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SHARPENING OF INEQUALITIES CONCERNING POLYNOMIALS

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Abstract. Let $P(z)=a_n\prod_{j=1}^{n}(z-z_j)$ be a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then Aziz [Proc. Am. Math. Soc., **89**, (1983) 259–266] proved

$$\max_{|z|=1}|P'(z)|\geqslant \frac{2}{1+k^n}\sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=1}|P(z)|.$$

In this paper, we prove a polar derivative extension which sharpens the above inequality. As a consequence, we also derive a result on Bernstein type inequality for the class of polynomials having all its zeros in $|z|\geqslant k$, $k\leqslant 1$.

1. Introduction

If $P(z)$ is a polynomial of degree n, then according to a well-known inequality due to Bernstein [3], we have

$$\max_{|z|=1}|P'(z)|\leqslant n\max_{|z|=1}|P(z)|. \tag{1}$$

Inequality (1) is best possible and equality holds if $P(z)=az^n$, $a\neq 0$.

If $P(z)$ has no zero in $|z|<1$, then

$$\max_{|z|=1}|P'(z)|\leqslant \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{2}$$

Equality in (2) holds if $P(z)=\alpha+\beta z^n$, $|\alpha|=|\beta|$.

Inequality (2) was conjectured by Erdös and later proved by Lax [16].

On the other hand, Turán [22] proved that if $P(z)$ has all its zeros in $|z|\leqslant 1$, then

$$\max_{|z|=1}|P'(z)|\geqslant \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{3}$$

It is of really interest to further notice about inequalities (2) and (3) that the restrictions imposed on the zeros of the polynomial concerned regarding the estimate of $\max_{|z|=1}|P'(z)|$

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as an upper or lower bound, accordingly as the zeros of the polynomial are contained respectively on and outside or on and inside the unit circle.

It is a common and natural interest to seek improvements, generalizations, extensions etc. of results existed in literature. In this regard, Malik [17] proved the following partial generalization of inequality (2) for polynomial $P(z)$ of degree n having no zero in $|z| < k$, $k \geqslant 1$

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1}|P(z)|. \tag{4}$$

Equality in (4) holds if $P(z) = (z+k)^n$.

The direct analogous inequality of (4) when the polynomial has no zero in $|z| < k$, $k \leqslant 1$, in general, does not seem to exist in literature till date. However, in an attempt to investigate the existence of this type of inequality, a special inequality was obtained by Govil [9] under a strong restriction on the moduli of the derivatives of the polynomial and its inversive polynomial that if $P(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leqslant 1$, then

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1}|P(z)|. \tag{5}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point on $|z| = 1$, where and throughout $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$.

In the same paper [17], as an application of his famous inequality (4), Malik [17] for the first time established a generalization of Turán's inequality (3) that if $P(z)$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then

$$\max_{|z|=1}|P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1}|P(z)|. \tag{6}$$

The result is sharp and extremal polynomial being $P(z) = (z+k)^n$.

Whereas the analogous inequality of (6) for $k \geqslant 1$ was proved by Govil [8] as

$$\max_{|z|=1}|P'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1}|P(z)|. \tag{7}$$

Inequality (7) is sharp and equality holds if $P(z) = z^n + k^n$.

It is noteworthy that Turán famous inequality (3) has been generalized in completion as regards the value of radius k of the closed disc $|z| \leqslant k$, referred to as the zero region.

By considering the locations of all the zeros of the polynomial, Aziz [2] improved inequality (7) and proved that if $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then

$$\max_{|z|=1}|P'(z)| \geqslant \frac{2}{1+k^n} \sum_{j=1}^{n} \frac{k}{k+|z_j|} \max_{|z|=1}|P(z)|. \tag{8}$$

If $P(z)$ be a polynomial of degree n and α is a complex number then the polar derivative of $P(z)$ with respect to α is given by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, and it generalizes ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z).$$

For more information on inequalities involving polar derivative and extensions of above inequalities, one can refer the following literature: Akhter et al. [1], Dewan et al. [4], Dewan et al. [5], Govil [10], Govil [12], Jain [14], Kumar [15], Milovanović et al. [18], Singh and Chanam [19], Singh et al. [20] and Singh et al. [21].

2. Lemmas

In order to prove our theorems, we shall make use of the following lemmas.

