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HÖLDER'S INEQUALITIES AND MULTILINEAR SINGULAR INTEGRALS ON GENERALIZED ORLICZ SPACES

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Abstract. Let $\varphi_j \in \Phi_w(\mathbb{R}^n)$, $j = 1,\dots,m$, be generalized Orlicz functions. We obtain a general version of the Hölder inequality on the generalized Orlicz spaces

$$\left\|\prod_{j=1}^{m} f_j\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant C\prod_{j=1}^{\infty}\|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)},$$

where $\varphi = (\prod_{j=1}^{m}\varphi_j^{-1})^{-1}$. If every φ_j satisfies the conditions, (A0), (A1), (A2), (aInc)$_{p_j}$ and (aDec)$_{q_j}$ with $1 < p_j, q_j < \infty$, then the multilinear sparse operators and multilinear Calderón–Zygmund operators are bounded from $L^{\varphi_1(\cdot)}(\mathbb{R}^n)\times\cdots\times L^{\varphi_m(\cdot)}(\mathbb{R}^n)$ to $L^{\varphi(\cdot)}(\mathbb{R}^n)$. We also establish the boundedness of the multilinear fractional integral operators over the generalized Orlicz spaces. These results are also new for classical Orlicz spaces as the special case.

1. Introduction

The classical Hölder's inequality was discovered, independently, by When Leonard James Rogers (1862–1933) and Otto Hölder (1859–1937). We refer to [37] for a detailed and historical exposition. There are so many versions of Hölder's inequalities associated to different function spaces, such as variable Lebesgue spaces [31], mixed Lebesgue spaces [6], generalized Orlicz spaces [16, 21] and so on. These versions of Hölder inequalities could be collected in the following inequality [7].

THEOREM 1.1. *Let X be a Banach function space with the associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathbb{R}^n}|f(x)g(x)|\,dx \leqslant \|f\|_X\|g\|_{X'},$$

where X' is the associate space (Köthe dual) of X defined by setting

$$X' := \left\{f \in \mathscr{M}(\mathbb{R}^n):\ \|f\|_{X'} := \sup\left\{\|fg\|_{L^1(\mathbb{R}^n)}:\ g \in X,\ \|g\|_X = 1\right\} < \infty\right\}.$$

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Without any doubt, this inequality as a fundamental tool has played a key role in Mathematical Analysis. For example, the classical Hölder inequality is used to prove Minkowski's inequality (the triangle inequality for L^p spaces with $p \in [1,\infty)$) and to establish that L^q is the dual space of L^p for $p \in [1,\infty)$, where $1/p+1/q=1$. According to our knowledge, however, people often use a generalized version of the Hölder inequality to study the mutlilinear theory in harmonic analysis, that is, there are suitable function spaces $X, X_1, \ldots, X_m$ such that then

$$\left\| \prod_{j=1}^{m} f_j \right\|_X \leqslant \prod_{j=1}^{m} \|f_j\|_{X_j}, \tag{1.1}$$

whenever $f_j \in X_j,\ j=1,\ldots,m$. Ones can obtain (1.1) by replacing with Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces or mixed Lebesgue spaces under suitable assumptions for the underlying spaces. Until now, we can not find a paper which has established this type of Hölder's inequality over the Orlicz spaces and more generalized Orlicz spaces. One of the motivations for us writing this paper is to establish the similar inequality (1.1) in the setting of generalized Orlicz spaces.

The classical Orlicz spaces are well known and have been studied for a long period, see for instance the monograph [45] and related references. The generalized Orlicz spaces, also called Musielak–Orlicz spaces and Nakano spaces, are a class of Banach function spaces that include a number of spaces of interest in harmonic analysis and PDEs as special cases. The basic example of a variable exponent space was introduced by Orlicz [43]. Following [43], these spaces were introduced by Nakano [41, 42] and others and a comprehensive synthesis of this earlier work is due to Musielak [40]. As mentioned in [21], the generalized Orlicz spaces are of interest as the natural generalization of some important function spaces such as Lebesgue spaces, weighted Lebesgue spaces, classical Orlicz spaces, variable Lebesgue spaces and so on. On the other hand, the generalized Orlicz spaces have appeared in many problems in PDEs and the calculus of variations [2, 3, 4, 5, 13, 17, 23, 27] and also have applications to image processing [1, 8, 24] and fluid dynamics [46]. Meanwhile, the generalized Orlicz spaces as the underlying spaces are drawing more and more people who are interested in functionals or partial differential equations with non-standard growth increase. Hästö and his collaborators [22, 23, 25, 26] have systematically studied the operators of classical harmonic analysis and generalized Sobolve spaces and established a very broad theory that unites and extends previous work. We refer to the recent book [21] for more details on this topic.

