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IMPROVEMENTS OF A–NUMERICAL RADIUS FOR SEMI–HILBERTIAN SPACE OPERATORS

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Abstract. Let A be a bounded positive operator on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. The semi-product $\langle x,y\rangle_A := \langle Ax,y\rangle, x,y\in H$, induces a semi-norm $\|\cdot\|_A$ on H. Let $\omega_A(T)$ and $\|T\|_A$ denote the A-numerical radius and the A-operator semi-norm of an operator T in semi-Hilbertian space $(H,\langle\cdot,\cdot\rangle_A)$, respectively. In this paper, some new bounds for the A-numerical radius of operators in semi-inner product space induced by A are derived. In particular, for $T\in\mathcal{B}_A(H)$ and $\alpha\geqslant 0$, we prove that

$$\omega_A^4(T)\leqslant\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\,\omega_A(T^2)$$

and

$$\omega_A^4(T)\leqslant\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(T^2).$$

It is worth noting that our results improve the existing A-numerical radius inequalities. Further, we also give a refinement inequality of A-operator semi-norm triangle inequality.

1. Introduction

Let H be a nontrivial complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{B}(H)$ denote the C^*-algebra of all bounded linear operators acting on H, and I stand for the identity operator on H. For any $T\in\mathcal{B}(H)$, the range, the null space and the adjoint of T are, respectively, denoted by $R(T)$, $N(T)$ and T^*. If M is a closure linear subspace of H, then P_M stands for the orthogonal projection onto M. Also $\overline{M}$ is the closure of linear subspace M with respect to the norm topology of H.

We assume operator $A\in\mathcal{B}(H)$ is positive in this paper, i.e. $\langle Ax,x\rangle\geqslant 0$ for any $x\in H$. The positive operator A induces the semi-inner product as $\langle x,y\rangle_A=\langle Ax,y\rangle$ for $x,y\in H$. $\|\cdot\|_A$ denotes the semi-norm on H, which satisfies $\|x\|_A=\sqrt{\langle x,x\rangle_A}$. For more properties of semi-norm $\|\cdot\|_A$, we refer readers to [3,4].

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For $T \in \mathcal{B}(H)$, $\|T\|_A$ stands for the A-operator semi-norm of T, and it is defined as

$$\|T\|_A := \sup_{\substack{x \in \overline{R(A)} \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A}.$$

Set $\mathcal{B}^A(H) = \{T \in \mathcal{B}(H) : \exists \lambda > 0, \|Tx\|_A \leqslant \lambda \|x\|_A, \forall x \in \overline{R(A)}\}$. It is easy to see that $\|T\|_A < \infty$ if and only if $T \in \mathcal{B}^A(H)$. It may happen that $\|T\|_A = +\infty$ for some $T \in \mathcal{B}(H)$, see [17]. Furthermore, some properties of the A-operator semi-norm $\|\cdot\|_A$ were studied in [3]. One of them gives a characterization for an operator T to be in $\mathcal{B}^A(H)$ as follows: if $T \in \mathcal{B}^A(H)$, then $A^{\frac{1}{2}}T(A^{\frac{1}{2}})^\dagger$ is a bounded operator. And it holds

$$\|T\|_A = \left\|A^{\frac{1}{2}}T(A^{\frac{1}{2}})^\dagger\right\| = \left\|\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^\dagger}\right\|, \tag{1.1}$$

where $\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^\dagger}$ is the unique bounded linear extension of $A^{\frac{1}{2}}T(A^{\frac{1}{2}})^\dagger$ to $\mathcal{B}(H)$.

An operator $S \in \mathcal{B}(H)$ is called an A-adjoint of T if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for any $x, y \in H$, that is $AS = T^*A$. Note that, an operator $T \in \mathcal{B}(H)$ may admit none, one or many A-adjoints. The set of all operators that admit A-adjoint is denoted by $\mathcal{B}_A(H)$. By Douglas theorem [12], it holds

$$\mathcal{B}_A(H) = \{T \in \mathcal{B}(H) : R(T^*A) \subseteq R(A)\}.$$

For $T \in \mathcal{B}_A(H)$, the solution of $AX = T^*A$ is unique and the unique solution is denoted by $T^{\sharp_A}$. Moreover, $T^{\sharp_A} = A^\dagger T^* A$, $N(T^{\sharp_A}) = N(T^*A)$ and $R(T^{\sharp_A}) \subseteq \overline{R(A)}$, where $A^\dagger$ is the Moore-Penrose inverse of A.

Recall that the set of all operators admitting $A^{\frac{1}{2}}$-adjoint is denoted by $\mathcal{B}_{A^{1/2}}(H)$, operator in $\mathcal{B}_{A^{1/2}}(H)$ is also called A-bounded operator. It could be deduced by Douglas theorem that

$$\mathcal{B}_{A^{1/2}}(H) = \{T \in \mathcal{B}(H) : \exists \lambda > 0, \|Tx\|_A \leqslant \lambda \|x\|_A, \forall x \in H\}.$$

If $T \in \mathcal{B}_{A^{1/2}}(H)$, we have $TN(A) \subseteq N(A)$. $\mathcal{B}_A(H)$ and $\mathcal{B}_{A^{1/2}}(H)$ are two subalgebras of $\mathcal{B}(H)$ and satisfy $\mathcal{B}_A(H) \subseteq \mathcal{B}_{A^{1/2}}(H) \subseteq \mathcal{B}^A(H)$, see [2, 4]. This indicates that if T admits A-adjoint, then T is A-bounded operator. For the A-operator semi-norm of A-bounded operators, it was proved in [13] that

$$\begin{aligned} \|T\|_A &= \sup\{\|Tx\|_A : x \in H, \|x\|_A = 1\} \\ &= \sup\{|\langle Tx, y\rangle_A| : x, y \in H, \|x\|_A = \|y\|_A = 1\}. \end{aligned}$$

Let $T, S \in \mathcal{B}_A(H)$, the following properties of A-operator semi-norm $\|\cdot\|_A$ are hold:

(1) $\|T^{\sharp_A}\|_A = \|T\|_A$, $\|TT^{\sharp_A}\|_A = \|T^{\sharp_A}T\|_A = \|T^{\sharp_A}\|_A^2 = \|T\|_A^2$;

(2) $\|Tx\|_A \leqslant \|T\|_A \|x\|_A$ for any $x \in H$;

(3) $\|ST\|_A \leqslant \|S\|_A \|T\|_A$.

In particular, an operator $T \in \mathcal{B}(H)$ is called A-selfadjoint if AT is selfadjoint. An operator $T \in \mathcal{B}(H)$ is A-positive if AT is positive. Obviously, an A-positive operator is always an A-selfadjoint operator and A-selfadjoint operators are always in $\mathcal{B}_A(H)$. It was shown in [18] that T is A-positive if and only if $\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}}$ is a positive operator. And it should be noted that $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are both A-positive. In addition, any operator $T \in \mathcal{B}_A(H)$ can be always represented as $T = B + \mathrm{i}C$, where

$$B = \frac{T + T^{\sharp_A}}{2} \quad \text{and} \quad C = \frac{T - T^{\sharp_A}}{2\mathrm{i}}.$$

It is worth noting that B and C are all A-selfadjoint operators.

