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FRACTIONAL INTEGRAL OPERATORS ON GRAND MORREY SPACES AND GRAND HARDY–MORREY SPACES

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Abstract. This paper establishes the mapping properties of the fractional integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces defined on the Euclidean spaces. We obtain our results by refining the Rubio de Francia extrapolation method as the existing extrapolation method cannot be directly applied to the grand Morrey spaces. This method also yields the mapping properties of nonlinear operators. In particular, we establish the Sobolev embedding, the Poincaré inequality and the mapping properties of the fractional geometric maximal functions on the grand Morrey spaces.

1. Introduction

This paper aims to study the fractional integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces on the Euclidean space $\mathbb{R}^n$.

The grand Morrey spaces and the grand Hardy-Morrey spaces are the extensions of the grand Lebesgue spaces, the Morrey spaces and the Hardy-Morrey spaces. The grand Lebesgue spaces were introduced by Iwaniec and Sbordone in [31] to study the integrability of the Jacobian. The grand Lebesgue spaces become one of the major function spaces in harmonic analysis and theory of function spaces. A number of important properties and applications for the grand Lebesgue spaces were given in [2, 5, 6, 16, 18, 19, 21, 22, 32].

The grand Lebesgue spaces were generalized to the grand Morrey spaces defined on finite measure spaces in [33, 37, 38, 48]. It has been recently extended to the grand Morrey spaces defined on the Euclidean space $\mathbb{R}^n$ in [30]. As the grand Morrey spaces in [30] are defined on $\mathbb{R}^n$, we can also introduce and study the grand Hardy-Morrey spaces in [30]. Especially, we obtain the boundedness of Calderón-Zygmund singular integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces. We also establish the boundedness of the linear and the nonlinear commutators of Calderón-Zygmund singular integral operators on the grand Morrey spaces in [30]. The mapping properties of the parametric Marcinkiewicz integrals on the grand Hardy-Morrey spaces were also obtained in [30]. We establish the above results by generalizing and refining the Rubio de Francia extrapolation method [43, 44, 45] to the grand Morrey spaces.

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The above results motivate us to investigate the mapping properties of the fractional integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces on $\mathbb{R}^n$. The fractional integral operators and its related operators are important operators in harmonic analysis and partial differential equations [1, 8, 9, 36, 52, 50]. In particular, the mapping properties of the fractional integral operators on Morrey type spaces and Hardy spaces were reported and established in [1, 4, 15, 29, 34, 41, 42, 47].

In this paper, we study the mapping properties of the fractional integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces by using the extrapolation method. Notice that the existing extrapolation method cannot be directly applied to obtain our desired result. The main obstacle is, roughly speaking, on the characterization of the p-power of the small block space where the small block space is a pre-dual of the grand Morrey space. To overcome this difficulty, we have to estimate the p-power of the blocks instead of the p-power of the small block space. To use the estimate for the p-power of the blocks, we need to use the p-power of the Hardy-Littlewood maximal operators instead of the Rubio de Francia operator for the extrapolation method of the grand Morrey spaces.

By using the above refinements and modifications of the Rubio de Francia method, we can establish the mapping properties of the fractional integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces. Furthermore, the mapping properties of the fractional integral operators yield the Sobolev embedding and the Poincaré inequality on the grand Morrey spaces. As the extrapolation method can also be applied to nonlinear operators, we also obtain the mapping properties of the fractional geometric maximal functions on the grand Morrey space defined on $\mathbb{R}$.

This paper is organized as follows. Section 2 recalls the definitions and some preliminary results for the grand Lebesgue spaces. It also contains the definitions and the duality results of the grand Morrey space and the small block space. The extrapolation method for the grand Morrey spaces are presented in Section 3. The mapping properties of the fractional integral operators, the Sobolev embedding, the Poincaré inequality and the mapping properties of the fractional geometric maximal functions on the grand Morrey spaces are established in Section 4. The mapping properties of the fractional integral operators on the grand Hardy-Morrey spaces are also presented in Section 4.

2. Preliminaries and definitions

Let $\mathscr{M}(\mathbb{R}^n)$ and L^1_{loc} be the class of Lebesgue measurable functions and the class of locally integrable functions on $\mathbb{R}^n$, respectively. For any Lebesgue measurable set F, the Lebesgue measure of F is denoted by $|F|$. For any $x \in \mathbb{R}^n$ and $r > 0$, define $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. Define $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$. For any $B = B(x,r) \in \mathbb{B}$ and $s \in (0,\infty)$, write $sB = B(x,sr)$. We denote the center of B by c_B.

We recall the definition and review some properties of the grand Lebesgue spaces in this section. For any $f \in L^1_{loc}$ and $B \in \mathbb{B}$, write

$$f_B = \fint_B f(x)dx = \frac{1}{|B|}\int_B f(x)dx.$$

For any $p \in [0,\infty)$ and $B \in \mathbb{B}$, $L^p(B)$ consists of all Lebesgue measurable function f satisfying

$$\|f\|_{L^p(B)} = \left(\fint_B |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

For any $p \in [1,\infty]$, let p' be the conjugate of p. That is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $B \in \mathbb{B}$, $f \in \mathscr{M}(\mathbb{R}^n)$ and $s > 0$. Define $d_{f,B}(s) = \frac{1}{|B|}|\{x \in B : |f(x)| > s\}$ and $f_B^*(t) = \inf\{s > 0 : d_{f,B}(s) \leqslant t\}$, $t > 0$.

DEFINITION 2.1. Let $p \in (0,\infty)$ and $B \in \mathbb{B}$. The grand Lebesgue spaces $L^{p)}(B)$ consists of all $f \in \mathscr{M}(\mathbb{R}^n)$ satisfying

$$\|f\|_{L^{p)}(B)} = \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_t^1 (f_B^*(s))^p ds \right)^{\frac{1}{p}} < \infty$$

The small Lebesgue space $L^{(p}(B)$ consists of all $f \in \mathscr{M}(\mathbb{R}^n)$ satisfying

$$\|f\|_{L^{(p}(B)} = \int_0^1 (1-\ln t)^{-\frac{1}{p}} \left(\int_0^t (f_B^*(s))^p ds \right)^{\frac{1}{p}} \frac{dt}{t} < \infty.$$

According to [18, Theorem 2.3], whenever $p \in (1,\infty)$, $L^{p)}(B)$ and $L^{(p}(B)$ are rearrangement-invariant Banach function spaces. When $p \in (0,1)$, the grand Lebesgue spaces and the small Lebesgue spaces are rearrangement-invariant quasi-Banach function spaces. The reader is referred to [25, Definition 2.1] for the definition of rearrangement-invariant quasi-Banach function spaces.