LEMMA 1. *If $P(z)=a_n\prod_{\nu=1}^{n}(z-z_\nu)$ is a polynomial of degree n having all its zeros in $|z|\leqslant 1$, then for every real or complex number α with $|\alpha|\geqslant 1$*

$$\max_{|z|=1}|D_\alpha P(z)|\geqslant(|\alpha|-1)\sum_{\nu=1}^{n}\frac{1}{1+|z_\nu|}\max_{|z|=1}|P(z)|. \tag{9}$$

This lemma is due to Giroux et al. [7].

LEMMA 2. *If $P(z)$ is a polynomial of degree n atmost, then*

$$\max_{|z|=1}|P'(z)|=\begin{cases} n\max\limits_{|z|=1}|P(z)|-\dfrac{2n}{n+2}|P(0)|, & \text{if } n\geqslant 2, \quad (10)\\ n\max\limits_{|z|=1}|P(z)|-|P(0)|, & \text{if } n=1. \quad (11)\end{cases}$$

LEMMA 3. *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree $n\geqslant 1$ and let $R\geqslant 1$, then*

$$M(P,R)\leqslant R^nM(P,1)-|P'(0)|(R^{n-1}-R^{n-3})(\sqrt{R^2+1}-1),\ \ n\geqslant 4, \tag{12}$$

$$M(P,R)\leqslant R^nM(P,1)-|P'(0)|(R^2-R)(\sqrt{R^2+R+1}-1),\ \ n=3, \tag{13}$$

$$M(P,R)\leqslant R^nM(P,1)-|P'(0)|R\left(\sqrt{\frac{R^2+1}{2}}-1\right),\ \ n=2, \tag{14}$$

$$M(P,R)\leqslant R^nM(P,1)-|P(0)|(R-1),\ \ n=1. \tag{15}$$

The above two lemmas are due to Frappier et al. [6].

LEMMA 4. *If* $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n \geqslant 2$, *and* $R \geqslant 1$, *then*

$$M(P,R) \leqslant R^n M(P,1) - \frac{2(R^n-1)}{(n+2)}|a_0| - 2|a_2|\left\{ (C_{n-2}(R) - C_{n-4}(R)) - \left(\frac{R^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3} \right) \right\}, \ n \geqslant 5, \quad (16)$$

$$M(P,R) \leqslant R^n M(P,1) - \frac{2(R^n-1)}{(n+2)}|a_0| - 2|a_2|\left\{ D_R - \left(\frac{R^3-1}{3} - \frac{R^2-1}{2} \right) \right\}, \ n=4, \quad (17)$$

$$M(P,R) \leqslant R^n M(P,1) - \frac{2(R^n-1)}{(n+2)}|a_0| - 2|a_2|\left\{ F_R - \left(\frac{R^2-1}{2} \right) \right\}, \ n=3, \quad (18)$$

$$M(P,R) \leqslant R^n M(P,1) - \frac{(R^n-1)}{2}|a_0| - |a_1|\frac{(R-1)^n}{2}, \ n=2. \quad (19)$$

where

$$C_t(R) = \int_1^R r^t \sqrt{r^2+1}\,dr, \ \ t>0, \quad (20)$$

$$D_R = \int_1^R (r^2-r)\sqrt{r^2+r+1}\,dr, \quad (21)$$

and

$$F_R = \int_1^R r\sqrt{\frac{r^2+1}{2}}\,dr. \quad (22)$$

Proof of Lemma 4. Let us assume that $P(z)$ is a polynomial of degree $n \geqslant 5$ so that $P'(z)$ is a polynomial of degree $(n-1) \geqslant 4$, applying inequality (12) of Lemma 3 to $P'(z)$, we get

$$\max_{|z|=r>1}|P'(z)| \leqslant r^{n-1}M(P',1) - |P''(0)|(r^{n-2}-r^{n-4})(\sqrt{r^2+1}-1). \quad (23)$$

Using inequality (10) of Lemma 2 in (23), we get

$$\begin{aligned}
&\max_{|z|=r}|P'(z)| \\
&\leqslant r^{n-1}\left\{ n\max_{|z|=1}|P(z)| - \frac{2n}{(n+2)}|a_0| \right\} - |P''(0)|(r^{n-2}-r^{n-4})(\sqrt{r^2+1}-1), \\
&= nr^{n-1}\max_{|z|=1}|P(z)| - \frac{2nr^{n-1}}{(n+2)}|a_0| - 2|a_2|(r^{n-2}-r^{n-4})(\sqrt{r^2+1}-1). \quad (24)
\end{aligned}$$