The purpose of this paper is to develop harmonic analysis on generalized Orlicz spaces by extending the theory of multilinear singular integrals to this setting. The theory of multilinear analysis related to the Calderón-Zygmund program originated in the work of Coifman and Meyer [9, 10, 11]. Its study has been attracting a lot of attention in the last few decades. A series of papers about this topic enriches this program, for example Christ and Journé [12], Kenig and Stein [30], and Grafakos and Torres [19, 20] on Lebesgue spaces, Wang and Yi [47] and Iida et al. [29] on Morrey spaces, Lerner et al [32], Li and Sun [36] and Li et al [34] and their referee on weighted Lebesgue spaces, Huang and Xu [28] on variable Lebesgue spaces. The boundness of the multilinear singular integrals heavily depends on the Hölder inequality (1.1) over the underlying spaces we mentioned above.

Applying the classical methods of proving the Hölder inequality on Lebesgue spaces, we obtain a version of the Hölder inequality over quasi-Banach function spaces X. Let X be a quasi-Banach function space equipped with the quasi-norm $\|\cdot\|_X$. Assume that $p, p_1, \ldots, p_m \in (0,\infty)$ satisfy $1/p = 1/p_1 + \cdots + 1/p_m$. By the p-convexification X^p of X, the Hölder inequality

$$\left\|\prod_{j=1}^{m} f_j\right\|_{X^p} \lesssim \prod_{j=1}^{m} \|f_j\|_{X^{p_j}}$$

and the weak Hölder inequality

$$\left\|\prod_{j=1}^{m} f_j\right\|_{wX^p} \lesssim \prod_{j=1}^{m} \|f_j\|_{wX^{p_j}}$$

hold true without additional assumptions. These inequalities generalize the classical Hölder inequalities on Lebesgue spaces, i.e., $X = L^1(\mathbb{R}^n)$. However, they cannot contain the Hölder inequalities built on variable Lebesgue spaces or mixed-norm Lebesgue spaces.

To prove a multilinear operator T is bounded from $X_1 \times \cdots \times X_m$ to X, we always hope that the Hölder inequality (1.1) holds true. However, in general, these function spaces X and X_j, $(j = 1, \ldots, m)$ may not be the p-convexifications of some function space. This motivates us to consider a more versatile Hölder inequality over the generalized Orlicz spaces. By using the remarkable work [14, 23, 25], we can pose some sufficient condition on the Φ-functions φ for the hypotheses of the Hölder inequality (1.1) replacing by the generalized Orlicz spaces to hold. These conditions which will appear in our main results are somewhat technical, however, we would like to emphasize that they are easy to check and sufficiently general. The details refer to [14, 21]. As immediate consequences of our Hölder inequality, we derive norm inequalities for a number of multilinear operators on generalized Orlicz spaces: in particular, for the multilinear maximal function, the multilinear Calderón–Zygmund singular integrals, the multilinear fractional integrals and the multilinear fractional maximal operators.

The remainder of this paper is organized as follows. In section 2 we recall the definitions of generalized Orlicz spaces and establish the Hölder inequalities over these spaces. Using the sparse operators and the duality, we get the boundedness of the multilinear Calderón–Zygmund singular integrals and the multilinear fractional integrals over the generalized Orlicz spaces in Section 3.

2. Φ-functions and generalized Orlicz spaces

Hereafter, we say that a function f is *almost increasing* if there exists $L \geqslant 1$ such that for all $s \leqslant t$, $f(s) \leqslant Lf(t)$. Almost decreasing is defined analogously. We say that f is increasing/decresing for $L = 1$.