When $p \in (1,\infty)$, the grand Lebesgue spaces and the small Lebesgue spaces are initially defined in terms of the following norms

$$\|f\|^*_{L^{p)}(B)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \fint_B |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

$$\|g\|^*_{L^{(p}(B)} = \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_B |g(x)|^{(p-\varepsilon)'} dx \right)^{\frac{1}{(p-\varepsilon)'}}.$$

In view of [20, Corollary 3.3 (23)] and [20, Theorem 4.2 (30)], whenever $p \in (1,\infty)$, $\|\cdot\|^*_{L^{p)}(B)}$ and $\|\cdot\|^*_{L^{(p}(B)}$ are equivalent norms of $\|\cdot\|_{L^{p)}(B)}$ and $\|\cdot\|_{L^{(p}(B)}$, respectively.

Since $\|\cdot\|_{L^{p)}(B)}$ and $\|\cdot\|^*_{L^{p)}(B)}$ are mutually equivalent, there is a constant $C > 0$ such that for any Lebesgue measurable set F

$$\|\chi_F\|_{L^{p)}(B)} \leqslant C\|\chi_F\|^*_{L^{p)}(B)} \leqslant C\|\chi_F\|_{L^p(B)} = C\left(\frac{|F \cap B|}{|B|}\right)^{\frac{1}{p}}. \tag{2.1}$$

Additionally, the embedding $L^{p-\varepsilon}(B) \hookrightarrow L^1(B)$, $\varepsilon \in (0, p-1)$ guarantees that there are constants $C_0, C_1 > 0$ independent of B such that for any $f \in L^{p)}(B)$,

$$\|f\|_{L^1(B)} \leqslant C_0\|f\|^*_{L^{p)}(B)} \leqslant C_1\|f\|_{L^{p)}(B)}. \tag{2.2}$$

We use the quasi-norms $\|\cdot\|_{L^{p)}(B)}$ and $\|\cdot\|_{L^{(p}(B)}$ instead of $\|\cdot\|^*_{L^{p)}(B)}$ and $\|\cdot\|^*_{L^{(p}(B)}$ to define the grand Lebesgue spaces and the small Lebesgue spaces because $\|\cdot\|_{L^{p)}(B)}$ and $\|\cdot\|_{L^{(p}(B)}$ are well defined for all $p\in(0,\infty)$.

Let $p,q\in(0,\infty)$. We have

$$\||f|^q\|^{\frac{1}{q}}_{L^{p)}(B)} = \sup_{0<t<1}(1-\ln t)^{-\frac{1}{pq}}\left(\int_t^1 (f^*_B(s))^{pq}ds\right)^{\frac{1}{pq}} = \|f\|_{L^{pq)}(B)}.$$

Consequently, the q-convexification of $L^{p)}(B)$ is $L^{pq)}(B)$.

The results in [16] and [18, Section 3] assert that the associate space of $L^{p)}(B)$ is $L^{(p'}(B)$ and vice versa. Particularly, we have the Hölder inequality [18, Theorem 2.5]

$$⨍_B |f(x)g(x)|dx \leqslant C\|f\|_{L^{p)}(B)}\|g\|_{L^{(p'}(B)} \tag{2.3}$$

and the norm conjugate formula [18, Corollary 2.10]

$$C_0\|f\|_{L^{p)}(B)} \leqslant \sup\left\{⨍_B |f(x)g(x)|dx : \|g\|_{L^{(p'}(B)}\leqslant 1\right\} \leqslant C_1\|f\|_{L^{p)}(B)} \tag{2.4}$$

for some $C_0,C_1>0$.

For any $f\in L^1_{loc}$, the Hardy-Littlewood maximal operator M is defined as

$$\mathrm{M}f(x) = \sup_{B\ni x}⨍_B |f(y)|dy, \quad x\in\mathbb{R}^n$$

where the supremum is taken over all ball B containing x. It is well known that for any $p\in(1,\infty)$, M is bounded on $L^p(\mathbb{R}^n)$, see [50, Chapter 1, Theorem 1].

We now state the definition of the grand Morrey space [30, Definition 3.1].

DEFINITION 2.2. Let $p\in(0,\infty)$ and $u:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. The grand Morrey space $M^{p)}_u(\mathbb{R}^n)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M^{p)}_u(\mathbb{R}^n)} = \sup_{B(x,r)\in\mathbb{B}}\frac{1}{u(x,r)}\|f\|_{L^{p)}(B(x,r))} < \infty.$$

For any $B=B(x,r)\in\mathbb{B}$, we also write $u(B)=u(x,r)$.

The grand Morrey space is an extension of the grand Lebesgue spaces and the classical Morrey spaces. The classical Morrey spaces were introduced by Morrey in [40] for the study of quasi-linear elliptic partial differential equations. Since then, a huge number of extensions of Morrey spaces has been given, see [1, 33, 37, 38, 41, 47, 48, 49].

The following proposition assures that the characteristic functions of balls belong to the grand Morrey space.

PROPOSITION 2.1. *Let* $p \in (0,\infty)$ *and* $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *be a Lebesgue measurable function. If there exists a constant* $C > 0$ *such that for any* $x \in \mathbb{R}^n$,

$$Cr^{-n/p} < u(x,r), \quad r > 1, \tag{2.5}$$

$$C \leqslant u(x,r), \quad r \leqslant 1, \tag{2.6}$$

then for any $B \in \mathbb{B}$, $\chi_B \in M_u^{p)}(\mathbb{R}^n)$.

For the proof of the above result, the reader is referred to [30, Proposition 3.1].

We recall the definition of small block space from [30, Definition 3.2].

DEFINITION 2.3. Let $p \in (1,\infty)$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. A Lebesgue measurable function b is a small (p,u)-block if there exists a $B \in \mathbb{B}$ such that $\operatorname{supp} b \subset B$ and

$$\|b\|_{L^{(p}(B)} \leqslant \frac{1}{u(B)|B|}. \tag{2.7}$$

We write $b \in \mathfrak{b}_u^{(p}$ if b is a small (p,u)-block.

The small block space $\mathfrak{B}_u^{(p}(\mathbb{R}^n)$ consists of all $f \in \mathscr{M}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathfrak{B}_u^{(p}(\mathbb{R}^n)} = \inf\left\{ \sum_{i=1}^{\infty} |\lambda_i| : f = \sum_{i=1}^{\infty} \lambda_i b_i,\ \{b_i\}_{i=1}^{\infty} \subset \mathfrak{b}_u^{(p} \right\} < \infty.$$

We have the following duality results for the grand Morrey space and the small block space.