Now, for each θ, $0 \leqslant \theta < 2\pi$, we have

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leqslant \int_1^R |P'(re^{i\theta})|dr, \ R>1,$$

which on using inequality (24) gives

$$
\begin{aligned}
|P(Re^{i\theta})-P(e^{i\theta})| \leqslant \max_{|z|=1}|P(z)|\int_1^R nr^{n-1}dr-\frac{2|a_0|}{(n+2)}\int_1^R nr^{n-1}dr \\
-2|a_2|\left\{\int_1^R (r^{n-2}-r^{n-4})\sqrt{r^2+1}dr \ -\int_1^R (r^{n-2}-r^{n-4})dr\right\}, \\
\leqslant \max_{|z|=1}|P(z)|(R^n-1)-\frac{2(R^n-1)}{(n+2)}|a_0|-2|a_2| \\
\times\left\{(C_{n-2}(R)-C_{n-4}(R)-\left(\frac{R^{n-1}-1}{n-1}-\frac{R^{n-3}-1}{n-3}\right)\right\}.
\end{aligned}
$$

Hence on $|z|=R\geqslant 1$, we have

$$
\begin{aligned}
|P(Re^{i\theta})| &\leqslant |P(Re^{i\theta})-P(e^{i\theta})|+|P(e^{i\theta})|, \\
&\leqslant R^n\max_{|z|=1}|P(z)|-\frac{2(R^n-1)}{(n+2)}|a_0|-2|a_2| \\
&\quad\times\left\{(C_{n-2}(R)-C_{n-4}(R))-\left(\frac{R^{n-1}-1}{n-1}-\frac{R^{n-3}-1}{n-3}\right)\right\},
\end{aligned}
$$

which prove inequality (16).

Now the proof of inequalities (17), (18) and (19) follow on the same lines as that of inequality (16), but instead of using inequality (12), we use respectively inequalities (13), (14) and (15) of Lemma 3. We omit the details. This completes the proof of Lemma 4. □

LEMMA 5. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then*

$$
\max_{|z|=1}|P'(z)| \leqslant \frac{n}{2}\left\{\max_{|z|=1}|P(z)|-\min_{|z|=1}|P(z)|\right\}. \tag{25}
$$

The above lemma is due to Govil [11].

LEMMA 6. *If $P(z)=\sum_{\nu=1}^{n}a_\nu z^\nu$ is a polynomial of degree $n\geqslant 3$ having all its zeros in $|z|\geqslant 1$, then for $R\geqslant 1$, we have*

$$
\begin{aligned}
M(P,R) \leqslant \left(\frac{R^n+1}{2}\right)M(P,1)-\left(\frac{R^n-1}{2}\right)\min_{|z|=1}|P(z)|-\frac{2|a_1|}{(n+1)} \\
\times\left\{\frac{(R^n-1)-n(R-1)}{n}\right\}-6|a_3|\Bigg[(R-1)(C_{n-3}(R)-C_{n-5}(R)) \\
-\Bigg\{\left(\frac{(R^{n-1}-1)-(n-1)(R-1)}{(n-1)(n-2)}\right) \\
-\left(\frac{(R^{n-3}-1)-(n-3)(R-1)}{(n-3)(n-4)}\right)\Bigg\}\Bigg],\ n>5,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
M(P,R) \leqslant & \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1}|P(z)| - \frac{2|a_1|}{(n+1)} \\
& \times \left\{\frac{(R^n-1)-n(R-1)}{n}\right\} - 6|a_3| \left[(R-1)D_R\right. \\
& \left. - \left\{\left(\frac{(R^4-1)-4(R-1)}{12}\right) - \left(\frac{(R^3-1)-3(R-1)}{6}\right)\right\}\right], \ n=5, \qquad (27) \\
M(P,R) \leqslant & \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1}|P(z)| - \frac{2|a_1|}{(n+1)} \\
& \times \left\{\frac{(R^n-1)-n(R-1)}{n}\right\} - 6|a_3| \left[(R-1)F_R\right. \\
& \left. - \left(\frac{(R^3-1)-3(R-1)}{6}\right)\right], \ n=4, \qquad (28) \\
M(P,R) \leqslant & \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1}|P(z)| - \frac{|a_1|}{2} \\
& \times \left\{\frac{(R^n-1)-n(R-1)}{n}\right\} - \frac{|a_2|}{n}(R-1)^n, \ n=3. \qquad (29)
\end{aligned}
$$

where $C_t(R)$, D_R and F_R are as defined in Lemma 4.

