ *does not form a Schauder basis of H.*

Proof. To prove the first statement note that only zero is orthogonal to all elements of the sequence (17).

Let $\{e_n^*\}_{n=1}^{\infty}\subset H$ be a sequence, biorthogonal to $\{e_n\}_{n=1}^{\infty}$ in H, i.e. $\langle e_n, e_m^*\rangle = \delta_n^m$. Then one can define the operator T^* on H and one can show that $e_n^*\in D\left((I-T^*)^{-k}\right)$

for any k. E.g. in the space ℓ_2 and for the case of canonical orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of ℓ_2 this fact follows immediately from representations of operators $(I-T)^{-k}$ and $(I-T^*)^{-k}$ in the form of infinite matrices, i.e.

$$(I-T)^{-k} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ C_k^1 & 1 & 0 & 0 & \dots \\ C_{k+1}^2 & C_k^1 & 1 & 0 & \dots \\ C_{k+2}^3 & C_{k+1}^2 & C_k^1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (I-T^*)^{-k} = \left((I-T)^{-k}\right)^T.$$

Thus

$$(I-T^*)^{-k} = \left((I-T)^{-k}\right)^T$$

is an unbounded operator. Nevertheless $e_n^* \in D\left((I-T^*)^{-k}\right)$ for any k and the formula (18) makes sense. It follows that the operator $(I-T^*)^{-k}$ has dense domain. The second statement follows from the fact that

$$\left\langle (I-T)^k e_n, \psi_m \right\rangle = \left\langle (I-T)^k e_n, (I-T^*)^{-k} e_m^* \right\rangle = \langle e_n, e_m^* \rangle = \delta_n^m$$

and the uniqueness of $\{\psi_n\}_{n\in\mathbb{N}}$ is guaranteed by the first statement.

The third statement is true since

$$\sup_n \left\| (I-T)^k e_n \right\| < \infty$$

while

$$\sup_n \|\psi_n\| = \infty.$$

Finally, the third statement clearly implies the fourth and the fourth yields the last. $\square$

REMARK 3. Since for all natural n,k

$$(I-T)^k e_n = \Delta^k e_n,$$

the sequence $\{\Delta^k e_n\}_{n\in\mathbb{N}}$ has the same properties in H as the sequence (17) in Proposition 2, i.e. the sequence $\{\Delta^k e_n\}_{n\in\mathbb{N}}$ is complete in H, minimal but not uniformly minimal in H. Note that some partial results of such kind for the case when $\{e_n\}_{n\in\mathbb{N}}$ is orthonormal basis in H were obtained by N. Bari [2] .

Applying the similar arguments as in the proof of Proposition 2 one obtains the following result on the properties of sequences of the form (17) in spaces ℓ_p, $p>1$.

PROPOSITION 3. *Let $\{e_n\}_{n\in\mathbb{N}}$ be a symmetric basis of ℓ_p, $p>1$, $T:\ell_p \mapsto \ell_p$ be the right shift operator, defined by* (16)*, associated with symmetric basis $\{e_n\}_{n\in\mathbb{N}}$, and $k\in\mathbb{N}$. Then the following is true.*

1. *$\overline{Lin}\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}} = \ell_p$;*

2. *$\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}}$ has a unique biorthogonal sequence*

$$\left\{\psi_n^* = (I-T^*)^{-k} e_n^*\right\}_{n\in\mathbb{N}},$$

   in ℓ_q, where $\{e_n^*\}_{n\in\mathbb{N}}$ is biorthogonal to $\{e_n\}_{n\in\mathbb{N}}$ basis of ℓ_q, $p^{-1}+q^{-1}=1$;

3. *$\{\psi_n^*\}_{n\in\mathbb{N}}$ is uniformly minimal in ℓ_q while the sequence (17) is minimal but not uniformly minimal in ℓ_p;*

4. *$\left\{(I-T)^k e_n\right\}_{n\in\mathbb{N}}$ does not form a Schauder basis of ℓ_p;*

5. *$\{\psi_n^*\}_{n\in\mathbb{N}}$ does not form a Schauder basis of ℓ_q.*

The references on equivalent definitions, properties and prehistory of symmetric bases can be found, e.g., in [10], [11].