PROPOSITION 2.2. *Let* $p \in (1,\infty)$ *and* $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *be a Lebesgue measurable function. If*

$$\sup\left\{ \int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in \mathfrak{b}_u^{(p'} \right\} < \infty,$$

then $f \in M_u^{p)}(\mathbb{R}^n)$.

PROPOSITION 2.3. *Let* $p \in (1,\infty)$ *and* $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *be a Lebesgue measurable function. There is a constant* $C > 0$ *such that for any* $f \in M_u^{p)}(\mathbb{R}^n)$ *and* $g \in \mathfrak{B}_u^{(p'}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant C\|f\|_{M_u^{p)}(\mathbb{R}^n)} \|g\|_{\mathfrak{B}_u^{(p'}(\mathbb{R}^n)}. \tag{2.8}$$

PROPOSITION 2.4. *Let* $p \in (1,\infty)$ *and* $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *be a Lebesgue measurable function. There are constants* $C_0, C_1 > 0$ *such that for any* $f \in M_u^{p)}(\mathbb{R}^n)$,

$$C_0\|f\|_{M_u^{p)}(\mathbb{R}^n)} \leqslant \sup\left\{ \int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in \mathfrak{b}_u^{(p'} \right\} \leqslant C_1\|f\|_{M_u^{p)}(\mathbb{R}^n)}. \tag{2.9}$$

The reader is referred to [30, Propositions 3.2, 3.3, 3.4] for the proofs of the above propositions.

The following proposition guarantees that the Hardy-Littlewood operator is well defined on the small block space [30, Proposition 3.5].

PROPOSITION 2.5. *Let $p \in (1,\infty)$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If u satisfies*

$$Cr^{-n/p'} < u(x,r), \quad r > 1, x \in \mathbb{R}^n \tag{2.10}$$

and (2.6), we have

$$\mathfrak{B}_u^{(p}(\mathbb{R}^n) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

The above results assure that the grand Morrey space is a ball Banach function space. For simplicity, we refer the reader to [46] for the definitions of the ball Banach function space. In addition, the results for the ball Banach function space are valid to the grand Morrey spaces such as the Brezis-Van Schaftingen-Yung formulae [14] and the compactness characterization of commutators [54].

According to [30, Theorem 3.1], the Hardy-Littlewood maximal operator is also bounded in the small block spaces.

THEOREM 2.6. *Let $p \in (1,\infty)$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If u satisfies (2.6), (2.10) and there is a constant $C > 0$ such that for any $B \in \mathbb{B}$*

$$\sum_{k=0}^{\infty} u(2^k B) \leqslant Cu(B), \tag{2.11}$$

then $\mathrm{M} : \mathfrak{B}_u^{(p}(\mathbb{R}^n) \to \mathfrak{B}_u^{(p}(\mathbb{R}^n)$ is bounded.

3. Extrapolation

We extend the Rubio de Francia extrapolation method to grand Morrey spaces in this section. Notice that we already have an extrapolation method for grand Morrey spaces in [30] but the results in [30] cannot directly be used for the fractional integral operators.

We first recall the definition of the Muckenhoupt weight functions.

DEFINITION 3.1. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if

$$[\omega]_{A_p} = \sup_{B \in \mathbb{B}} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if for any $B \in \mathbb{B}$

$$\frac{1}{|B|}\int_B \omega(y)dy \leqslant C\omega(x), \quad \text{a.e. } x \in B$$

for some constants $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \cup_{p\geqslant 1} A_p$.

We need some definitions for our main results.

For any $\theta \in [1,\infty)$ and locally integrable function f, define

$$\mathrm{M}_\theta f = \left(\mathrm{M}(|f|^\theta)\right)^{1/\theta}.$$

DEFINITION 3.2. Let $\alpha > 0$. For any $u : \mathbb{R} \times (0,\infty) \to (0,\infty)$, define $u_\alpha(B) = u(B)|B|^{\alpha/n}$.

3.1. Main result

We are now ready to establish the extrapolation theory for the grand Morrey spaces.

THEOREM 3.1. *Let $\alpha \in [0,\infty)$, $0 \leqslant p_0 \leqslant q_0 < \infty$ satisfying $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. Let $p_0 < p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\theta \in (1,(q/q_0)')$.*

Suppose that u satisfies

$$C < (u(x,r)r^\alpha)^{q_0\theta} r^{n(\theta-1)}, \quad r \leqslant 1, x \in \mathbb{R}^n, \tag{3.1}$$

$$Cr^{-nq_0/q} < (u(x,r)r^\alpha)^{q_0\theta} r^{n(\theta-1)}, \quad r > 1, x \in \mathbb{R}^n, \tag{3.2}$$

$$\sum_{k=0}^{\infty}(u(2^kB)|2^kB|^{\alpha/n})^{p_0}|2^kB|^{(p_0/q_0)/\theta'} \tag{3.3}$$

$$\leqslant C(u(B)|B|^{\alpha/n})^{p_0}|B|^{(p_0/q_0)/\theta'}, \quad \forall B \in \mathbb{B}$$

for some $C > 0$.

Let $f \in M_u^{p)}(\mathbb{R}^n)$ and g be a Lebesgue measurable. If for any

$$\omega \in \left\{\mathrm{M}_\theta h : h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}\right\},$$

we have constant $C > 0$

$$\left(\int_{\mathbb{R}^n}|g(x)|^{q_0}\omega(x)dx\right)^{1/q_0} \leqslant C\left(\int_{\mathbb{R}^n}|f(x)|^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0} < \infty, \tag{3.4}$$

then $g \in M_{u_\alpha}^{q)}(\mathbb{R}^n)$ and

$$\|g\|_{M_{u_\alpha}^{q)}(\mathbb{R}^n)} \leqslant C_0\|f\|_{M_u^{p)}(\mathbb{R}^n)}$$

for some $C_0 > 0$.