Proof of Lemma 6. We assume that $P(z)$ is a polynomial of degree $n \geqslant 5$. For each θ, $0 \leqslant \theta < 2\pi$ and for $R \geqslant 1$, we have

$$
|P(Re^{i\theta}) - P(e^{i\theta})| \leqslant \int_1^R |P'(re^{i\theta})| dr. \qquad (30)
$$

Since $P'(z)$ is a polynomial of degree $(n-1) \geqslant 4$. Using inequality (16) of Lemma 4 in (30), we get

$$
\begin{aligned}
& |P(Re^{i\theta}) - P(e^{i\theta})| \\
& \leqslant \max_{|z|=1}|P'(z)| \int_1^R r^{n-1} dr - \frac{2|a_1|}{(n+1)} \int_1^R (r^{n-1}-1) dr \\
& \quad - 6|a_3| \left\{\int_1^R (C_{n-3}(R) - C_{n-5}(R)) dr - \int_1^R \left(\frac{r^{n-2}-1}{n-2} - \frac{r^{n-4}-1}{n-4}\right) dr\right\}, \\
& = \max_{|z|=1}|P'(z)| \left(\frac{R^n-1}{n}\right) - \frac{2|a_1|}{(n+1)} \left\{\frac{(R^n-1)-n(R-1)}{n}\right\} \\
& \quad - 6|a_3| \left[(R-1)(C_{n-3}(R) - C_{n-5}(R)) - \left\{\left(\frac{(R^{n-1}-1)-(n-1)(R-1)}{(n-1)(n-2)}\right)\right.\right. \\
& \quad \left.\left. - \left(\frac{(R^{n-3}-1)-(n-3)(R-1)}{(n-3)(n-4)}\right)\right\}\right]. \qquad (31)
\end{aligned}
$$

Using Lemma 5 in (31), we get

$$\begin{aligned}
&|P(Re^{i\theta})-P(e^{i\theta})|\\
&\leqslant\left(\frac{R^n-1}{2}\right)\left\{\max_{|z|=1}|P(z)|-\min_{|z|=1}|P(z)|\right\}-\frac{2|a_1|}{(n+1)}\left\{\frac{(R^n-1)-n(R-1)}{n}\right\}\\
&\quad-6|a_3|\left[(R-1)(C_{n-3}(R)-C_{n-5}(R))-\left\{\left(\frac{(R^{n-1}-1)-(n-1)(R-1)}{(n-1)(n-2)}\right)\right.\right.\\
&\quad\left.\left.-\left(\frac{(R^{n-3}-1)-(n-3)(R-1)}{(n-3)(n-4)}\right)\right\}\right].
\end{aligned}$$

Hence on $|z|=R\geqslant 1$, we have

$$\begin{aligned}
&|P(Re^{i\theta})|\\
&\leqslant\left(\frac{R^n+1}{2}\right)\max_{|z|=1}|P(z)|-\left(\frac{R^n-1}{2}\right)\min_{|z|=1}|P(z)|\\
&\quad-\frac{2|a_1|}{(n+1)}\left\{\frac{(R^n-1)-n(R-1)}{n}\right\}-6|a_3|\left[(R-1)(C_{n-3}(R)-C_{n-5}(R))\right.\\
&\quad\left.-\left\{\left(\frac{(R^{n-1}-1)-(n-1)(R-1)}{(n-1)(n-2)}\right)-\left(\frac{(R^{n-3}-1)-(n-3)(R-1)}{(n-3)(n-4)}\right)\right\}\right],
\end{aligned}$$

which prove the proof of inequality (26).

The proof of inequalities (27), (28) and (29) follow on the same lines as that of inequality (26), but instead of using inequality (16), we use respectively inequalities (17), (18) and (19) of Lemma 4. We omit the details. This completes the proof of Lemma 6. □

LEMMA 7. *If $P(z)$ is a polynomial of degree n, then on $|z|=1$*

$$|P'(z)|+|Q'(z)|\leqslant n\max_{|z|=1}|P(z)|.$$

The above result is due to Govil and Rahman [13].

3. Main result

We begin by presenting the following extension of inequality (8) to the polar derivative by considering the locations of all the zeros and some coefficients of the polynomial at the same time, our result sharpens inequality (8).