Recently, the numerical radius was extended to the semi-inner product space induced by positive operator A, which is called A-numerical radius. It was introduced in [26] as

$$\omega_A(T) := \sup\{|\langle Tx, x\rangle_A| : x \in H, \|x\|_A = 1\}.$$

It should be mentioned that it may happen $\omega_A(T) = +\infty$ for some $T \in \mathcal{B}(H)$, see [22]. However, $\omega_A(T)$ defines a semi-norm on $\mathcal{B}_A(H)$ which is equivalent to $\|T\|_A$. More precisely, for $T \in \mathcal{B}_A(H)$, it holds

$$\frac{1}{2}\|T\|_A \leqslant \omega_A(T) \leqslant \|T\|_A. \tag{1.2}$$

Recently, several refinements of the inequalities in (1.2) have been proved by many researchers. For example, it was shown by the authors in [15, 28] that for $T \in \mathcal{B}_A(H)$, then

$$\frac{1}{4}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A \leqslant \omega_A^2(T) \leqslant \frac{1}{2}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A. \tag{1.3}$$

Moreover, a refinement of the second inequality in (1.3) was also proved in [24], which asserts

$$\omega_A^4(T) \leqslant \frac{3}{16}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A^2 + \frac{1}{8}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A \omega_A(T^2). \tag{1.4}$$

For an account of the recent results for $\omega_A(\cdot)$, we refer the readers to [1, 7, 8, 10, 14, 19–21] and the references therein.

In this paper, the main task is to derive several refinements of the inequalities (1.3) and (1.4). The structure is as follows. In section 2, the preliminary lemmas of this paper are shown. In section 3, we present some upper bounds for A-numerical radius of semi-Hilbertian space operators. In particular, for $T \in \mathcal{B}_A(H)$ and $\alpha \geqslant 0$, we prove that

$$\omega_A^4(T) \leqslant \frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \omega_A(T^2)$$

and

$$\omega_A^4(T) \leqslant \frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^2(T^2).$$

In section 4, some lower bounds of A-numerical radius are also obtained. Particularly, if $T \in \mathcal{B}_A(H)$, we show that

$$\begin{aligned}\frac{1}{4}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A \leqslant& \frac{1}{8}(\|T + T^{\sharp_A}\|_A^2 + \|T - T^{\sharp_A}\|_A^2) + \frac{1}{16}\Big|\|T + T^{\sharp_A}\|_A^2 - \|T - T^{\sharp_A}\|_A^2\Big| \\ \leqslant& \omega_A^2(T).\end{aligned}$$

In section 5, an inequality of A-operator semi-norm for $T+S$ is given. Namely, if $T,S \in \mathcal{B}_A(H)$,

$$\|T+S\|_A^2 \leqslant \|T^{\sharp_A}T+S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A}+SS^{\sharp_A}\|_A^{\frac{1}{2}}+\|T\|_A\|S\|_A+\omega_A(S^{\sharp_A}T).$$

It should be mentioned that the numerical radius inequalities of this paper improve the existing ones in [10, 24, 28], and the A-operator semi-norm inequality for the sum of two operators refines the triangle inequality. Moreover, if taking $A=I$, we would obtain the refined bounds of classical numerical radius and classical norm.

2. Preliminaries

To prove the results of this paper, we need the following lemmas. The first lemma is established in [27].

LEMMA 2.1. *Let $a,b,e \in H$ with $\|e\|_A = 1$, then*

$$|\langle a,e\rangle_A\langle e,b\rangle_A| \leqslant \frac{1}{2}\left(\|a\|_A\|b\|_A+|\langle a,b\rangle_A|\right).$$

LEMMA 2.2. *Let $a,b,e \in H$ with $\|e\|_A = 1$ and $\alpha \geqslant 0$. Then*

$$|\langle a,e\rangle_A\langle e,b\rangle_A|^2 \leqslant \frac{1}{4}\left(\frac{1+2\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2+\frac{3+2\alpha}{1+\alpha}\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|\right).$$

Proof. By the Cauchy-Schwarz inequality, it holds

$$\begin{aligned}|\langle a,b\rangle_A|^2 \leqslant&\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|\\ \leqslant&\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|+\alpha\left(\|a\|_A^2\|b\|_A^2-|\langle a,b\rangle_A|^2\right).\end{aligned}$$

Thus, we can deduce that

$$|\langle a,b\rangle_A|^2 \leqslant \frac{\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2+\frac{1}{1+\alpha}\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|. \tag{2.1}$$

Moreover, Lemma 2.1 yields

$$|\langle a,e\rangle_A\langle e,b\rangle_A|^2 \leqslant \frac{1}{4}\left(\|a\|_A^2\|b\|_A^2+2\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|+|\langle a,b\rangle|_A^2\right).$$

Together with inequality (2.1), it can be established that

$$\begin{aligned}&|\langle a,e\rangle_A\langle e,b\rangle_A|^2\\ \leqslant&\frac{1}{4}\left(\|a\|_A^2\|b\|_A^2+2\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|+|\langle a,b\rangle|_A^2\right)\\ \leqslant&\frac{1}{4}\left(\|a\|_A^2\|b\|_A^2+2\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|+\frac{\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2+\frac{1}{1+\alpha}\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|\right)\\ =&\frac{1}{4}\left(\frac{1+2\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2+\frac{3+2\alpha}{1+\alpha}\|a\|_A\|b\|_A\Big|\langle a,b\rangle_A\Big|\right).\end{aligned}$$

This completes the proof. □

REMARK 2.1. It was shown in [24] that for any $a,b,e \in H$ with $\|e\|_A = 1$, it holds

$$|\langle a,e\rangle_A\langle e,b\rangle_A|^2 \leqslant \frac{1}{4}\left(3\|a\|_A^2\|b\|_A^2 + \|a\|_A\|b\|_A\left|\langle a,b\rangle_A\right|\right). \tag{2.2}$$

Note that Lemma 2.2 is sharper than the inequality (2.2). As a matter of fact,

$$\begin{aligned}
&|\langle a,e\rangle_A\langle e,b\rangle_A|^2\\
\leqslant&\frac{1}{4}\left(\frac{1+2\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \frac{3+2\alpha}{1+\alpha}\|a\|_A\|b\|_A\left|\langle a,b\rangle_A\right|\right)\\
\leqslant&\frac{1}{4}\left(\frac{1+2\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \frac{2+\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \|a\|_A\|b\|_A\left|\langle a,b\rangle_A\right|\right)\\
=&\frac{1}{4}\left(3\|a\|_A^2\|b\|_A^2 + \|a\|_A\|b\|_A\left|\langle a,b\rangle_A\right|\right).
\end{aligned}$$