Proof. Let $h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}$. We find that

$$|h| \leqslant \mathrm{M}_\theta h. \tag{3.5}$$

As $h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}$, $|h|^\theta \in \mathfrak{b}_v^{((q/q_0)'/\theta}$ where

$$v(B) = u_\alpha(B)^{q_0\theta}|B|^{\theta-1} = (u(B)|B|^{\alpha/n})^{q_0\theta}|B|^{\theta-1}.$$

Since $p_0/q_0 < 1 < \theta$, according to (3.3), we obtain

$$\begin{aligned}\left(\sum_{k=0}^{\infty}\frac{v(2^kB)}{v(B)}\right)^{(p_0/q_0)/\theta} &\leqslant \sum_{k=0}^{\infty}\left(\frac{v(2^kB)}{v(B)}\right)^{(p_0/q_0)/\theta}\\ &= \sum_{k=0}^{\infty}\frac{(u(2^kB)|2^kB|^{\alpha/n})^{p_0}|2^kB|^{(p_0/q_0)/\theta'}}{(u(B)|B|^{\alpha/n})^{p_0}|B|^{(p_0/q_0)/\theta'}} < C.\end{aligned}$$

Consequently, v satisfies (2.11). In addition, (3.1) and (3.2) assure that v fulfills (2.6), (2.10) and $(q/q_0)'/\theta > 1$, in view of [30, (3.11)], we have

$$\lambda_k = C_0\frac{v(2^{k+1}B)}{v(B)}$$

for some $C_0 > 0$ and $d_k = \lambda_k^{-1}\chi_{2^{k+1}B\setminus 2^kB}\mathrm{M}(|h|^\theta)$ such that $\{d_k\}_{k=0}^\infty \subset \mathfrak{b}_v^{((q/q_0)'/\theta}$ and

$$\mathrm{M}(|h|^\theta) = \sum_{k=0}^{\infty}\lambda_k d_k.$$

As $(p_0/q_0)/\theta < 1$, we find that

$$\begin{aligned}(\mathrm{M}_\theta h)^{p_0/q_0} = (\mathrm{M}(|h|^\theta))^{(p_0/q_0)/\theta} &= \left(\sum_{k=0}^{\infty}\lambda_k d_k\right)^{(p_0/q_0)/\theta}\\ &\leqslant \sum_{k=0}^{\infty}|\lambda_k|^{(p_0/q_0)/\theta}|d_k|^{(p_0/q_0)/\theta}.\end{aligned}$$

Notice that $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}$ yields

$$\left(\frac{q}{q_0}\right)'\frac{q_0}{p_0} = \frac{q}{q-q_0}\frac{q_0}{p_0} = \frac{p}{p-p_0} = \left(\frac{p}{p_0}\right)'. \tag{3.6}$$

Furthermore, the definition of u_α and $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$ give

$$\begin{aligned}v(B)^{(p_0/q_0)/\theta}|B|^{(p_0/q_0)/\theta} &= u_\alpha(B)^{p_0}|B|^{(\theta-1)\frac{p_0}{q_0\theta}}|B|^{(p_0/q_0)/\theta}\\ &= u_\alpha(B)^{p_0}|B|^{\frac{p_0}{q_0}}\\ &= u(B)^{p_0}|B|^{\frac{\alpha p_0}{n}+\frac{p_0}{q_0}}\\ &= u(B)^{p_0}|B|.\end{aligned} \tag{3.7}$$

As $d_k \in \mathfrak{b}_v^{((q/q_0)'/\theta}$, (3.6) guarantees that

$$\||d_k|^{(p_0/q_0)/\theta}\|_{L^{((p/p_0)'}(B)} = \|d_k\|^{(p_0/q_0)/\theta}_{L^{((p/p_0)'(p_0/q_0)/\theta}(B)}$$
$$= \|d_k\|^{(p_0/q_0)/\theta}_{L^{((q/q_0)'/\theta}(B)} \leqslant \frac{1}{(v(B)|B|)^{(p_0/q_0)/\theta}}.$$

Consequently, (3.7) yields
$$|d_k|^{(p_0/q_0)/\theta} \in \mathfrak{b}_{u^{p_0}}^{((p/p_0)'}.$$

Inequality (3.3) assures that

$$\sum_{k=0}^{\infty} |\lambda_k|^{(p_0/q_0)/\theta} = \sum_{k=0}^{\infty} \left(\frac{(u_\alpha(2^{k+1}B))^{q_0\theta}|2^{k+1}B|^{\theta-1}}{(u_\alpha(B))^{q_0\theta}|B|^{\theta-1}} \right)^{(p_0/q_0)/\theta}$$
$$\leqslant \sum_{k=0}^{\infty} \frac{(u(2^kB)|2^kB|^{\alpha/n})^{p_0}|2^kB|^{(p_0/q_0)/\theta'}}{(u(B)|B|^{\alpha/n})^{p_0}|B|^{(p_0/q_0)/\theta'}} < C$$

for some $C > 0$ independent of h.

Therefore, $(\mathrm{M}_\theta h)^{p_0/q_0} \in \mathfrak{B}_{u^{p_0}}^{((p/p_0)'}(\mathbb{R}^n)$ with

$$\|(\mathrm{M}_\theta h)^{p_0/q_0}\|_{\mathfrak{B}_{u^{p_0}}^{((p/p_0)'}(\mathbb{R}^n)} < C \tag{3.8}$$

for some $C > 0$ independent of h.

According to (3.5) and (3.4), we find that

$$\int_{\mathbb{R}^n} |g(x)|^{q_0}|h(x)|dx \leqslant \int_{\mathbb{R}^n} |g(x)|^{q_0} \mathrm{M}_\theta h(x) dx$$
$$\leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^{p_0}(\mathrm{M}_\theta h(x))^{p_0/q_0} dx \right)^{q_0/p_0}$$

for some $C > 0$. Consequently, Proposition 2.3, (3.8) and [30, (3.1)] give

$$\int_{\mathbb{R}^n} |g(x)|^{q_0}|h(x)|dx \leqslant C\||f|^{p_0}\|^{q_0/p_0}_{M_{u^{p_0}}^{p/p_0)}(\mathbb{R}^n)} \|(\mathrm{M}_\theta h)^{p_0/q_0}\|^{q_0/p_0}_{\mathfrak{B}_{u^{p_0}}^{((p/p_0)'}(\mathbb{R}^n)}$$
$$\leqslant C\|f\|^{q_0}_{M_u^{p)}(\mathbb{R}^n)}.$$

By taking the supremum over $h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}$, we obtain

$$\sup\left\{ \int_{\mathbb{R}^n} |g(x)|^{q_0}|h(x)|dx : h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'} \right\} \leqslant C\|f\|^{q_0}_{M_u^{p)}(\mathbb{R}^n)} < \infty.$$

Therefore, Proposition 2.2 asserts that $|g|^{q_0} \in M_{u_\alpha^{q_0}}^{q/q_0)}(\mathbb{R}^n)$ and

$$\||g|^{q_0}\|_{M_{u_\alpha^{q_0}}^{q/q_0)}(\mathbb{R}^n)} \leqslant C\|f\|^{q_0}_{M_u^{p)}(\mathbb{R}^n)}.$$

In view of [30, (3.1)], we get

$$\|g\|^{q_0}_{M^{q)}_{u\alpha}(\mathbb{R}^n)} = \||g|^{q_0}\|_{M^{q/q_0)}_{u^{q_0}_{\alpha}}(\mathbb{R}^n)}.$$

Thus, we have

$$\|g\|_{M^{q)}_{u\alpha}(\mathbb{R}^n)} \leqslant C\|f\|_{M^{p)}_{u}(\mathbb{R}^n)}$$

for some $C > 0$. □

Notice that in [30], we use the Rubio de Francia operator $\mathscr{R}_{p,u}$ [30, Definition 4.2] to obtain the extrapolation method in [30, Theorem 4.2]. This operator cannot be used in the above result as the above result involves the p-power of the small (p,u)-block. Therefore, we use the operator M_θ instead of $\mathscr{R}_{p,u}$. As observed in the proof of the above theorem, the estimate of $|\mathrm{M}_\theta h|^r$, $r \in (0,1)$, when h is a small (p,u)-block can be controlled by the r-inequality.