THEOREM 1. *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu=a_n\prod_{j=1}^{n}(z-z_j)$, a_0, $a_n\neq 0$, is a polynomial of degree $n\geqslant 3$ having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then for every complex number α*

with $|\alpha| \geqslant k$

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant \frac{2(|\alpha|-k)}{k^n+1}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right)\left[\max_{|z|=1}|P(z)|+\frac{k^n-1}{2k^n}\min_{|z|=k}|P(z)|\right.$$
$$\left.+\frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\}+\frac{6|a_{n-3}|}{k^3}\phi(k)\right]$$
$$+\frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|na_0+\alpha a_1|+\frac{1}{k^{n-1}}|2(n-2)a_1+6\alpha a_3|\psi(k),\ n\geqslant 4, \tag{32}$$

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant \frac{2(|\alpha|-k)}{k^n+1}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right)\left[\max_{|z|=1}|P(z)|+\frac{k^n-1}{2k^n}\min_{|z|=k}|P(z)|\right.$$
$$\left.+\frac{|a_{n-1}|}{2k}\left\{\frac{(k^n-1)-n(k-1)}{n}\right\}+\frac{|a_{n-2}|}{k^2}\frac{(k-1)^n}{n}\right]$$
$$+\frac{(k^{n-1}-1)}{2k^{n-1}}|na_0+\alpha a_1|+\frac{(k-1)^{n-1}}{2k^{n-1}}|(n-1)a_1+2\alpha a_2|,\ n=3, \tag{33}$$

where

$$\phi(k)=\begin{cases}(k-1)(C_{n-3}(k)-C_{n-5}(k))-\left\{\left(\frac{(k^{n-1}-1)-(n-1)(k-1)}{(n-1)(n-2)}\right)\right. \\ \qquad \left.-\left(\frac{(k^{n-3}-1)-(n-3)(k-1)}{(n-3)(n-4)}\right)\right\}, & \text{if } n\geqslant 6, \quad (34)\\ (k-1)D_k-\left\{\left(\frac{(k^4-1)-4(k-1)}{12}\right)\right. \\ \qquad \left.-\left(\frac{(k^3-1)-3(k-1)}{6}\right)\right\}, & \text{if } n=5, \quad (35)\\ (k-1)F_k-\left(\frac{(k^3-1)-3(k-1)}{6}\right), & \text{if } n=4, \quad (36)\end{cases}$$

$$\psi(k)=\begin{cases}(C_{n-3}(k)-C_{n-5}(k))-\left(\frac{(k^{n-2}-1)}{n-2}-\frac{(k^{n-4}-1)}{n-4}\right), & \text{if } n\geqslant 6, \ (37)\\ D_k-\left(\frac{k^3-1}{3}-\frac{k^2-1}{2}\right), & \text{if } n=5, \ (38)\\ F_k-\left(\frac{k^2-1}{2}\right), & \text{if } n=4, \ (39)\end{cases}$$

$C_t(k)$, D_k *and* F_k *are as defined in Lemma 4.*

Proof of Theorem 1. First, suppose that $P(z)$ is a polynomial of degree $n \geqslant 6$. Since the polynomial $P(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, the polynomial $T(z) = P(kz)$ has all its zeros in $|z| \leqslant 1$. Applying inequality (9) of Lemma 1 to the polynomial $T(z)$, we have for $|\frac{\alpha}{k}| \geqslant 1$

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}T(z)| \geqslant \left(\frac{|\alpha|}{k}-1\right)\sum_{j=1}^{n}\frac{1}{1+\frac{|z_j|}{k}}\max_{|z|=1}|T(z)|,$$

or

$$\max_{|z|=1}\left|nP(kz)+\left(\frac{\alpha}{k}-z\right)kP'(kz)\right| \geqslant \left(\frac{|\alpha|}{k}-1\right)\sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=1}|P(kz)|,$$

which is equivalent to

$$\max_{|z|=k}\left|nP(z)+\left(\frac{\alpha}{k}-\frac{z}{k}\right)kP'(z)\right| \geqslant \left(\frac{|\alpha|}{k}-1\right)\sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=k}|P(z)|,$$

or

$$\max_{|z|=k}|D_\alpha P(z)| \geqslant \left(\frac{|\alpha|-k}{k}\right)\sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=k}|P(z)|. \tag{40}$$

Since the polynomial $P(z)$ is of degree $n \geqslant 6$, $D_\alpha P(z)$ is a polynomial of degree $(n-1) \geqslant 5$. Thus, applying inequality (16) of Lemma 4 to $D_\alpha P(z)$ with $R = k \geqslant 1$, we get

$$\begin{aligned}\max_{|z|=k}|D_\alpha P(z)| \leqslant k^{n-1}\max_{|z|=1}|D_\alpha P(z)| &- \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1| \\ &- |2(n-2)a_1+6\alpha a_3|\Bigg\{(C_{n-3}(k)-C_{n-5}(k)) \\ &- \left(\frac{k^{n-2}-1}{n-2}-\frac{k^{n-4}-1}{n-4}\right)\Bigg\}.\end{aligned} \tag{41}$$