LEMMA 2.3. *Let $a,b,e \in H$ with $\|e\|_A = 1$ and $\alpha \geqslant 0$. Then*

$$|\langle a,e\rangle_A\langle e,b\rangle_A|^2 \leqslant \frac{1}{2}\left(\frac{1+2\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \frac{1}{1+\alpha}|\langle a,b\rangle_A|^2\right).$$

Proof. By similar discussion with Lemma 2.2, we have

$$|\langle a,b\rangle_A|^2 \leqslant |\langle a,b\rangle_A|^2 + \alpha\left(\|a\|_A^2\|b\|_A^2 - |\langle a,b\rangle_A|^2\right).$$

This indicates that

$$|\langle a,b\rangle_A|^2 \leqslant \frac{\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \frac{1}{1+\alpha}|\langle a,b\rangle_A|^2. \tag{2.3}$$

On the other hand, combine Lemma 2.1 and the convexity of the function $f(t) = t^2$, it can be obtained

$$\begin{aligned}
|\langle a,e\rangle_A\langle e,b\rangle_A|^2 &\leqslant \left(\frac{(\|a\|_A\|b\|_A + |\langle a,b\rangle_A|)}{2}\right)^2\\
&\leqslant \frac{1}{2}(\|a\|_A^2\|b\|_A^2 + |\langle a,b\rangle_A|^2).
\end{aligned} \tag{2.4}$$

Then, according to the inequalities (2.3) and (2.4), one has

$$\begin{aligned}
|\langle a,e\rangle_A\langle e,b\rangle_A|^2 &\leqslant \frac{1}{2}(\|a\|_A^2\|b\|_A^2 + |\langle a,b\rangle_A|^2)\\
&\leqslant \frac{1}{2}(\|a\|_A^2\|b\|_A^2 + \frac{\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \frac{1}{1+\alpha}|\langle a,b\rangle_A|^2)\\
&= \frac{1}{2}\left(\frac{1+2\alpha}{1+\alpha}\|a\|_A^2\|b\|_A^2 + \frac{1}{1+\alpha}|\langle a,b\rangle_A|^2\right).
\end{aligned}$$

This completes the proof. □

REMARK 2.2. The inequality in Lemma 2.3 is sharper than the inequality (2.2) with $\alpha = 1$. To see this, note that

$$\begin{aligned} |\langle a,e\rangle_A \langle e,b\rangle_A|^2 &\leqslant \frac{1}{2}\left(\frac{3}{2}\|a\|_A^2\|b\|_A^2 + \frac{1}{2}|\langle a,b\rangle_A|^2\right) \\ &= \frac{1}{4}\left(3\|a\|_A^2\|b\|_A^2 + |\langle a,b\rangle_A|^2\right) \\ &\leqslant \frac{1}{4}\left(3\|a\|_A^2\|b\|_A^2 + \|a\|_A\|b\|_A\left|\langle a,b\rangle_A\right|\right). \end{aligned}$$

The following lemma can be found in [24].

LEMMA 2.4. *Let $T \in \mathcal{B}_A(H)$. Then for any $x,y \in H$ with $\|x\|_A = \|y\|_A = 1$, we have*

$$|\langle Tx,y\rangle_A|^2 \leqslant \sqrt{\langle T^{\sharp_A}Tx,x\rangle_A}\sqrt{\langle TT^{\sharp_A}y,y\rangle_A}.$$

The next lemma is proved in [4].

LEMMA 2.5. *Let $T \in \mathcal{B}_A(H)$. Then, $T = T^{\sharp_A}$ if and only if T is an A-selfadjoint operator and $R(T) \subseteq \overline{R(A)}$.*

REMARK 2.3. By Lemma 2.5, it can be deduced that $(T^{\sharp_A}T)^{\sharp_A} = T^{\sharp_A}T$.

LEMMA 2.6. *Let $T,S \in \mathcal{B}(H)$ be A-positive operators. Then*

$$\|T+S\|_A \leqslant \max\{\|T\|_A, \|S\|_A\} + \|TS\|_A^{\frac{1}{2}}.$$

Proof. In [11], it was proved that if B and C are positive operators on a Hilbert space, then

$$\|B+C\| \leqslant \max\{\|B\|, \|C\|\} + \|BC\|^{\frac{1}{2}}.$$

Since T and S are A-positive, we have $\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}}$ and $\overline{A^{\frac{1}{2}}S(A^{\frac{1}{2}})^{\dagger}}$ are positive operators. By equation (1.1) and $TN(A) \subseteq N(A)$, one gives

$$\begin{aligned} \|T+S\|_A &= \left\|\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}} + \overline{A^{\frac{1}{2}}S(A^{\frac{1}{2}})^{\dagger}}\right\| \\ &\leqslant \max\left\{\left\|\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}}\right\|, \left\|\overline{A^{\frac{1}{2}}S(A^{\frac{1}{2}})^{\dagger}}\right\|\right\} + \left\|\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}A^{\frac{1}{2}}S(A^{\frac{1}{2}})^{\dagger}}\right\|^{\frac{1}{2}} \\ &= \max\left\{\left\|\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}}\right\|, \left\|\overline{A^{\frac{1}{2}}S(A^{\frac{1}{2}})^{\dagger}}\right\|\right\} + \left\|\overline{A^{\frac{1}{2}}TS(A^{\frac{1}{2}})^{\dagger}}\right\|^{\frac{1}{2}} \\ &= \max\{\|T\|_A, \|S\|_A\} + \|TS\|_A^{\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

3. Some upper bounds of A-numerical radius

The main goal of this section is to derive several upper bounds for A-numerical radius which are refinements of some existing ones.

THEOREM 3.1. *Let $T \in \mathcal{B}_A(H)$ and $\alpha \geqslant 0$. Then*

$$\omega_A^4(T) \leqslant \frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \omega_A(T^2).$$

Proof. Let $x \in H$ with $\|x\|_A = 1$. Then

$$\begin{aligned}
&|\langle Tx,x\rangle_A|^4\\
=&|\langle Tx,x\rangle_A \langle x, T^{\sharp_A}x\rangle_A|^2\\
\leqslant&\frac{1}{4}\left(\frac{1+2\alpha}{1+\alpha}\|Tx\|_A^2\|T^{\sharp_A}x\|_A^2 + \frac{3+2\alpha}{1+\alpha}\|Tx\|_A\|T^{\sharp_A}x\|_A\left|\langle Tx, T^{\sharp_A}x\rangle_A\right|\right) \quad \text{(Lemma 2.2)}\\
=&\frac{1+2\alpha}{4(1+\alpha)}\left(\sqrt{\langle T^{\sharp_A}Tx,x\rangle_A\langle TT^{\sharp_A}x,x\rangle_A}\right)^2\\
&+\frac{3+2\alpha}{4(1+\alpha)}\sqrt{\langle T^{\sharp_A}Tx,x\rangle_A\langle TT^{\sharp_A}x,x\rangle_A}\left|\langle Tx, T^{\sharp_A}x\rangle_A\right|\\
\leqslant&\frac{1+2\alpha}{16(1+\alpha)}\langle (T^{\sharp_A}T + TT^{\sharp_A})x,x\rangle_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\langle (T^{\sharp_A}T + TT^{\sharp_A}x,x\rangle_A\left|\langle T^2x,x\rangle_A\right|\\
&\text{(by the arithmetic-geometric mean inequality)}\\
\leqslant&\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A\omega_A(T^2).
\end{aligned}$$