DEFINITION 2.1. Let $\varphi : [0,\infty) \to [0,\infty]$ be an increasing function such that $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Such a function φ is called a Φ-*prefunction*. For a Φ-prefunction, φ

(1) if $t \to \frac{\varphi(t)}{t}$ is almost increasing on $(0,\infty)$, we call that φ is a *weak* Φ*-function*;

(2) if it is a left continuous and convex, we call that φ is a *convex* Φ*-function*;

(3) if it is continuous in $\overline{\mathbb{R}^n}$ and convex, we call that φ is a strong Φ-function.

The set of weak, convex and strong Φ-function are denoted by Φ_w, Φ_c and Φ_s respectively.

From the definition, it follows that $\Phi_s \subset \Phi_c \subset \Phi_w$.

DEFINITION 2.2. Given two functions φ and ψ on $[0,\infty)$, we say that they are *equivalent*, $\varphi \simeq \psi$, if there exists $L \in [1,\infty)$ such that, for any $t \in [0,\infty)$, $\varphi(t/L) \leqslant \psi(t) \leqslant \varphi(Lt)$.

While it is common in the literature to work with Φ-functions, it is also convenient to work at times with either weak or strong Φ-functions. Ones can do so since every weak Φ-function is equivalent to a strong one, which was obtained in [23, Proposition 2.3].

LEMMA 2.3. *Every weak* Φ*-function is equivalent to a strong* Φ*-function.*

Two Φ-(pre)functions $\varphi \approx \psi$, if there exists $K \geqslant 1$ such that, for any $t \in [0,\infty)$, $K^{-1}\varphi(t) \leqslant \psi(t) \leqslant K\varphi(t)$. We say that φ is doubling if there exists a positive constant A such that $\varphi(2t) \leqslant A\varphi(t)$ for every $t \geqslant 0$. For doubling Φ functions, $\simeq$ and $\approx$ are equivalent.

Since the weak Φ-function are not bijections, they are not strictly speaking invertible. However, we can define a left-inverse. This notion is a key for us to establish the Hölder inequality (1.1) replacing by generalized Orlicz spaces.

DEFINITION 2.4. Given $\varphi \in \Phi_w$, by φ^{-1} we denote the *left-inverse* of φ

$$\varphi(\tau) := \inf\{t \geqslant 0 : \varphi(t) \geqslant \tau\}.$$

By the definition of the left inverse, we know that φ^{-1} is increasing and $\varphi^{-1} \circ \varphi(t) \leqslant t$ and if φ is left-continuous, then $\varphi \circ \varphi^{-1}(\tau) \leqslant \tau$.

LEMMA 2.5. *Let* $\varphi_j \in \Phi_w,\ j=1\dots,m$ *and define* $\varphi = (\prod_{j=1}^m \varphi_j^{-1})^{-1}$. *Then* φ *is increasing, left-continuous and* $\varphi^{-1} = \prod_{j=1}^m \varphi_j^{-1}$.

Proof. From [21, Lemma 2.3.9 (a) and (c)], it follows that every φ_j^{-1} is increasing and left-continuous, for $j=1,\dots,m$, and so is $\prod_{j=1}^m \varphi_j^{-1}$. This, together with [21, Lemma 2.3.9(c) and Lemma 2.3.11], implies that φ is increasing, left-continuous and $\varphi^{-1} = \prod_{j=1}^m \varphi_j^{-1}$, which finishes the proof of Lemma 2.5. □

Now we give the Young inequality associated to Φ-functions.

THEOREM 2.6. *Let* $\varphi_j \in \Phi_w,\ j=1\dots,m$ *and define* $\varphi = (\prod_{j=1}^m \varphi_j^{-1})^{-1}$. *Assume that every* φ_j *satisfies that* $\varphi_j^{-1}(\varphi_j(t)) = t,\ j \in \{1,\dots,m\}$, *for any* $t \in (0,\infty)$. *Then we have the following Young inequality*

$$\varphi\left(\prod_{j=1}^m a_j\right) \leqslant \sum_{j=1}^m \varphi_j(a_j), \tag{2.1}$$

for any $a_j \in (0,\infty),\ j=1,\dots,m$.