4. Properties of functions $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$, where $\{f(n)\}_{n=1}^{\infty}\in\mathscr{F}$

This section is aimed at the discussion of the property of completeness of systems of functions $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ in spaces $C[-a,a]$ and $L_p(-a,a)$, $a>0$. Recall that the set of elements of Banach space is called linked if each vector of the set belongs to the closed linear span of the others. Obviously, linked system cannot be minimal. E.g., it can be shown that the sequence $\{t^n\}_{n\in\mathbb{N}}$ is linked in $L_2(0,1)$. Consider $\{f(n)\}_{n=1}^{\infty}\in\mathscr{F}$, where the set $\mathscr{F}$ is defined by (12), namely

$$\mathscr{F}=\left\{\{f(n)\}_{n=1}^{\infty}\subset\mathbb{R}:\lim_{n\to\infty}f(n)=+\infty,\ \lim_{n\to\infty}|\Delta f(n)|=0\right\}.$$

PROPOSITION 4. *Let $\{f(n)\}_{n=1}^{\infty}\in\mathscr{F}$ and $a>0$. Then*

1. *The system $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ is complete in $C[-a,a]$, i.e. the completeness radius of $\{f(n)\}_{n=1}^{\infty}$ equals to ∞.*

2. *The system $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ is complete in $L_p(-a,a)$, $1\leqslant p<\infty$.*

3. *The completeness property of the system $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ in $L_p(-a,a)$, $1\leqslant p<\infty$, or in space $C[-a,a]$, is unaffected if some $f(n)$ is replaced by another number.*

4. *The system $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ is not minimal in $C[-a,a]$.*

5. *The system $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ is linked in $C[-a,a]$.*

6. *The system* $\left\{e^{it(f(n)+i\mu_n)}\right\}_{n\in\mathbb{N}}$, *where* $\mu_n \in \mathbb{R}$ *is chosen such that*

$$\sup_n |\mu_n| < \infty,$$

is complete in $L_2(-\pi,\pi)$.

7. *The system* $\{e^{it\lambda_n}\}_{n\in\mathbb{N}}$, *where* $\lambda_n \in \mathbb{R}$ *is chosen such that*

$$\sum_{n\in\mathbb{N}} |f(n)-\lambda_n| < \infty,$$

is complete in $L_p(-a,a)$, $1 \leqslant p < \infty$.

Proof. The first statement clearly follows from [23], Theorem 2, p. 97, and the second obviously follows from the first.

The third statement follows from [23], Theorem 7, p. 108.

The fourth statement follows from the first and [23], Theorem 9, p. 109, and the fifth follows from the previous and [23], Theorem 10, p. 109.

The 6^{th} statement follows from the second and [23], Theorem 12, p. 112.

The 7^{th} statement follows from the second and [23], Theorem 11, p. 111. □

5. Concluding remarks

1) The right-hand side of (3) under the condition $\{f(n)\}_{n=1}^{\infty} \in \mathscr{S}_k$ was first obtained and proved completely in [13].

2) In the context of Section 2 the following natural question of generalization of splitting inequalities from Theorem 1 arises. Is it possible to apply under the more general condition

$$\{f(n)\}_{n=1}^{\infty} \in \mathscr{F}$$

the scheme of the proof of the right-hand side of (3) from [13], presented in Section 2 of [19]? Proposition 1 or Corollary 3 could serve as the base case of induction in this scheme. If so, one can hope to use splitting inequalities from the above in order to prove splitting inequalities from the below, similarly to that was done in the proof of Theorem 1.

3) It is shown in Section 3 that if $\{e_n\}_{n\in\mathbb{N}}$ is a Riesz basis of a Hilbert space H, then for any $k \geqslant 1$ the system

$$\Delta^k e_n, \quad n \in \mathbb{N},$$

is complete, minimal but not uniformly minimal in H. How to construct a generator of unbounded C_0-group in H with eigenvectors $\{\Delta^k e_n\}_{n\in\mathbb{N}}$?

4) By Proposition 4 the system of funtions $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$ is complete in $C[-a,a]$ and $L_p(-a,a)$, $1 \leqslant p < \infty$, for any $a > 0$, but linked. Is it possible to construct the generator of unbounded C_0-group in these spaces with eigenvectors $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$?

5) Since by Remark 3 for any k the sequence $\left\{(-1)^k\Delta^k e_{n+k}\right\}_{n\in\mathbb{N}}$ is complete in H, minimal but not uniformly minimal (and hence, does not form a Schauder basis) and by Proposition 4 the sequence $\left\{e^{itf(n)}\right\}_{n\in\mathbb{N}}$, where $\{f(n)\}_{n=1}^{\infty}\in\mathscr{F}$, also is complete in any $L_p(-a,a)$, $1\leqslant p<\infty$, $a>0$, but linked (hence, not even minimal), it is interesting to study these properties for the differences of exponentials

$$\left\{\Delta^k e^{itf(n)}\right\}_{n\in\mathbb{N}} \tag{19}$$

in Banach function spaces. Since the Müntz-Szász theorem is in fact true as well for complex exponentials, see [23], Theorem 10, p. 109 and Theorem 15, p. 117, it is expected that properties for the differences of exponentials (19) are similar to the corresponding properties of differences of polynomials

$$\left\{\Delta^k t^n\right\}_{n\in\mathbb{N}}.$$

6) In 1971 V. Katsnel'son [7] proved quite general sufficient condition in terms of zeros of a sine-type entire function for the sequence of complex exponentials to form a Riesz basis of Hilbert space $L_2(-a,a)$, $a>0$. Other sufficient conditions for Riesz basis property of complex exponentials can be found, e.g., in [23], [8]. However, the question on the existence of conditional (or non-Riesz) basis of complex exponentials in $L_2(-\pi,\pi)$ is still open, see [23], p. 165, and [14], [15] for the discussion of this sophisticated question.