Taking the supremum over $x \in H$ with $\|x\|_A = 1$, the result can be naturally established as

$$\omega_A^4(T) \leqslant \frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \omega_A(T^2).$$

This completes the proof. □

REMARK 3.1. Theorem 3.1 is a refinement of the inequality (1.4). Indeed, since $\omega_A(T^2) \leqslant \omega_A^2(T)$, we can deduce that

$$\begin{aligned}
&\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A\omega_A(T^2)\\
\leqslant&\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 + \frac{2+\alpha}{8(1+\alpha)}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A\omega_A^2(T)\\
&+\frac{1}{8}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A\omega_A(T^2)
\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{2+\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\\&\quad+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2)\\&=\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2).\end{aligned}$$

Thus

$$\begin{aligned}\omega_A^4(T)&\leqslant \frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2)\\&\leqslant \frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2).\end{aligned}$$

To show that Theorem 3.1 is a nontrivial improvement of the inequalities (1.3) and (1.4), we give the following example.

EXAMPLE 3.1. Let

$$T=\begin{bmatrix}0&2&0\\0&0&1\\0&0&0\end{bmatrix}\quad\text{and}\quad A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}.$$

Then by elementary calculations, we have

$$\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2)=\frac{4+3\sqrt{3}+4\alpha}{9(1+\alpha)}$$

and

$$\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2)=\frac{12+\sqrt{3}}{9}=\frac{(12+\sqrt{3})(1+\alpha)}{9(1+\alpha)}.$$

Thus

$$\begin{aligned}&\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2)\\&<\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2).\end{aligned}$$

THEOREM 3.2. *Let $T\in\mathcal{B}_A(H)$ and $\alpha\geqslant 0$. Then*

$$\omega_A^4(T)\leqslant \frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(T^2).$$

Proof. Let $x\in H$ with $\|x\|_A=1$. Then

$$\begin{aligned}&|\langle Tx,x\rangle_A|^4\\&=|\langle Tx,x\rangle_A\langle x,T^{\sharp_A}x\rangle_A|^2\\&\leqslant\frac{1}{2}\left(\frac{1+2\alpha}{1+\alpha}\|Tx\|_A^2\|T^{\sharp_A}x\|_A^2+\frac{1}{1+\alpha}|\langle Tx,T^{\sharp_A}x\rangle_A|^2\right)\quad\text{(Lemma 2.3)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1+2\alpha}{2(1+\alpha)}\left(\sqrt{\langle T^{\sharp_A}Tx,x\rangle_A\langle TT^{\sharp_A}x,x\rangle_A}\right)^2 + \frac{1}{2(1+\alpha)}|\langle Tx,T^{\sharp_A}x\rangle_A|^2\\
&\leqslant \frac{1+2\alpha}{8(1+\alpha)}\langle (T^{\sharp_A}T+TT^{\sharp_A})x,x\rangle_A^2 + \frac{1}{2(1+\alpha)}|\langle T^2x,x\rangle_A|^2\\
&\quad \text{(by the arithmetic-geometric mean inequality)}\\
&\leqslant \frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^2(T^2).
\end{aligned}
$$

Taking the supremum over $x \in H$ with $\|x\|_A = 1$, we deduce that

$$
\omega_A^4(T) \leqslant \frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^2(T^2).
$$

This completes the proof. □

REMARK 3.2. Theorem 3.2 is sharper than the inequality (1.3). As a matter of fact,

$$
\begin{aligned}
&\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^2(T^2)\\
\leqslant &\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^4(T)\\
\leqslant &\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\\
= &\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2.
\end{aligned}
$$

Now, we give an example to show that Theorem 3.2 is a nontrivial improvement of inequality (1.3).

EXAMPLE 3.2. Let T and A be the same as described in Example 3.1. Then it can be checked that

$$
\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^2(T^2) = \frac{19+32\alpha}{18(1+\alpha)}
$$

and

$$
\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 = \frac{16}{9} = \frac{32+32\alpha}{18(1+\alpha)}.
$$

Therefore

$$
\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2(1+\alpha)}\omega_A^2(T^2) < \frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2.
$$

REMARK 3.3. If taking $\alpha = 0$ in Theorem 3.2, we will obtain

$$
\omega_A^4(T) \leqslant \frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2 + \frac{1}{2}\omega_A^2(T^2),
$$

which is sharper than the inequality (1.4).

In fact

$$\begin{aligned}
&\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2}\omega_A^2(T^2)\\
\leqslant&\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{4}\omega_A^4(T)+\frac{1}{4}\omega_A^2(T)\omega_A(T^2)\\
\leqslant&\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\omega_A(T^2)\\
=&\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\omega_A(T^2).
\end{aligned}$$

Therefore, if taking $\alpha=0$, it holds

$$\begin{aligned}
\omega_A^4(T)\leqslant&\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2}\omega_A^2(T^2)\\
\leqslant&\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\omega_A(T^2).
\end{aligned}$$

The following example shows that Theorem 3.2 is a nontrivial improvement of inequality (1.4) with $\alpha=0$.

EXAMPLE 3.3. Let T and A be the same as the matrices in Example 3.1. Then it can be checked that

$$\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2}\omega_A^2(T^2)=\frac{19}{18}$$

and

$$\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\omega_A(T^2)=\frac{12+\sqrt{3}}{9}.$$

So

$$\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2}\omega_A^2(T^2)<\frac{3}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{8}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\omega_A(T^2).$$

The following two theorems give the new upper bounds for $\omega_A^4(S^{\sharp_A}T)$.