3.2. Examples

We now give a function u that satisfies (2.5)–(2.6) and (3.1)–(3.3). Let $\alpha \in (0,\infty)$, $p \in (1, \frac{n}{\alpha})$, $\beta \in (0,1)$ and $u(B(x,r)) = u(x,r) = r^{-\frac{n}{p}\beta}$. It is easy to see that it satisfies (2.5)–(2.6). Proposition 2.1 guarantees that $M^{p)}_{\mu}(\mathbb{R}^n)$ is non-trivial.

We select a θ such that

$$1 < \theta < \frac{q}{q-q_0} = \left(\frac{q}{q_0}\right)'. \tag{3.9}$$

As $q_0 < q$, θ is well defined.

As $\theta < (q/q_0)'$, we find that $\theta(1 - \frac{q_0}{q}) < 1$. In view of $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, we obtain

$$\theta - 1 < \frac{q_0}{q}\theta = \left(\frac{1}{p} - \frac{\alpha}{n}\right) q_0\theta.$$

Hence,

$$p\left(\frac{\theta-1}{q_0\theta} + \frac{\alpha}{n}\right) < 1. \tag{3.10}$$

When β satisfies

$$p\left(\frac{\theta-1}{q_0\theta} + \frac{\alpha}{n}\right) < \beta < 1, \tag{3.11}$$

we have

$$\theta - 1 < \left(\frac{1}{p}\beta - \frac{\alpha}{n}\right) q_0\theta. \tag{3.12}$$

Since

$$(u(x,r)r^{\alpha})^{q_0\theta} r^{n(\theta-1)} = r^{(-\frac{n}{p}\beta+\alpha)q_0\theta+n(\theta-1)}, \tag{3.13}$$

(3.12) asserts that $u(x,r) = r^{-\frac{n}{p}\beta}$ satisfies (3.1).

Next, we consider (3.2). Since $\theta > 1$, we find that

$$\left(n - \frac{n}{q}q_0\right)\theta > \frac{n}{(q/q_0)'}.$$

As $\frac{n}{q}q_0 = \left(\frac{n}{p} - \alpha\right)q_0 > \left(\frac{n}{p}\beta - \alpha\right)q_0$, we have

$$\left(n - \left(\frac{n}{p}\beta - \alpha\right)q_0\right)\theta > \frac{n}{(q/q_0)'} = n - \frac{nq_0}{q}.$$

Thus,

$$\left(-\frac{n}{p}\beta + \alpha\right)q_0\theta + n(\theta - 1) + \frac{nq_0}{q} > 0. \tag{3.14}$$

In view of (3.13), we verify that $u(x,r) = r^{-\frac{n}{p}\beta}$ satisfies (3.2).

We now deal with (3.3). We find that

$$(u(B)|B|^{\alpha/n})^{p_0}|B|^{(p_0/q_0)/\theta'} = r^{(-\frac{n}{p}\beta+\alpha)p_0+n\frac{p_0/q_0}{\theta'}}.$$

As $\theta < (q/q_0)'$, we have $\frac{q}{q_0} < \theta'$. Hence, $\frac{n}{q} = \frac{n}{p} - \alpha$ gives

$$\frac{n(1/q_0)}{\theta'} < \frac{n}{q} = \frac{n}{p} - \alpha.$$

That is,

$$\frac{p}{n}\left(\frac{n(1/q_0)}{\theta'} + \alpha\right) < 1. \tag{3.15}$$

When β satisfies (3.11), we have

$$\frac{p}{n}\left(\frac{n(1/q_0)}{\theta'} + \alpha\right) < \beta.$$

Consequently,

$$\frac{n(1/q_0)}{\theta'} < \left(\frac{n}{p}\beta - \alpha\right).$$

As $p_0 > 0$, we find that

$$\left(-\frac{n}{p}\beta + \alpha\right)p_0 + n\frac{p_0/q_0}{\theta'} < 0.$$

Therefore, $u(x,r) = r^{-\frac{n}{p}\beta}$ fulfills (3.3). Furthermore, we have $u_\alpha(x,r) = C_0 r^{-\frac{n}{p}\beta+\alpha}$ where $C_0 > 0$. Since $\frac{n}{q} = \frac{n}{p} - \alpha > \frac{n}{p}\beta - \alpha$, we see that

$$Cr^{-\frac{n}{q}} < C_0 r^{-\frac{n}{p}\beta+\alpha} = u_\alpha(x,r), \quad r > 1$$

for some $C>0$. In addition, according to (3.12), we obtain

$$-\frac{n}{p}\beta+\alpha<\frac{1-\theta}{q_0\theta}<0$$

because $\theta>1$. Thus,

$$C<C_0 r^{-\frac{n}{p}\beta+\alpha}=u_\alpha(x,r), \quad r<1$$

for some $C>0$. Consequently, Proposition 2.1 guarantees that $M^{q)}_{u_\alpha}(\mathbb{R}^n)$ is non-trivial. In conclusion, for any given θ satisfying (3.9), when β satisfies (3.11), $M^{p)}_u(\mathbb{R}^n)$, $M^{q)}_{u_\alpha}(\mathbb{R}^n)$ are nontrivial and we are allowed to apply Theorem 3.1 for $f\in M^{p)}_u(\mathbb{R}^n)$.

Applications for some concrete operators by Theorem 3.1 are presented in the following section.

4. Applications

In this section, we apply Theorem 3.1 to obtain the mapping properties of the fractional integral operators on the grand Morrey spaces and the grand Hardy-Morrey spaces. The Sobolev embedding, the Poincaré inequality and the mapping properties of the fractional geometric maximal functions on the grand Morrey spaces are also obtained.

4.1. Fractional integral operators

Let $\alpha\in(0,n)$. For any locally integrable function f, the fractional integral operator I_α is defined as

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

We recall the weighted norm inequalities for fractional integral operators from [39].