Proof. Since, for any $j \in \{1,\dots,m\}$, φ_j^{-1} is increasing, we have, for any $a_j \in (0,\infty)$,

$$a_j \leqslant \varphi_j^{-1}(\varphi_j(a)) \leqslant \varphi_j^{-1}\left(\sum_{j=1}^m \varphi_j(a_j)\right).$$

By Lemma 2.5 and [21, Lemma 2.3.9(b)], we know that φ is increasing and $\varphi(\varphi^{-1}(t)) \leqslant t$. Thus

$$\varphi\left(\prod_{j=1}^m a_j\right) \leqslant \varphi\left(\varphi^{-1}\left(\sum_{j=1}^m \varphi_j(a_j)\right)\right) \leqslant \sum_{j=1}^m \varphi_j(a_j),$$

which completes the proof of Theorem 2.6. □

To define generalized Orlicz spaces, we first recall the definition of generalized Φ-functions.

DEFINITION 2.7. A function $\varphi:\ \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *generalized* Φ*-function* if

(1) $\varphi(x,\cdot)$ is a Φ-function for every $x \in \mathbb{R}^n$;

(2) $x \mapsto \varphi(x,|f(x)|)$ is measurable for every measurable function $f(x)$.

The set $\Phi(\mathbb{R}^n)$ is the family of all generalized Φ-functions. The families $\Phi_w(\mathbb{R}^n)$, $\Phi_c(\mathbb{R}^n)$ and $\Phi_s(\mathbb{R}^n)$ are defined analogously.

Now we recall generalized Orlicz spaces and refer to [14, Definition 2.6].

DEFINITION 2.8. Let $\varphi \in \Phi_w(\mathbb{R}^n)$. For any measurable function f on $\mathbb{R}^n$, we define the semimodular $\rho_{\varphi(\cdot)}$ by

$$\rho_{\varphi(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx.$$

The *generalized Orlicz space* is defined as the set $L^{\varphi(\cdot)}(\mathbb{R}^n)$:

$$L^{\varphi(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ measurable}:\ \rho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) < \infty, \text{ for some } \lambda > 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} := \inf\left\{ \lambda > 0:\ \rho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1 \right\}.$$

The following two properties are collected in [21, Lemma 3.2.3 and Proposition 3.2.4]

LEMMA 2.9. *Let* $\varphi \in \Phi_w(\mathbb{R}^n)$. *Then*

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} < 1 \Rightarrow \rho_{\varphi(\cdot)}(f) \leqslant 1 \Rightarrow \|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant 1.$$

If φ *is left continuous, then* $\rho_{\varphi(\cdot)}(f) \leqslant 1 \Leftrightarrow \|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant 1$.

The following lemma shows that equivalent Φ-functions give rise to the same space; see [21, Proposition 3.2.4].

LEMMA 2.10. *Let* $\varphi, \psi \in \Phi_w(\mathbb{R}^n)$. *If* $\varphi \simeq \psi$, *then* $L^{\varphi(\cdot)}(\mathbb{R}^n) = L^{\psi(\cdot)}(\mathbb{R}^n)$ *and the norms are comparable.*

REMARK 2.11. From Lemmas 2.3 and 2.10, we can always suppose that $\varphi \in \Phi_s(\mathbb{R}^n)$. Then by [21, Corollary 2.3.4], we have $\varphi^{-1}(\varphi(x,t)) = t$, for almost $x \in \mathbb{R}^n$ and any $t \in (0,\infty)$.

THEOREM 2.12. *Let* $\varphi_1, \ldots, \varphi_m \in \Phi_w(\mathbb{R}^n)$ *and define* $\varphi = (\prod_{j=1}^m \varphi_j^{-1})^{-1}$. *Then*

$$\left\| \prod_{j=1}^m f_j \right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant m \prod_{j=1}^m \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

Proof. By Theorem 2.6, we know that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\prod_{j=1}^m |f_j(x)|}{\prod_{j=1}^m \|f_j\|_{L^{\varphi_j}(\mathbb{R}^n)}}\right) dx \leqslant \sum_{j=1}^m \int_{\mathbb{R}^n} \varphi_j\left(x, \frac{|f_j(x)|}{\|f_j\|_{L^{\varphi_j}(\mathbb{R}^n)}}\right) dx \leqslant m.$$

This shows that by Lemma 2.9

$$\left\| \prod_{j=1}^m f_j \right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant m \prod_{j=1}^m \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}. \qquad \square$$

DEFINITION 2.13. Let $\varphi \in \Phi_w(\mathbb{R}^n)$. We denote by φ^* the conjugate function of φ which is defined, for $u \geqslant 0$, by

$$\varphi^*(x,u) := \sup_{t \geqslant 0}(tu - \varphi(x,t)).$$

By definition of φ^*, for any $t,u \geqslant 0$,

$$tu \leqslant \phi(x,t) + \phi^*(x,u). \tag{2.2}$$

The following general norm conjugate formula, in a sense the opposite of Hölder's inequality, holds true for weak Φ-functions, which was proved in [14, Lemma 2.7].