THEOREM 3.3. *Let $T,S\in\mathcal{B}_A(H)$ and $\alpha\geqslant 0$. Then*

$$\begin{aligned}
\omega_A^4(S^{\sharp_A}T)\leqslant&\frac{1+2\alpha}{16(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2\\
&+\frac{3+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T).
\end{aligned}$$

Proof. Let $x\in H$ with $\|x\|_A=1$. Then

$$\begin{aligned}
&4|\langle Wx,x\rangle_A\langle Rx,x\rangle_A|^2\\
\leqslant&\frac{1+2\alpha}{1+\alpha}\|Wx\|_A^2\|R^{\sharp_A}x\|_A^2+\frac{3+2\alpha}{1+\alpha}\|Wx\|_A\|R^{\sharp_A}x\|_A|\langle Wx,R^{\sharp_A}x\rangle_A|\quad\text{(by Lemma 2.2)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1+2\alpha}{1+\alpha}\left(\sqrt{\langle W^{\sharp_A}Wx,x\rangle_A\langle RR^{\sharp_A}x,x\rangle_A}\right)^2 \\
&\quad + \frac{3+2\alpha}{1+\alpha}\sqrt{\langle W^{\sharp_A}Wx,x\rangle_A\langle RR^{\sharp_A}x,x\rangle_A}\,|\langle RWx,x\rangle_A| \\
&\leqslant \frac{1+2\alpha}{4(1+\alpha)}\left(\langle W^{\sharp_A}Wx,x\rangle_A+\langle RR^{\sharp_A}x,x\rangle_A\right)^2 \\
&\quad + \frac{3+2\alpha}{2(1+\alpha)}\left(\langle W^{\sharp_A}Wx,x\rangle_A+\langle RR^{\sharp_A}x,x\rangle_A\right)|\langle RWx,x\rangle_A| \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1+2\alpha}{4(1+\alpha)}\langle (W^{\sharp_A}W+RR^{\sharp_A})x,x\rangle_A^2 + \frac{3+2\alpha}{2(1+\alpha)}\langle (W^{\sharp_A}W+RR^{\sharp_A})x,x\rangle_A|\langle RWx,x\rangle_A|.
\end{aligned}
$$

By replacing $W = T^{\sharp_A}T$ and $R = S^{\sharp_A}S$ in the above inequality, Remark 2.3 indicates $(T^{\sharp_A}T)^{\sharp_A} = T^{\sharp_A}T$ and $(S^{\sharp_A}S)^{\sharp_A} = S^{\sharp_A}S$. Thus, it can be deduced that

$$
\begin{aligned}
&|\langle T^{\sharp_A}Tx,x\rangle_A\langle S^{\sharp_A}Sx,x\rangle_A|^2 \\
&\leqslant \frac{1+2\alpha}{16(1+\alpha)}\langle\left[(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\right]x,x\rangle_A^2 \\
&\quad + \frac{3+2\alpha}{8(1+\alpha)}\langle\left[(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\right]x,x\rangle_A|\langle S^{\sharp_A}ST^{\sharp_A}Tx,x\rangle_A| \\
&\leqslant \frac{1+2\alpha}{16(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2 + \frac{3+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T).
\end{aligned}
$$

In addition, by utilizing the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|\langle x,S^{\sharp_A}Tx\rangle_A|^4 &= |\langle Tx,Sx\rangle_A|^4 \\
&\leqslant \|Tx\|_A^4\|Sx\|_A^4 \\
&= \langle Tx,Tx\rangle_A^2\langle Sx,Sx\rangle_A^2 \\
&= \langle x,T^{\sharp_A}Tx\rangle_A^2\langle x,S^{\sharp_A}Sx\rangle_A^2 \\
&= |\langle T^{\sharp_A}Tx,x\rangle_A\langle S^{\sharp_A}Sx,x\rangle_A|^2.
\end{aligned} \tag{3.1}
$$

Combining above two inequalities, we can obtain

$$
\begin{aligned}
|\langle x,S^{\sharp_A}Tx\rangle_A|^4 \leqslant &\frac{1+2\alpha}{16(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2 \\
&+ \frac{3+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T).
\end{aligned}
$$

Taking the supremum over $x \in H$ with $\|x\|_A = 1$ will produce

$$
\begin{aligned}
\omega_A^4(S^{\sharp_A}T) \leqslant &\frac{1+2\alpha}{16(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2 \\
&+ \frac{3+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T).
\end{aligned}
$$

This completes the proof. □

REMARK 3.4. In [24], H. Qiao et. al. proved that

$$\omega_A^4(S^{\sharp_A}T) \leqslant \frac{3}{16}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{8}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T). \quad (3.2)$$

It should be mentioned that Theorem 3.3 is sharper than the inequalities (3.2). To prove this assertion, we first to deduce that

$$\omega_A(S^{\sharp_A}ST^{\sharp_A}T) \leqslant \frac{1}{2}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A. \quad (3.3)$$

In fact, for $x \in H$ with $\|x\|_A = 1$, it holds

$$\begin{aligned}
|\langle S^{\sharp_A}ST^{\sharp_A}Tx,x\rangle_A| &= |\langle T^{\sharp_A}Tx, S^{\sharp_A}Sx\rangle_A| \\
&\leqslant \|T^{\sharp_A}Tx\|_A\|S^{\sharp_A}Sx\|_A \\
&= \sqrt{\langle T^{\sharp_A}Tx, T^{\sharp_A}Tx\rangle_A\langle S^{\sharp_A}Sx, S^{\sharp_A}Sx\rangle_A} \\
&\leqslant \frac{1}{2}\langle\left[(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\right]x,x\rangle_A \\
&\leqslant \frac{1}{2}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A.
\end{aligned}$$

Thus, taking the supremum over $x \in H$ with $\|x\|_A = 1$ will obtain inequality (3.3). Then, with similar discussion of Remark 3.1 and using inequality (3.3), it can be established

$$\begin{aligned}
\omega_A^4(S^{\sharp_A}T) \leqslant& \frac{1+2\alpha}{16(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2 \\
&+\frac{3+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T) \\
\leqslant& \frac{3}{16}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{8}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T).
\end{aligned}$$

THEOREM 3.4. *Let $T,S \in \mathcal{B}_A(H)$ and $\alpha \geqslant 0$. Then*

$$\omega_A^4(S^{\sharp_A}T) \leqslant \frac{1+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(S^{\sharp_A}ST^{\sharp_A}T).$$

Proof. For $x \in H$ with $\|x\|_A = 1$, it can be established that

$$\begin{aligned}
&2|\langle Wx,x\rangle_A\langle Rx,x\rangle_A|^2 \\
\leqslant& \frac{1+2\alpha}{1+\alpha}\|Wx\|_A^2\|R^{\sharp_A}x\|_A^2+\frac{1}{1+\alpha}|\langle Wx,R^{\sharp_A}x\rangle_A|^2 \quad \text{(by Lemma 2.3)} \\
=& \frac{1+2\alpha}{1+\alpha}\left(\sqrt{\langle W^{\sharp_A}Wx,x\rangle_A\langle RR^{\sharp_A}x,x\rangle_A}\right)^2+\frac{1}{1+\alpha}|\langle RWx,x\rangle_A|^2
\end{aligned}$$

$$\leqslant \frac{1+2\alpha}{4(1+\alpha)}\left(\langle W^{\sharp_A}Wx,x\rangle_A+\langle RR^{\sharp_A}x,x\rangle_A\right)^2+\frac{1}{1+\alpha}|\langle RWx,x\rangle_A|^2$$

(by the arithmetic-geometric mean inequality)

$$=\frac{1+2\alpha}{4(1+\alpha)}\langle (W^{\sharp_A}W+RR^{\sharp_A})x,x\rangle_A^2+\frac{1}{1+\alpha}|\langle RWx,x\rangle_A|^2.$$