THEOREM 4.1. *Let $\alpha\in(0,n)$, $p_0\in(1,\frac{n}{\alpha})$, $\frac{1}{p_0}=\frac{1}{q_0}+\frac{\alpha}{n}$ and $\omega\in A_1$. There exists a constant $C>0$ such that*

$$\left(\int_{\mathbb{R}^n}|I_\alpha f(x)|^{q_0}\omega(x)dx\right)^{1/q_0}\leqslant C\left(\int_{\mathbb{R}^n}|f(x)|^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0}.$$

For the proof of the above theorem, the reader is referred to [39, Theorem 4]. Notice that the weighted norm inequalities for fractional integral operators obtained in [39, Theorem 4] are valid for a larger class of weight functions. As A_1 is a subclass of this class of weight functions, therefore, we have the above results.

THEOREM 4.2. *Let $\alpha\in(0,n)$, $p\in(1,\frac{n}{\alpha})$, $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}$ and $u:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. If there exist $q_0\in(1,q)$ and $\theta\in(1,(q/q_0)')$ such that u satisfies (3.1)–(3.3), then there exists a constant $C>0$ such that for any $f\in M^{p)}_u(\mathbb{R}^n)$*

$$\|I_\alpha f\|_{M^{q)}_{u_\alpha}(\mathbb{R}^n)}\leqslant C\|f\|_{M^{p)}_u(\mathbb{R}^n)}.$$

Proof. Define p_0 by $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$. As $q_0 \in (1,q)$ and $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$, we have $p_0 < p < \frac{n}{\alpha}$.

Let $h \in \mathfrak{b}^{((q/q_0)'}_{u_\alpha^{q_0}}$ and $f \in M^{p)}_u(\mathbb{R}^n)$. We may assume that $f \geqslant 0$ by considering $|f|$ instead of f if necessary. The Hölder inequality yields

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} f(x)^{p_0} (\mathrm{M}_\theta h(x))^{p_0/q_0} dx\right)^{1/p_0} \\
&\leqslant C \|f^{p_0}\|^{1/p_0}_{M^{p/p_0)}_{u^{p_0}}(\mathbb{R}^n)} \|(\mathrm{M}_\theta h)^{p_0/q_0}\|^{1/p_0}_{\mathfrak{B}^{((p/p_0)'}_{u^{p_0}}(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{M^{p)}_u(\mathbb{R}^n)}.
\end{aligned}$$

Consequently, we have

$$M^{p)}_u(\mathbb{R}^n) \hookrightarrow \bigcap_{h \in \mathfrak{b}^{((q/q_0)'}_{u_\alpha^{q_0}}} L^{p_0}((\mathrm{M}_\theta h)^{p_0/q_0}). \tag{4.1}$$

For any $h \in \mathfrak{b}^{((q/q_0)'}_{u_\alpha^{q_0}}$, [24, Theorem 9.2.8] assures that $\mathrm{M}_\theta h \in A_1$. Theorem 4.1 and (4.1) guarantee that for any $f \in M^{p)}_u(\mathbb{R}^n)$, we have

$$\left(\int_{\mathbb{R}^n} (I_\alpha f(x))^{q_0} \mathrm{M}_\theta h(x) dx\right)^{1/q_0} \leqslant C \left(\int_{\mathbb{R}^n} f(x)^{p_0} (\mathrm{M}_\theta h(x))^{p_0/q_0} dx\right)^{1/p_0}.$$

Thus, Theorem 3.1 yields a constant $C > 0$ such that for any $f \in M^{p)}_u(\mathbb{R}^n)$, $I_\alpha f \in M^{q)}_{u_\alpha}(\mathbb{R}^n)$ and

$$\|I_\alpha f\|_{M^{q)}_{u_\alpha}(\mathbb{R}^n)} \leqslant C\|f\|_{M^{p)}_u(\mathbb{R}^n)}. \qquad \square$$

The boundedness of the fractional integral operators on Morrey spaces were obtained by Spanne [42], which is reported by Peetre in [42], and Adams [1]. For the history on the boundedness of the fractional integral operators on Morrey spaces, the reader is referred to [49]. The above result is an extension of the boundedness of the fractional integral operators from Morrey spaces to grand Morrey spaces.

We remark that the proof of Theorem 4.2 does not rely on the linearity of I_α. The proof is valid for any operator, no matter whether it is linear or not.

COROLLARY 4.3. *Let* $\alpha \in (0,n)$, $p \in (1, \frac{n}{\alpha})$, $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$. *Let* $\beta \in (\frac{\alpha p}{n}, 1)$, $\mu(x,r) = r^{-\frac{n}{p}\beta}$ *and* $\mu_\alpha(x,r) = \mu(x,r) r^\alpha$, $r > 0$ *and* $x \in \mathbb{R}^n$. *There exists a constant* $C > 0$ *such that for any* $f \in M^{p)}_\mu(\mathbb{R}^n)$

$$\|I_\alpha f\|_{M^{q)}_{\mu_\alpha}(\mathbb{R}^n)} \leqslant C\|f\|_{M^{p)}_\mu(\mathbb{R}^n)}. \tag{4.2}$$

Proof. As $\beta > p\frac{\alpha}{n}$, we have a $\theta \in (1,(q/q_0)')$ such that

$$p\left(\frac{1}{\theta'}+\frac{\alpha}{n}\right) < \beta.$$

Since $q_0 > 1$, we find that

$$p\left(\frac{1}{q_0\theta'}+\frac{\alpha}{n}\right) < p\left(\frac{1}{\theta'}+\frac{\alpha}{n}\right) < \beta.$$

There exist θ and β satisfying (3.9) and (3.11). Therefore, $M^{p)}_{\mu}(\mathbb{R}^n)$ and $M^{q)}_{\mu_\alpha}(\mathbb{R}^n)$ are nontrivial and (4.2) is valid. $\square$

We use the above result to obtain the Sobolev embedding and the Poincaré inequality for the grand Morrey spaces. We first establish the Sobolev embedding on $M^{p)}_{u}(\mathbb{R}^n)$.

THEOREM 4.4. *Let $n > 1$, $p \in (1,n)$, $\frac{1}{p} = \frac{1}{q}+\frac{1}{n}$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. If there exist $q_0 \in (1,q)$ and $\theta \in (1,(q/q_0)')$ such that u satisfies (3.1)–(3.3), then there exists a constant $C > 0$ such that for any compactly supported Lipschitz functions f on $\mathbb{R}^n$, we have*

$$\|f\|_{M^{q)}_{u_\alpha}(\mathbb{R}^n)} \leqslant C\|\nabla f\|_{M^{p)}_{u}(\mathbb{R}^n)}$$

where $u_\alpha(B) = u(B)|B|^{\frac{\alpha}{n}}$.