LEMMA 2.14. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$, and let $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$, $g \in L^{\varphi^*(\cdot)}(\mathbb{R}^n)$. Then*

$$c(\varphi)\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant \sup_{\|g\|_{L^{\varphi^*(\cdot)}(\mathbb{R}^n)} \leqslant 1} \int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leqslant 2\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}. \tag{2.3}$$

REMARK 2.15. Let $\varphi \in \Phi_w(\mathbb{R}^n)$. From [14, Lemma 2.3], we know that $\varphi^{-1}(x,t)(\varphi^*)^{-1}(x,t) \approx t$. By this, Lemma 2.10 and Definition 2.8, the right-hand inequality of (2.3) is a consequence of Theorem 2.12.

The remainder of this section is to give the Hölder inequality on the Banach function spaces.

DEFINITION 2.16. A Banach space $X \subset \mathscr{M}(\mathbb{R}^n)$ is called a *Banach function space* if it satisfies

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \leqslant |f|$ almost everywhere implies that $\|g\|_X \leqslant \|f\|_X$;

(iii) $0 \leqslant f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;

(iv) for any finite set $E \subset \mathbb{R}^n$ implies that $\mathbf{1}_E \in X$;

(v) for any finite set $E \in \mathbb{R}^n$, there exists a positive constant $C_{(E)}$, depending on E, such that, for any $f \in X$,

$$\int_E |f(x)|\,dx \leqslant C_{(E)}\|f\|_X.$$

The *p-convexification* X^p of X is defined by setting $X^p := \{f \in \mathscr{M}(\mathbb{R}^n) : |f|^p \in X\}$ equipped with the quasi-norm $\|f\|_{X^p} := \||f|^p\|_X^{1/p}$.

DEFINITION 2.17. Let X be a Banach function space. The *weak Banach function space* WX is defined to be the set of all measurable functions f satisfying

$$\|f\|_{WX} := \sup_{\alpha \in (0,\infty)} \left\{\alpha \left\|\mathbf{1}_{\{x \in \mathbb{R}^n :\, |f(x)| > \alpha\}}\right\|_X\right\} < \infty. \tag{2.4}$$

REMARK 2.18. (i) Let X be a Banach function space. For any $f\in X$ and $\alpha\in(0,\infty)$, we have $\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)|>\alpha\}}(x)\leqslant|f(x)|/\alpha$ for any $x\in\mathbb{R}^n$, which, together with Definition 2.16(ii), further implies that $\sup_{\alpha\in(0,\infty)}\left\{\alpha\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)|>\alpha\}}\|_X\right\}\leqslant\|f\|_X$. This shows that $X\subset WX$.

(ii) Let $f,\ g\in WX$ with $|f|\leqslant|g|$. By Definition 2.16(ii), we conclude that $\|f\|_{WX}\leqslant\|g\|_{WX}$. And, for any $p\in(0,\infty)$, $WX^p=[WX]^p$.

THEOREM 2.19. *Let X be a Banach function space. Let $p,p_1,\cdots,p_m\in(0,\infty)$ satisfy $1/p=1/p_1+\cdots+1/p_m$.*

(1) If $f_j\in X^{p_j}$ and $j\in\{1,\cdots,m\}$, then

$$\left\|\prod_{j=1}^m f_j\right\|_{X^p}\leqslant m\prod_{j=1}^m\|f_j\|_{WX^{p_j}}$$

(2) If $f_j\in WX^{p_j}$ and $j\in\{1,\cdots,m\}$, then

$$\left\|\prod_{j=1}^m f_j\right\|_{WX^p}\leqslant C\prod_{j=1}^m\|f_j\|_{WX^{p_j}}$$

Proof. Without loss of generality, we assume that $m=2$.