Let $W=T^{\sharp_A}T$ and $R=S^{\sharp_A}S$ in the above inequality, it yields

$$\begin{aligned}&|\langle T^{\sharp_A}Tx,x\rangle_A\langle S^{\sharp_A}Sx,x\rangle_A|^2\\ \leqslant&\frac{1+2\alpha}{8(1+\alpha)}\langle\left[(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\right]x,x\rangle_A^2+\frac{1}{2(1+\alpha)}|\langle S^{\sharp_A}ST^{\sharp_A}Tx,x\rangle_A|^2\\ \leqslant&\frac{1+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(S^{\sharp_A}ST^{\sharp_A}T).\end{aligned}$$

Together with inequality (3.1) and taking the supremum over $x\in H$ with $\|x\|_A=1$, the result will be deduced as

$$\omega_A^4(S^{\sharp_A}T)\leqslant\frac{1+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(S^{\sharp_A}ST^{\sharp_A}T).$$

This completes the proof. □

REMARK 3.5. It follows from Theorem 2.7 in [10] that if $T,S\in\mathcal{B}_A(H)$, then

$$\omega_A^2(S^{\sharp_A}T)\leqslant\frac{1}{2}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A. \tag{3.4}$$

Theorem 3.4 improves inequality (3.4). Consequently, by similar discussion of Remark 3.2 will deduce

$$\begin{aligned}\omega_A^2(S^{\sharp_A}T)\leqslant&\sqrt{\frac{1+2\alpha}{8(1+\alpha)}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(S^{\sharp_A}ST^{\sharp_A}T)}\\ \leqslant&\frac{1}{2}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A.\end{aligned}$$

REMARK 3.6. For $\alpha=0$, Theorem 3.4 is sharper than the inequality (3.2). Utilizing inequality (3.3) and with a similar discussion of Remark 3.3, we would obtain

$$\begin{aligned}\omega_A^2(S^{\sharp_A}T)\leqslant&\sqrt{\frac{1}{8}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{2}\omega_A^2(S^{\sharp_A}ST^{\sharp_A}T)}\\ \leqslant&\frac{3}{16}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A^2+\frac{1}{8}\|(T^{\sharp_A}T)^2+(S^{\sharp_A}S)^2\|_A\omega_A(S^{\sharp_A}ST^{\sharp_A}T).\end{aligned}$$

4. Some lower bounds of A-numerical radius

The main task of this section is to derive several lower bounds for A-numerical radius. Begin this section, it is worth noting that if $T\in\mathcal{B}_A(H)$ with $T=B+\mathrm{i}C$, it holds

$$\|TT^{\sharp_A}+T^{\sharp_A}T\|_A=2\|B^2+C^2\|_A, \tag{4.1}$$

where $B = \frac{T+T^{\sharp_A}}{2}$ and $C = \frac{T-T^{\sharp_A}}{2\mathrm{i}}$. Thus, we can deduce our next refinement.

THEOREM 4.1. *Let $T \in \mathcal{B}_A(H)$. Then*

$$\begin{aligned}\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A \leqslant & \frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2)+\frac{1}{16}\left|\|T+T^{\sharp_A}\|_A^2-\|T-T^{\sharp_A}\|_A^2\right| \\ \leqslant & \omega_A^2(T).\end{aligned}$$

Proof. Follows from the identity (4.1) and the first inequality in (1.3), it is obvious that

$$\begin{aligned}\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A = & \frac{1}{2}\|B^2+C^2\|_A \\ \leqslant & \frac{1}{2}\left(\max\{\|B\|_A^2,\|C\|_A^2\}+\|B^2C^2\|_A^{\frac{1}{2}}\right) \quad \text{(by Lemma 2.6)} \\ \leqslant & \frac{1}{2}\left(\max\{\|B\|_A^2,\|C\|_A^2\}+\|B\|_A\|C\|_A\right) \\ \leqslant & \frac{1}{2}\left(\max\{\|B\|_A^2,\|C\|_A^2\}+\frac{1}{2}(\|B\|_A^2+\|C\|_A^2)\right) \\ & \text{(by the arithmetic-geometric mean inequality)} \\ = & \frac{1}{2}\left((\|B\|_A^2+\|C\|_A^2)+\frac{1}{2}\left|\|B\|_A^2-\|C\|_A^2\right|\right) \\ & \left(\text{by } \max\{a,b\}=\frac{1}{2}(a+b+|a-b|)\right).\end{aligned}$$

Since $\|B\|_A \leqslant \omega_A(T)$ and $\|C\|_A \leqslant \omega_A(T)$, we can obtain

$$\frac{1}{2}\left(\max\{\|B\|_A^2,\|C\|_A^2\}+\frac{1}{2}(\|B\|_A^2+\|C\|_A^2)\right) \leqslant \omega_A^2(T).$$

This indicates that

$$\frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2)+\frac{1}{16}\left|\|T+T^{\sharp_A}\|_A^2-\|T-T^{\sharp_A}\|_A^2\right| \leqslant \omega_A^2(T).$$

This completes the proof. □

In [9], the authors shows that

$$\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A \leqslant \frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2) \leqslant \omega_A^2(T). \tag{4.2}$$

Obviously, the inequalities in Theorem 4.1 refine the inequality (4.2). In order to appreciate our inequalities, we give the following example, which shows that our inequalities are non-trivial improvements of the inequality (4.2).

EXAMPLE 4.1. Consider

$$A=\begin{bmatrix}1&0\\0&1\end{bmatrix} \text{ and } T=\begin{bmatrix}2+\mathrm{i}&0\\0&1+3\mathrm{i}\end{bmatrix}.$$

Then by elementary calculations, it holds

$$\omega_A^2(T)=10,\ \frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A=5,\ \frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2)=6.5,$$

$$\frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2)+\frac{1}{16}\left|\|T+T^{\sharp_A}\|_A^2-\|T-T^{\sharp_A}\|_A^2\right|=7.75.$$

Thus

$$\begin{aligned}\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A <&\frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2)\\ <&\frac{1}{8}(\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2)+\frac{1}{16}\left|\|T+T^{\sharp_A}\|_A^2-\|T-T^{\sharp_A}\|_A^2\right|\\ <&\omega_A^2(T).\end{aligned}$$

The following corollary can be immediately obtained by Theorem 4.1.

COROLLARY 4.1. *Let $T\in\mathcal{B}_A(H)$. Then*

$$\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A+\frac{1}{16}\left|\|T+T^{\sharp_A}\|_A^2-\|T-T^{\sharp_A}\|_A^2\right|\leqslant\omega_A^2(T).$$

In the next theorem, we obtain an A-norm inequality which refines the triangle inequality.