Combining (40) and (41), gives

$$\begin{aligned}k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1| - |2(n-2)a_1+6\alpha a_3| \\ \times\left\{(C_{n-3}(k)-C_{n-5}(k)) - \left(\frac{k^{n-2}-1}{n-2}-\frac{k^{n-4}-1}{n-4}\right)\right\} \\ \geqslant \left(\frac{|\alpha|-k}{k}\right)\sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=k}|P(z)|.\end{aligned} \tag{42}$$

Let $q(z) = z^nP(\frac{1}{z})$. Since $P(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, it follows that the polynomial $q(\frac{z}{k})$ has all its zeros in $|z| \geqslant 1$ and is of degree $n \geqslant 6$. Applying inequality

(26) of Lemma 6 to $q(\frac{z}{k})$ for $R = k \geqslant 1$, we get

$$\begin{aligned}\max_{|z|=k}\left|q\left(\frac{z}{k}\right)\right| \leqslant \left(\frac{k^n+1}{2}\right)\max_{|z|=1}\left|q\left(\frac{z}{k}\right)\right| - \left(\frac{k^n-1}{2}\right)\min_{|z|=1}\left|q\left(\frac{z}{k}\right)\right| \\ - \frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\} - \frac{6|a_{n-3}|}{k^3} \\ \times\left[(k-1)(C_{n-3}(k)-C_{n-5}(k)) - \left\{\left(\frac{(k^{n-1}-1)-(n-1)(k-1)}{(n-1)(n-2)}\right)\right.\right. \\ \left.\left. - \left(\frac{(k^{n-3}-1)-(n-3)(k-1)}{(n-3)(n-4)}\right)\right\}\right],\end{aligned}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=1}|P(z)| \leqslant \left(\frac{k^n+1}{2k^n}\right)\max_{|z|=k}|P(z)| - \left(\frac{k^n-1}{2k^n}\right)\min_{|z|=k}|P(z)| \\ - \frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\} - \frac{6|a_{n-3}|}{k^3} \\ \times\left[(k-1)(C_{n-3}(k)-C_{n-5}(k)) - \left\{\left(\frac{(k^{n-1}-1)-(n-1)(k-1)}{(n-1)(n-2)}\right)\right.\right. \\ \left.\left. - \left(\frac{(k^{n-3}-1)-(n-3)(k-1)}{(n-3)(n-4)}\right)\right\}\right],\end{aligned}$$

which simplifies to

$$\begin{aligned}\max_{|z|=k}|P(z)| \geqslant \left(\frac{2k^n}{k^n+1}\right)\left[\max_{|z|=1}|P(z)| + \left(\frac{k^n-1}{2k^n}\right)\min_{|z|=k}|P(z)|\right. \\ + \frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\} + \frac{6|a_{n-3}|}{k^3} \\ \times\left[(k-1)(C_{n-3}(k)-C_{n-5}(k)) - \left\{\left(\frac{(k^{n-1}-1)-(n-1)(k-1)}{(n-1)(n-2)}\right)\right.\right. \\ \left.\left.\left. - \left(\frac{(k^{n-3}-1)-(n-3)(k-1)}{(n-3)(n-4)}\right)\right\}\right]\right]. \qquad (43)\end{aligned}$$

Combining (42) and (43), we get

$$\begin{aligned}k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1| - |2(n-2)a_1+6\alpha a_3| \\ \times\left\{(C_{n-3}(k)-C_{n-5}(k)) - \left(\frac{k^{n-2}-1}{n-2} - \frac{k^{n-4}-1}{n-4}\right)\right\}\end{aligned}$$

$$\geqslant \frac{2k^{n-1}(|\alpha|-k)}{k^n+1}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right)\left[\max_{|z|=1}|P(z)|+\left(\frac{k^n-1}{2k^n}\right)\min_{|z|=k}|P(z)|\right.$$
$$+\frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\}+\frac{6|a_{n-3}|}{k^3}$$
$$\times\left[(k-1)(C_{n-3}(k)-C_{n-5}(k))-\left\{\left(\frac{(k^{n-1}-1)-(n-1)(k-1)}{(n-1)(n-2)}\right)\right.\right.$$
$$\left.\left.\left.-\left(\frac{(k^{n-3}-1)-(n-3)(k-1)}{(n-3)(n-4)}\right)\right\}\right]\right],$$