(1) Obviously, we only need to show the inequality for $\|f_j\|_{X^{p_j}}>0,\ j=1,2$. Let $q_1=p_1/p$ and $q_2=p_2/p$. Then $1/q_1+1/q_2=1$. Substituting $a=|f_1|/\|f_1\|_{X^{p_1}}$, $b=|f_2|/\|f_2\|_{X^{p_2}}$, $q=q_1$ and $q'=q_2$ in the well-known Young inequality $ab\leqslant a^q/q+b^{q'}/q'$, we have

$$\begin{aligned}\left\|\frac{|f_1f_2|}{\|f_1\|_{X^{p_1}}\|f_2\|_{X^{p_2}}}\right\|_{X^p}&\leqslant\left\|\frac{1}{q}\left(\frac{|f_1|}{\|f_1\|_{X^{p_1}}}\right)^{q_1}+\frac{1}{q'}\left(\frac{|f_2|}{\|f_2\|_{X^{p_2}}}\right)^{q_2}\right\|_{X^p}\\&\leqslant\left\|\left[\frac{1}{q}\left(\frac{|f_1|}{\|f_1\|_{X^{p_1}}}\right)^{q_1}+\frac{1}{q'}\left(\frac{|f_2|}{\|f_2\|_{X^{p_2}}}\right)^{q_2}\right]^p\right\|_X^{1/p}\\&\leqslant2\left\|\frac{1}{q}\left(\frac{|f_1|}{\|f_1\|_{X^{p_1}}}\right)^{p_1}+\frac{1}{q'}\left(\frac{|f_2|}{\|f_2\|_{X^{p_2}}}\right)^{p_2}\right\|_X^{1/p}\\&\leqslant2\left[\left\|\frac{1}{q}\left(\frac{|f_1|}{\|f_1\|_{X^{p_1}}}\right)^{p_1}\right\|_X+\left\|\frac{1}{q'}\left(\frac{|f_2|}{\|f_2\|_{X^{p_2}}}\right)^{p_2}\right\|_X\right]^{1/p}\\&\leqslant2.\end{aligned}$$

Thus,

$$\|f_1f_2\|_{X^p}\leqslant2\|f_1\|_{X^{p_1}}\|f_2\|_{X^{p_2}},$$

which completes the proof of part (1).

(2) We also assume that $\|f_j\|_{WX^{p_j}}=1$, $j=1,2$. For any $\lambda>0$, let positive numbers λ_1 and λ_2 such that $\lambda=\lambda_1^{-1}\lambda_2^{-1}$. By Definition 2.17

$$\lambda_1^{-1}\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_1(x)|>\lambda_1^{-1}\}}\right\|_X^{1/p_1}=\lambda_1^{-1}\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_1(x)|>\lambda_1^{-1}\}}\right\|_{X^{p_1}}\leqslant 1$$

and

$$\lambda_2^{-1}\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_2(x)|>\lambda_2^{-1}\}}\right\|_X^{1/p_2}=\lambda_2^{-1}\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_2(x)|>\lambda_2^{-1}\}}\right\|_{X^{p_2}}\leqslant 1.$$

Since $\{x\in\mathbb{R}^n:\ |f_1(x)f_2(x)|>\lambda_1^{-1}\lambda_2^{-1}\}\subset\{x\in\mathbb{R}^n:\ |f_1(x)|>\lambda_1^{-1}\}\cup\{x\in\mathbb{R}^n:\ |f_2(x)|>\lambda_2^{-1}\}$, we have

$$\begin{aligned}\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_1(x)f_2(x)|>\lambda\}}\right\|_{X^p}&\leqslant\left[\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_1(x)|>\lambda_1^{-1}\}}\right\|_X+\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_2(x)|>\lambda_2^{-1}\}}\right\|_X\right]^{1/p}\\&\leqslant\left[\lambda_1^{p_1}+\lambda_2^{p_2}\right]^{1/p}.\end{aligned}$$

Let $\lambda_1^{p_1}=\frac{1}{p_1\lambda^p}p_1^{\frac{p}{p_1}}p_2^{\frac{p}{p_2}}$ and $\lambda_2^{p_2}=\frac{p_1}{p_2}\lambda_1^{p_1}$, then we get