THEOREM 4.2. *Let $T,S\in\mathcal{B}_A(H)$. Then*

$$\begin{aligned}\|T+S\|_A\leqslant&\sqrt{\|T\|_A^2+\|S\|_A^2+2\min\{\omega_A(ST^{\sharp_A}),\omega_A(T^{\sharp_A}S)\}}\\ \leqslant&\|T\|_A+\|S\|_A.\end{aligned}$$

Proof. Let $x\in H$ with $\|x\|_A=1$. Then we have

$$\begin{aligned}\|(T+S)^{\sharp_A}x\|_A^2=&\langle(T+S)^{\sharp_A}x,(T+S)^{\sharp_A}x\rangle_A\\ =&\langle T^{\sharp_A}x,T^{\sharp_A}x\rangle_A+\langle S^{\sharp_A}x,S^{\sharp_A}x\rangle_A+\langle ST^{\sharp_A}x,x\rangle_A+\langle TS^{\sharp_A}x,x\rangle_A\\ \leqslant&\|T^{\sharp_A}x\|_A^2+\|S^{\sharp_A}x\|_A^2+|\langle ST^{\sharp_A}x,x\rangle_A|+|\langle TS^{\sharp_A}x,x\rangle_A|\\ \leqslant&\|T\|_A^2+\|S\|_A^2+\omega_A(ST^{\sharp_A})+\omega_A(TS^{\sharp_A})\\ =&\|T\|_A^2+\|S\|_A^2+2\omega_A(ST^{\sharp_A}).\end{aligned}$$

Taking the supremum over $\|x\|_A=1$ and combine with $\|T\|_A=\|T^{\sharp_A}\|_A$, we have

$$\|T+S\|_A\leqslant\sqrt{\|T\|_A^2+\|S\|_A^2+2\omega_A(ST^{\sharp_A})}.$$

On the other hand, it was shown in [23] that

$$\|T+S\|_A \leqslant \sqrt{\|T\|_A^2+\|S\|_A^2+2\omega_A(T^{\sharp_A}S)}.$$

The above two inequalities indicate that

$$\|T+S\|_A \leqslant \sqrt{\|T\|_A^2+\|S\|_A^2+2\min\{\omega_A(ST^{\sharp_A}),\omega_A(T^{\sharp_A}S)\}}.$$

Moreover, we observe that $\min\{\omega_A(ST^{\sharp_A}),\omega_A(T^{\sharp_A}S)\} \leqslant \|S\|_A\|T\|_A$, this implies that

$$\begin{aligned}\|T+S\|_A &\leqslant \sqrt{\|T\|_A^2+\|S\|_A^2+2\min\{\omega_A(ST^{\sharp_A}),\omega_A(T^{\sharp_A}S)\}}\\ &\leqslant \sqrt{\|T\|_A^2+\|S\|_A^2+2\|S\|_A\|T\|_A}\\ &= \|T\|_A+\|S\|_A.\end{aligned}$$

This completes the proof. □

We now start to deduce our second theorem which refines the first inequality in (1.2).

THEOREM 4.3. *Let $T\in\mathcal{B}_A(H)$. Then*

$$\begin{aligned}\frac{1}{2}\|T\|_A &\leqslant \frac{1}{4}\left[\|T+T^{\sharp_A}\|_A^2+\|T-T^{\sharp_A}\|_A^2+2\omega_A\big((T^{\sharp_A}+T)(T^{\sharp_A}-T)\big)\right]^{\frac{1}{2}}\\ &\leqslant \omega_A(T).\end{aligned}$$

Proof. Let $T=B+\mathrm{i}C$. Then

$$\begin{aligned}\frac{1}{2}\|T\|_A &= \frac{1}{2}\|B+\mathrm{i}C\|_A\\ &\leqslant \frac{1}{2}\left[\|B\|_A^2+\|C\|_A^2+2\omega_A\big(BC^{\sharp_A}\big)\right]^{\frac{1}{2}} \quad \text{(by Theorem 4.2)}.\end{aligned}$$

Since $\omega_A(T)=\omega_A(T^{\sharp_A})$, it is easy to check that

$$\omega_A\big(BC^{\sharp_A}\big)=\omega_A\big((T^{\sharp_A}+T)(T^{\sharp_A}-T)\big).$$

This implies the first inequality of the theorem.

Now we prove the second inequality. Observe that $\|B\|_A\leqslant\omega_A(T)$ and $\|C\|_A\leqslant\omega_A(T)$, we can obtain

$$\begin{aligned}\frac{1}{2}\left[\|B\|_A^2+\|C\|_A^2+2\omega_A\big(BC^{\sharp_A}\big)\right]^{\frac{1}{2}} &\leqslant \frac{1}{2}\left[\|B\|_A^2+\|C\|_A^2+2\|BC^{\sharp_A}\|_A\right]^{\frac{1}{2}}\\ &\leqslant \frac{1}{2}\left[\|B\|_A^2+\|C\|_A^2+2\|B\|_A\|C\|_A\right]^{\frac{1}{2}}\\ &= \frac{1}{2}(\|B\|_A+\|C\|_A)\\ &\leqslant \omega_A(T).\end{aligned}$$

Thus, we get the second inequality in theorem. This completes the proof. □

The next refinement of the first inequality in (1.3) is as follows.

THEOREM 4.4. *Let $T \in \mathcal{B}_A(H)$. Then*

$$\begin{aligned}\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A \leqslant & \frac{1}{8}\left[\|T+T^{\sharp_A}\|_A^4+\|T-T^{\sharp_A}\|_A^4+2\omega_A\big((T^{\sharp_A}+T)^2(T^{\sharp_A}-T)^2\big)\right]^{\frac{1}{2}} \\ \leqslant & \omega_A^2(T).\end{aligned}$$

Proof. Let $T = B + \mathrm{i}C$. Since $\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A = \frac{1}{2}\|B^2+C^2\|_A$, then it holds

$$\begin{aligned}\frac{1}{4}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A =& \frac{1}{2}\|B^2+C^2\|_A \\ \leqslant & \frac{1}{2}\left[\|B^2\|_A^2+\|C^2\|_A^2+2\omega_A\big(B^2(C^2)^{\sharp_A}\big)\right]^{\frac{1}{2}} \\ & \text{(by Theorem 4.2)} \\ \leqslant & \frac{1}{2}\left[\|B\|_A^4+\|C\|_A^4+2\omega_A\big(B^2(C^2)^{\sharp_A}\big)\right]^{\frac{1}{2}}\end{aligned}$$

Since $\omega_A(T) = \omega_A(T^{\sharp_A})$, it is easy to check that

$$\omega_A\big(B^2(C^2)^{\sharp_A}\big) = \omega_A\big((T^{\sharp_A}+T)^2(T^{\sharp_A}-T)^2\big).$$

This implies the first inequality of the theorem.