which on simplification yields

$$\max_{|z|=1}|D_\alpha P(z)|$$
$$\geqslant \frac{2(|\alpha|-k)}{k^n+1}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right)\left[\max_{|z|=1}|P(z)|+\left(\frac{k^n-1}{2k^n}\right)\min_{|z|=k}|P(z)|\right.$$
$$+\frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\}+\frac{6|a_{n-3}|}{k^3}$$
$$\times\left[(k-1)(C_{n-3}(k)-C_{n-5}(k))-\left\{\left(\frac{(k^{n-1}-1)-(n-1)(k-1)}{(n-1)(n-2)}\right)\right.\right.$$
$$\left.\left.\left.-\left(\frac{(k^{n-3}-1)-(n-3)(k-1)}{(n-3)(n-4)}\right)\right\}\right]\right]$$
$$+\frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|na_0+\alpha a_1|+\frac{1}{k^{n-1}}|2(n-2)a_1+6\alpha a_3|$$
$$\times\left\{(C_{n-3}(k)-C_{n-5}(k))-\left(\frac{k^{n-2}-1}{n-2}-\frac{k^{n-4}-1}{n-4}\right)\right\},$$

which prove the desired result for $n\geqslant 6$.

The proof for the cases $n=5$, $n=4$ and $n=3$ follow on the same line as that of $n\geqslant 6$, but instead of using inequality (16) of Lemma 4 and (26) of Lemma 6, we use respectively inequalities (17), (18) and (19) of Lemma 4 and (27), (28) and (29) of Lemma 6. We omit the details. □

REMARK 1. Dividing both sides of inequalities (44) and (45) of Theorem 1 by $|\alpha|$ and taking $|\alpha|\to\infty$, we get the following result.

COROLLARY 1. *If* $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu=a_n\prod_{j=1}^{n}(z-z_j)$, a_0, $a_n\neq 0$, *is a polynomial* *of degree $n \geqslant 3$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=1}|P'(z)| \geqslant \frac{2}{k^n+1}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right)\left[\max_{|z|=1}|P(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|P(z)| + \frac{2|a_{n-1}|}{k}\left\{\frac{(k^n-1)-n(k-1)}{n(n+1)}\right\} + \frac{6|a_{n-3}|}{k^3}\phi(k)\right] + \frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|a_1| + \frac{6}{k^{n-1}}|a_3|\psi(k),\ n \geqslant 4, \tag{44}$$

$$\max_{|z|=1}|P'(z)| \geqslant \frac{2}{k^n+1}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right)\left[\max_{|z|=1}|P(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|P(z)| + \frac{|a_{n-1}|}{2k}\left\{\frac{(k^n-1)-n(k-1)}{n}\right\} + \frac{|a_{n-2}|}{k^2}\frac{(k-1)^n}{n}\right] + \frac{(k^{n-1}-1)}{2k^{n-1}}|a_1| + \frac{(k-1)^{n-1}}{k^{n-1}}|a_2|,\ n=3, \tag{45}$$

where $\phi(k)$ and $\psi(k)$ are as defined in Theorem 1.

As an application of Theorem 1, we prove the following improved Bernstein type inequality for polynomials having all its zeros in $|z| \geqslant k$, $k \leqslant 1$.

THEOREM 2. *Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{j=1}^{n}(z-z_j)$, a_0, $a_n \neq 0$, be a polynomial of degree $n \geqslant 3$ having all its zeros in $|z| \geqslant k$, $k \leqslant 1$. If $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point on $|z|=1$, then*

$$\max_{|z|=1}|P'(z)| \leqslant \left[n - \frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\right]\max_{|z|=1}|P(z)| - \frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\left[\frac{(1-k^n)}{2k^n}\min_{|z|=k}|P(z)| + \frac{2|a_1|}{k^{n-1}}\left\{\frac{(1-k^n)-n(1-k)k^{n-1}}{n(n+1)}\right\} + 6|a_3|k^3\phi\left(\frac{1}{k}\right)\right] - \frac{2\left(1-k^{n-1}\right)}{(n+1)}|a_{n-1}| - 6|a_{n-3}|k^{n-1}\psi\left(\frac{1}{k}\right),\ n \geqslant 4, \tag{46}$$