Proof. In view of [17, (4.3.6)], we have

$$|f(x)| \leqslant C\int_{\mathbb{R}^n}\frac{|\nabla f(y)|}{|x-y|^{n-1}}dy = CI_1(|\nabla f|)(x), \quad x \in \mathbb{R}^n.$$

Therefore, Theorem 4.2 yields

$$\|f\|_{M^{q)}_{u_\alpha}(\mathbb{R}^n)} \leqslant C\|I_1(|\nabla f|)\|_{M^{q)}_{u_\alpha}(\mathbb{R}^n)} = C\|\nabla f\|_{M^{p)}_{u}(\mathbb{R}^n)}. \quad \square$$

The reader is referred to [17, p. 4–5] for the definition of the gradient of a compactly supported Lipschitz function.

Next, we present the Poincaré inequality on the grand Morrey spaces.

THEOREM 4.5. *Let $n > 1$, $p \in (1,n)$, $\frac{1}{p} = \frac{1}{q}+\frac{1}{n}$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. If there exist $q_0 \in (1,q)$ and $\theta \in (1,(q/q_0)')$ such that u satisfies (3.1)–(3.3), then there exists a constant $C > 0$ such that for any $f \in C^1(\mathbb{R}^n)$ and $B \in \mathbb{B}$, we have*

$$\|(f-f_B)\chi_B\|_{M^{q)}_{u_\alpha}(\mathbb{R}^n)} \leqslant C\|\chi_B\nabla f\|_{M^{p)}_{u}(\mathbb{R}^n)}.$$

Proof. According to [17, (4.3.5)], we have

$$|f(x)-f_B| \leqslant C\int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}}dy = CI_1(\chi_B|\nabla f|)(x), \quad x\in\mathbb{R}^n.$$

Consequently, Theorem 4.2 gives

$$\|(f-f_B)\chi_B\|_{M^{q)}_{u\alpha}(\mathbb{R}^n)} \leqslant C\|\chi_B\nabla f\|_{M^{p)}_{u}(\mathbb{R}^n)}. \qquad \square$$

Let $\beta\in(\frac{p}{n},1)$ and $\mu(x,r)=r^{-\frac{n}{p}\beta}$. The above results and Corollary 4.3 show that for any compactly supported Lipschitz functions f on $\mathbb{R}^n$

$$\|f\|_{M^{q)}_{\mu\alpha}(\mathbb{R}^n)} \leqslant C\|\nabla f\|_{M^{p)}_{\mu}(\mathbb{R}^n)}.$$

Moreover, for any $f\in C^1(\mathbb{R}^n)$ and $B\in\mathbb{B}$, we have

$$\|(f-f_B)\chi_B\|_{M^{q)}_{\mu\alpha}(\mathbb{R}^n)} \leqslant C\|\chi_B\nabla f\|_{M^{p)}_{\mu}(\mathbb{R}^n)}.$$

4.2. Fractional geometric maximal functions

We now state the definitions of the fractional geometric maximal functions. Let $\alpha\geqslant 0$. For any $f\in\mathscr{M}(\mathbb{R})$, the fractional geometric maximal function $M_{\alpha,0}f$ is defined as

$$M_{\alpha,0}f(x)=\sup_{I\ni x}|I|^{\alpha}\exp\left(\frac{1}{|I|}\int_I \log|f(y)|dy\right)$$

where the notation $\sup_{I\ni x}$ means that the supremum is taken over all interval I containing x.

Moreover, for any $f\in\mathscr{M}(\mathbb{R})$, $M^*_{\alpha,0}$ is defined by

$$M^*_{\alpha,0}f(x)=\lim_{r\downarrow 0}\sup_{I\ni x}|I|^{\alpha}\left(\frac{1}{|I|}\int_I|f(y)|^r dy\right)^{1/r}.$$

When $\alpha=0$, the fractional geometric maximal functions $M_{\alpha,0}$ and $M^*_{\alpha,0}$ become the geometric maximal functions M_0 and M^*_0 [11, 12, 57], respectively.

The following weighted norm inequalities for the fractional geometric maximal functions are given in [13, Theorems 2, 4 and 8].

THEOREM 4.6. *Let $\alpha\in[0,\infty)$, $0<p\leqslant q<\infty$ satisfying $\frac{1}{p}-\frac{1}{q}=\alpha$ and $\omega\in A_\infty$. There exists a constant $C_0>0$ such that for any $f\in L^p(\omega^{p/q})$, we have*

$$\left(\int_{\mathbb{R}}(M_{\alpha,0}f(x))^q\omega(x)dx\right)^{1/q} \leqslant C_0\left(\int_{\mathbb{R}}|f(x)|^p\omega(x)^{p/q}dx\right)^{1/p},$$

$$\left(\int_{\mathbb{R}}(M^*_{\alpha,0}f(x))^q\omega(x)dx\right)^{1/q} \leqslant C_0\left(\int_{\mathbb{R}}|f(x)|^p\omega(x)^{p/q}dx\right)^{1/p}.$$

The reader is referred to [11, 12, 57] for the weighted norm inequalities for $M_{0,0}$ and $M^*_{0,0}$.

THEOREM 4.7. *Let* $\alpha \in [0,\infty)$, $p \in (0,\infty)$, $\frac{1}{p} = \frac{1}{q} + \alpha$ *and* $u : \mathbb{R} \times (0,\infty) \to (0,\infty)$. *If there exist* $q_0 \in (0,q)$ *and* $\theta \in (1,(q/q_0)')$ *such that* u *satisfies* (3.1)–(3.3), *then there exists a constant* $C > 0$ *such that for any* $f \in M^{p)}_u(\mathbb{R})$

$$\|M_{\alpha,0} f\|_{M^{q)}_{u^{\alpha}}(\mathbb{R})} \leqslant C\|f\|_{M^{p)}_u(\mathbb{R})},$$
$$\|M^*_{\alpha,0} f\|_{M^{q)}_{u^{\alpha}}(\mathbb{R})} \leqslant C\|f\|_{M^{p)}_u(\mathbb{R})}$$

As remarked after Theorem 4.2, the proof of Theorem 4.2 does not rely on the linearity of I_α. Thus, the proof is also valid for the nonlinear operators $M_{\alpha,0}$ and $M^*_{\alpha,0}$. With some obvious modifications, the proof of the above result is the same as the proof of Theorem 4.2. For simplicity, we skip the details.