On the other hand, combine with $\|B\|_A \leqslant \omega_A(T)$ and $\|C\|_A \leqslant \omega_A(T)$, one gives

$$\begin{aligned}\frac{1}{2}\left[\|B\|_A^4+\|C\|_A^4+2\omega_A\big(B^2(C^2)^{\sharp_A}\big)\right]^{\frac{1}{2}} \leqslant & \frac{1}{2}\left[\|B\|_A^4+\|C\|_A^4+2\|B^2(C^2)^{\sharp_A}\|_A\right]^{\frac{1}{2}} \\ \leqslant & \frac{1}{2}\left[\|B\|_A^4+\|C\|_A^4+2\|B\|_A^2\|C\|_A^2\right]^{\frac{1}{2}} \\ =& \frac{1}{2}(\|B\|_A^2+\|C\|_A^2) \\ \leqslant & \omega_A^2(T).\end{aligned}$$

Thus, the second inequality of theorem is established. This completes the proof. □

5. An improvement of the triangle inequality

The final section yields an improvement of the triangle inequality for the A-operator semi-norm.

THEOREM 5.1. *Let $T,S \in \mathcal{B}_A(H)$. Then*

$$\|T+S\|_A^2 \leqslant \|T^{\sharp_A}T+S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A}+SS^{\sharp_A}\|_A^{\frac{1}{2}}+\|T\|_A\|S\|_A+\omega_A(S^{\sharp_A}T).$$

Proof. Let $x,y \in H$ with $\|x\|_A = \|y\|_A = 1$, then using the Cauchy inequality

$$(ab+cd)^2 \leqslant (a^2+c^2)(b^2+d^2), a,b,c,d \in \mathbb{R}, \tag{5.1}$$

there is the following inequality established:

$$\begin{aligned}
&|\langle (T+S)x,y\rangle_A|^2\\
\leqslant &(|\langle Tx,y\rangle_A| + |\langle Sx,y\rangle_A|)^2\\
= &|\langle Tx,y\rangle_A|^2 + |\langle Sx,y\rangle_A|^2 + 2|\langle Tx,y\rangle_A||\langle Sx,y\rangle_A|\\
\leqslant &\left(\sqrt{\langle T^{\sharp_A}Tx,x\rangle_A}\sqrt{\langle TT^{\sharp_A}y,y\rangle_A} + \sqrt{\langle S^{\sharp_A}Sx,x\rangle_A}\sqrt{\langle SS^{\sharp_A}y,y\rangle_A}\right) + 2|\langle Tx,y\rangle_A||\langle Sx,y\rangle_A|\\
&\text{(by Lemma 2.4)}\\
\leqslant &\left(\langle T^{\sharp_A}Tx,x\rangle_A + \langle S^{\sharp_A}Sx,x\rangle_A\right)^{\frac{1}{2}}\left(\langle TT^{\sharp_A}x,x\rangle_A + \langle SS^{\sharp_A}x,x\rangle_A\right)^{\frac{1}{2}} + 2|\langle Tx,y\rangle_A||\langle Sx,y\rangle_A|\\
&\text{(by inequality (5.1))}\\
= &\langle (T^{\sharp_A}T + S^{\sharp_A}S)x,x\rangle_A^{\frac{1}{2}}\langle (TT^{\sharp_A} + SS^{\sharp_A})y,y\rangle_A^{\frac{1}{2}} + 2|\langle Tx,y\rangle_A||\langle Sx,y\rangle_A|\\
\leqslant &\|T^{\sharp_A}T + S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A} + SS^{\sharp_A}\|_A^{\frac{1}{2}} + 2|\langle Tx,y\rangle_A||\langle Sx,y\rangle_A|.
\end{aligned}$$

On the other hand, by using Lemma 2.1, we observe that

$$\begin{aligned}
2|\langle Tx,y\rangle_A||\langle Sx,y\rangle_A| &= 2|\langle Tx,y\rangle_A||\langle y,Sx\rangle_A|\\
&\leqslant \|Tx\|_A\|Sx\|_A + |\langle Tx,Sx\rangle_A|\\
&= \|Tx\|_A\|Sx\|_A + |\langle S^{\sharp_A}Tx,x\rangle_A|\\
&\leqslant \|T\|_A\|S\|_A + \omega_A(S^{\sharp_A}T).
\end{aligned}$$

Combining the above two inequalities and taking the supremum over $x,y \in H$ with $\|x\|_A = \|y\|_A = 1$, we get

$$\|T+S\|_A^2 \leqslant \|T^{\sharp_A}T + S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A} + SS^{\sharp_A}\|_A^{\frac{1}{2}} + \|T\|_A\|S\|_A + \omega_A(S^{\sharp_A}T).$$

This completes the proof. □

REMARK 5.1. Theorem 5.1 is sharper than triangle inequality. To see this, we note that

$$\begin{aligned}
&\|T^{\sharp_A}T + S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A} + SS^{\sharp_A}\|_A^{\frac{1}{2}} + \|T\|_A\|S\|_A + \omega_A(S^{\sharp_A}T)\\
\leqslant &(\|T^{\sharp_A}T\|_A + \|S^{\sharp_A}S\|_A)^{\frac{1}{2}}(\|TT^{\sharp_A}\|_A + \|SS^{\sharp_A}\|_A)^{\frac{1}{2}} + \|T\|_A\|S\|_A + \omega_A(S^{\sharp_A}T)\\
\leqslant &(\|T\|_A^2 + \|S\|_A^2)^{\frac{1}{2}}(\|T\|_A^2 + \|S\|_A^2)^{\frac{1}{2}} + \|T\|_A\|S\|_A + \|S^{\sharp_A}T\|_A\\
\leqslant &\|T\|_A^2 + \|S\|_A^2 + \|T\|_A\|S\|_A + \|S^{\sharp_A}\|_A\|T\|_A\\
= &(\|T\|_A + \|S\|_A)^2.
\end{aligned}$$

Namely

$$\sqrt{\|T^{\sharp_A}T+S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A}+SS^{\sharp_A}\|_A^{\frac{1}{2}}+\|T\|_A\|S\|_A+\omega_A(S^{\sharp_A}T)} \leqslant \|T\|_A+\|S\|_A.$$

Now, an example is given to explain that Theorem 5.1 is a nontrivial improvement of triangle inequality.

EXAMPLE 5.1. Let

$$T=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad S=\begin{bmatrix}1&0\\0&0\end{bmatrix}\ \text{ and }\ A=\begin{bmatrix}1&0\\0&2\end{bmatrix},$$

we can get

$$\|T^{\sharp_A}T+S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A}+SS^{\sharp_A}\|_A^{\frac{1}{2}}+\|T\|_A\|S\|_A+\omega_A(S^{\sharp_A}T)=\frac{2\sqrt{6}+3\sqrt{2}}{4}$$

and

$$(\|T\|_A+\|S\|_A)^2=\frac{3+2\sqrt{2}}{2}.$$

Therefore, we conclude that

$$\sqrt{\|T^{\sharp_A}T+S^{\sharp_A}S\|_A^{\frac{1}{2}}\|TT^{\sharp_A}+SS^{\sharp_A}\|_A^{\frac{1}{2}}+\|T\|_A\|S\|_A+\omega_A(S^{\sharp_A}T)} < \|T\|_A+\|S\|_A.$$

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