and

$$\max_{|z|=1}|P'(z)| \leqslant \left[n - \frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\right]\max_{|z|=1}|P(z)| - \frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\left[\frac{(1-k^n)}{2k^n}\min_{|z|=k}|P(z)|\right.$$

$$+\frac{|a_1|}{2k^{n-1}}\left\{\frac{(1-k^n)-n(1-k)k^{n-1})}{n(n+1)}\right\}+\frac{|a_2|}{k^{n-2}}\frac{(1-k)^n}{n}\Bigg]$$
$$-\frac{(1-k^{n-1})}{2}|a_{n-1}|-(1-k)^{n-1}|a_{n-2}|,\ n=3, \tag{47}$$

where $\phi(k)$ *and* $\psi(k)$ *are as defined in Theorem* 1.

Proof of Theorem 2. Since $P(z)$ is a polynomial of degree $n \geqslant 4$ and has all its zeros in $|z| \geqslant k$, $k \leqslant 1$, the polynomial

$$Q(z) = z^n P(\frac{1}{\bar{z}}) = \bar{a}_n + \bar{a}_{n-1} + \ldots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n = \bar{a}_n \prod_{j=1}^{n}(1 - z\bar{z}_j)$$

has all its zeros in $|z| \leqslant \frac{1}{k}$, $\frac{1}{k} \geqslant 1$. Applying inequality (44) of Corollary 1 to $Q(z)$ and using the facts that $\max\limits_{|z|=1}|P(z)| = \max\limits_{|z|=1}|Q(z)|$ and $\min\limits_{|z|=\frac{1}{k}}|Q(z)| = \dfrac{1}{k^n}\min\limits_{|z|=k}|P(z)|$, we have

$$\max_{|z|=1}|Q'(z)| \geqslant \frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\Bigg[\max_{|z|=1}|P(z)| + \frac{(1-k^n)}{2k^n}\min_{|z|=k}|P(z)|$$
$$+\frac{2|a_1|}{k^{n-1}}\left\{\frac{(1-k^n)-n(1-k)k^{n-1})}{n(n+1)}\right\}+6|a_3|k^3\phi\left(\frac{1}{k}\right)\Bigg]$$
$$+\frac{2\left(1-k^{n-1}\right)}{(n+1)}|a_{n-1}|+6|a_{n-3}|k^{n-1}\psi\left(\frac{1}{k}\right). \tag{48}$$

By Lemma 7, we have, on $|z| = 1$

$$|P'(z)| + |Q'(z)| \leqslant n\max_{|z|=1}|P(z)|. \tag{49}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point, then

$$\max_{|z|=1}\left\{|P'(z)| + |Q'(z)|\right\} = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{50}$$

Combining (48), (49) and (50), yields

$$\max_{|z|=1}|P'(z)| + \frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\Bigg[\max_{|z|=1}|P(z)| + \frac{(1-k^n)}{2k^n}\min_{|z|=k}|P(z)|$$
$$+\frac{2|a_1|}{k^{n-1}}\left\{\frac{(1-k^n)-n(1-k)k^{n-1})}{n(n+1)}\right\}+6|a_3|k^3\phi\left(\frac{1}{k}\right)\Bigg]$$
$$+\frac{2\left(1-k^{n-1}\right)}{(n+1)}|a_{n-1}|+6|a_{n-3}|k^{n-1}\psi\left(\frac{1}{k}\right)$$
$$\leqslant n\max_{|z|=1}|P(z)|, \tag{51}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=1}|P'(z)| \leqslant & \left[n-\frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\right]\max_{|z|=1}|P(z)| \\ & -\frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{|z_j|}{k+|z_j|}\right)\left[\frac{(1-k^n)}{2k^n}\min_{|z|=k}|P(z)|\right. \\ & \left.+\frac{2|a_1|}{k^{n-1}}\left\{\frac{(1-k^n)-n(1-k)k^{n-1}}{n(n+1)}\right\}+6|a_3|k^3\phi\left(\frac{1}{k}\right)\right] \\ & -\frac{2\left(1-k^{n-1}\right)}{(n+1)}|a_{n-1}|-6|a_{n-3}|k^{n-1}\psi\left(\frac{1}{k}\right),\end{aligned}$$

which prove the theorem for $n \geqslant 4$.

The proof for the case $n=3$ follows on the same line as that of $n \geqslant 4$, but instead of using inequality (44), we use inequality (45) of Corollary 1. We omit the details. □

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