4.3. Fractional integral operators on grand Hardy-Morrey spaces

We define the grand Hardy-Morrey spaces by using the grand maximal function. Let $\mathscr{F} = \{\|\cdot\|_{\alpha_i,\beta_i}\}$ be any finite collection of semi-norms on $\mathscr{S}$ and

$$\mathscr{S}_{\mathscr{F}} = \{\psi \in \mathscr{S} : \|\psi\|_{\alpha_i,\beta_i} \leqslant 1, \text{for all} \|\cdot\|_{\alpha_i,\beta_i} \in \mathscr{F}\}.$$

For any $f \in \mathscr{S}'$, write

$$\mathscr{M}_{\mathscr{F}} f(x) = \sup_{\psi \in \mathscr{S}_{\mathscr{F}}} \sup_{t>0} |(f * \psi_t)(x)|$$

where for any $t > 0$, write $\psi_t(x) = t^{-n}\psi(x/t)$.

DEFINITION 4.1. Let $p \in (0,1]$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. The grand Hardy-Morrey space $HM^{p)}_u(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'$ satisfying

$$\|f\|_{HM^{p)}_u(\mathbb{R}^n)} = \|\mathscr{M}_{\mathscr{F}} f\|_{M^{p)}_u(\mathbb{R}^n)} < \infty.$$

Let $p \in (0,\infty)$ and $\omega : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function, the weighted Hardy space $H^p(\omega)$ consists of all $f \in \mathscr{S}'$ satisfying

$$\|f\|_{H^p(\omega)} = \left(\int_{\mathbb{R}^n} \mathscr{M}_{\mathscr{F}} f(x)^p \omega(x) dx\right)^{1/p} < \infty.$$

For the details of the weighted Hardy spaces, such as the atomic decomposition, the reader is referred to [23, 51].

As the grand Morrey space is a ball Banach function space, the general result for the Hardy spaces built on ball quasi-Banach function spaces are valid for the grand Hardy-Morrey spaces such as the Littlewood-Paley characterization [7, 55] and the intrinsic square function characterization [56].

We now present the mapping properties of the fractional integral operators on weighted Hardy spaces [38, 39, 52].

THEOREM 4.8. *Let* $0 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then,* $v^{q/p} \in A_\infty$ *if and only if*

$$\|I_\alpha f\|_{H^q(v^{q/p})} \leqslant C\|f\|_{H^p(v)}$$

for some $C > 0$.

For the proof of the preceding theorem, the reader may consult [52, Corollary 6.2 and Theorem 8.1]. The result obtained in [52, Theorem 8.1] is for the case when f is an (∞, N) atom. As stated at the beginning of [52, p. 295], by using [52, Lemmas 2.1 and 2.2], the results in [52, Theorem 8.1] can be extended to obtain the boundedness of the fractional integral operators on the weighted Hardy spaces.

THEOREM 4.9. *Let* $\alpha \in (0,n)$, $p \in (0, \frac{n}{\alpha})$, $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$ *and* $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *If there exist* $q_0 \in (0,q)$ *and* $\theta \in (1,(q/q_0)')$ *such that* u *satisfies* (3.1)–(3.3), *then there exists a constant* $C > 0$ *such that for any* $f \in M_u^{p)}(\mathbb{R}^n)$

$$\|I_\alpha f\|_{HM_{u_\alpha}^{q)}(\mathbb{R}^n)} \leqslant C\|f\|_{HM_u^{p)}(\mathbb{R}^n)}.$$

Proof. Define p_0 by $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$. As $q_0 \in (1,q)$ and $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$, we have $p_0 < p < \frac{n}{\alpha}$.

Let $h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}$ and $f \in HM_u^{p)}(\mathbb{R}^n)$. The Hölder inequality yields

$$\left(\int_{\mathbb{R}^n} \mathcal{M}_{\mathscr{F}} f(x)^{p_0} (\mathrm{M}_\theta\, h(x))^{p_0/q_0} dx\right)^{1/p_0}$$
$$\leqslant C\|(\mathcal{M}_{\mathscr{F}} f)^{p_0}\|_{M_{u^{p_0}}^{p/p_0)}(\mathbb{R}^n)}^{1/p_0} \|(\mathrm{M}_\theta\, h)^{p_0/q_0}\|_{\mathfrak{B}_{u^{p_0}}^{((p/p_0)'}(\mathbb{R}^n)}^{1/p_0} \leqslant C\|f\|_{HM_u^{p)}(\mathbb{R}^n)}.$$

That is

$$HM_u^{p)}(\mathbb{R}^n) \hookrightarrow \bigcap_{h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}} H^{p_0}((\mathrm{M}_\theta\, h)^{p_0/q_0}). \tag{4.3}$$

For any $h \in \mathfrak{b}_{u_\alpha^{q_0}}^{((q/q_0)'}$, [24, Theorem 9.2.8] assures that $\mathrm{M}_\theta\, h \in A_1 \subset A_\infty$. Theorem 4.8 and (4.3) assert that for any $f \in HM_u^{p)}(\mathbb{R}^n)$, we have

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_{\mathscr{F}} I_\alpha f(x))^{q_0}\, \mathrm{M}_\theta\, h(x) dx\right)^{1/q_0}$$
$$\leqslant C\left(\int_{\mathbb{R}^n} (\mathcal{M}_{\mathscr{F}} f(x))^{p_0} (\mathrm{M}_\theta\, h(x))^{p_0/q_0} dx\right)^{1/p_0}.$$

Thus, (3.4) is fulfilled with the pair $(\mathcal{M}_{\mathscr{F}} I_\alpha f, \mathcal{M}_{\mathscr{F}} f)$. Theorem 3.1 yields a constant $C > 0$ such that for any $f \in HM_u^{p)}(\mathbb{R}^n)$, $I_\alpha f \in HM_{u_\alpha}^{q)}(\mathbb{R}^n)$ and

$$\|I_\alpha f\|_{HM_{u_\alpha}^{q)}(\mathbb{R}^n)} \leqslant C\|f\|_{HM_u^{p)}(\mathbb{R}^n)}. \qquad \square$$

For the mapping properties of the fractional integral operators on the Hardy type spaces, see [26, 27, 28, 35, 53].

We have the following application of Theorem 4.9 on $HM^{p)}_{\mu}(\mathbb{R}^n)$.

COROLLARY 4.10. *Let* $\alpha\in(0,n)$, $p\in(0,\frac{n}{\alpha})$, $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}$. *Let* $\beta\in(\frac{\alpha p}{n},1)$, $\mu(x,r)=r^{-\frac{n}{p}\beta}$ *and* $\mu_\alpha(x,r)=\mu(x,r)r^{\alpha}$, $r>0$ *and* $x\in\mathbb{R}^n$. *There exists a constant* $C>0$ *such that for any* $f\in HM^{p)}_{\mu}(\mathbb{R}^n)$

$$\|I_\alpha f\|_{HM^{q)}_{\mu_\alpha}(\mathbb{R}^n)}\leqslant C\|f\|_{HM^{p)}_{\mu}(\mathbb{R}^n)}.$$

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