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REFERENCES

[1] R. BALAN, *Stability theorems for Fourier frames and wevelet Riesz bases*, J. Fourier Anal. Appl. **3** (5) (1997), 499–504.

[2] N. K. BARI, *Biorthogonal systems and bases in Hilbert space*, Moskov. Gos. Univ. Učenye Zapiski. Matematika **148** (1951), 69–107.

[3] P. G. CASAZZA, O. CHRISTENSEN, *Perturbation of operators and applications to frame theory*, J. Fourier Anal. Appl. **3** (1997), 543–557.

[4] O. CHRISTENSEN, *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhäuser, Basel, 2016.

[5] R. F. CURTAIN, H. J. ZWART, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Texts in Applied Mathematics, vol. 21, Springer-Verlag, New York, 1995.

[6] T. KATO, *Similarity for sequences of projections*, Bull. Amer. Math. Soc. **73** (1967), 904–905.

[7] V. E. KATSNEL'SON, *Exponential bases in L^2*, Functional Analysis and Its Applications **5** (1971), 31–38.

[8] N. LEV, *Riesz bases of exponentials on multiband spectra*, Proc. Amer. Math. Soc. **140** (2012), 3127–3132.

[9] N. LEVINSON, *Gap and Density Theorems*, Colloquium Publications, vol. 26, Amer. Math. Soc., New York, 1940.

[10] V. MARCHENKO, *Isomorphic Schauder decompositions in certain Banach spaces*, Open Mathematics **12** (2014), 1714–1732.

[11] V. MARCHENKO, *Stability of unconditional Schauder decompositions in ℓ_p spaces*, Bull. Aust. Math. Soc. **92** (2015), 444–456.

[12] V. MARCHENKO, *Stability of Riesz bases*, Proc. Amer. Math. Soc. **146** (8) (2018), 3345–3351.
[13] V. MARCHENKO, *On spectral basis properties of operators of evolution equations*, PhD Thesis for the degree of candidate in physical and mathematical sciences, Kharkiv, (2016).
[14] A. NAKAMURA, *Basis properties and complements of complex exponential systems*, Hokkaido Math. J. **36** (2007), 193–206.
[15] A. NAKAMURA, *On the stability of conditional bases in* $L^2[-\pi,\pi]$, Tokyo J. Math. **32** (1) (2009), 237–245.
[16] G. SKLYAR, V. MARCHENKO, *Resolvent of the generator of the* C_0*-group with non-basis family of eigenvectors and sharpness of the XYZ theorem*, J. Spectr. Theory **11** (2021), 369–386.
[17] G. SKLYAR, V. MARCHENKO, *Hardy inequality and the construction of infinitesimal operators with non-basis family of eigenvectors*, J. Funct. Anal. **272** (3) (2017), 1017–1043.
[18] G. SKLYAR, V. MARCHENKO, *Hardy inequality and the construction of the generator of a* C_0*-group with eigenvectors not forming a basis*, Dopov. Nac. akad. nauk Ukr. **9** (2015), 13–17.
[19] G. SKLYAR, V. MARCHENKO, P. POLAK, *Sharp polynomial bounds for certain* C_0*-groups generated by operators with non-basis family of eigenvectors*, J. Funct. Anal. **280** (7) (2021), 108864.
[20] G. SKLYAR, V. MARCHENKO, P. POLAK, *One class of linearly growing* C_0*-groups*, J. Math. Phys. Anal. Geom. **17** (4) (2021), 509–520.
[21] R. E. A. C. PALEY, N. WIENER, *Fourier Transforms in the Complex Domain*, Colloquium Publications, vol. 19, Amer. Math. Soc., New York, 1934.
[22] G. Q. XU, S. P. YUNG, *The expansion of a semigroup and a Riesz basis criterion*, J. Differential Equations **210** (2005), 1–24.
[23] R. M. YOUNG, *An Introduction to Nonharmonic Fourier Series*, revised first edition, Academic Press, San Diego, 2001.
[24] H. ZWART, *Riesz basis for strongly continuous groups*, J. Differential Equations **249** (2010), 2397–2408.

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