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_1(x)f_2(x)|>\lambda\}}\right\|_{X^p}\leqslant\left[\lambda_1^{p_1}+\lambda_2^{p_2}\right]^{1/p}=\frac{1}{\lambda}p_1^{\frac{1}{p_1}}p_2^{\frac{1}{p_2}}\frac{1}{p^{1/p}},$$

which shows that $\lambda\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f_1(x)f_2(x)|>\lambda\}}\|_{X^p}\leqslant p_1^{\frac{1}{p_1}}p_2^{\frac{1}{p_2}}\frac{1}{p^{1/p}}\|f_1\|_{WX^{p_1}}\|f_2\|_{WX^{p_2}}$ and we finish the proof of the second part. □

Rescaling. Let $\varphi\in\Phi_w(\mathbb{R}^n)$ and $p\in(0,\infty)$. Define $\varphi_p(x,t):=\varphi(x,t^{1/p})$. Since $\varphi\in\Phi_w(\mathbb{R}^n)$, there exists $p_0\in[1,\infty)$ such that φ satisfies $(\mathrm{aInc})_{p_0}$. Then for any $p\in(0,p_0]$, by [14, p. 4329], φ_p is also a weak Φ-funciton and

$$\|f^p\|_{L^{\varphi_p(\cdot)}(\mathbb{R}^n)}=\|f\|^p_{L^{\varphi(\cdot)}(\mathbb{R}^n)}.$$

COROLLARY 2.20. *Let $\varphi\in\Phi_w(\mathbb{R}^n)$. Assume that $p,p_1,\cdots,p_m\in(0,\infty)$ satisfy $1/p=1/p_1+\cdots+1/p_m$. Then*

(1) If $f_j\in L^{\varphi_{p_j}(\cdot)}$ and $j\in\{1,\cdots,m\}$, then

$$\left\|\prod_{j=1}^m f_j\right\|_{L^{\varphi_p(\cdot)}}\leqslant m\prod_{j=1}^m\|f_j\|_{L^{\varphi_{p_j}(\cdot)}}$$

(2) If $f_j\in WL^{\varphi_{p_j}(\cdot)}$ and $j\in\{1,\cdots,m\}$, then

$$\left\|\prod_{j=1}^m f_j\right\|_{WL^{\varphi_p(\cdot)}}\leqslant C\prod_{j=1}^m\|f_j\|_{WL^{\varphi_{p_j}(\cdot)}}$$

3. Multiliear singular integrals on generalized Orlicz spaces

Hardy–Littlewood maximal function. For any $x \in \mathbb{R}^n$ and any locally integrable function f, the *Hardy–Littlewood maximal function* f is defined by

$$M(f)(x) := \sup_{Q \text{cube: } Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

We also recall a family of hypotheses that are closely related to the boundedness of the Hardy–Littlewood maximal function on generalized Orilcz spaces, which were introduced in [14, Definition 3.1].

DEFINITION 3.1. Given $\varphi \in \Phi_w(\mathbb{R}^n)$ and $0 < p < \infty$, we say that φ satisfies:

(A0), if there exists $\alpha \in (0,1]$ such that $\alpha \leqslant \varphi^{-1}(x,1) \leqslant \alpha^{-1}$ for almost $x \in \mathbb{R}^n$; Here and hereafter, $\varphi^{-1}(x,t) := \inf\{\tau \in [0,\infty) : \ \varphi(x,\tau) \geqslant t\}$.

(A1), if there exists $\beta \in (0,1)$ such that $\beta\varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$ for every $t \in [1, \frac{1}{|x-y|^n}]$ and every $x, y \in \mathbb{R}^n$ with $|x-y| \leqslant 1$.

(A2), if for every $s > 0$ there exist $\beta \in (0,1]$ and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\beta\varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$ for almost every $x, y \in \mathbb{R}^n$ and every $t \in [h(x)+h(y), s]$.

$(\mathrm{Inc})_p$, if $t \to t^{-p}\varphi(x,t)$ is increasing for almost $x \in \mathbb{R}^n$.

$(\mathrm{aInc})_p$, if $t \to t^{-p}\varphi(x,t)$ is almost increasing for almost $x \in \mathbb{R}^n$.

$(\mathrm{Dec})_p$, if $t \to t^{-p}\varphi(x,t)$ is decreasing for almost $x \in \mathbb{R}^n$.

$(\mathrm{aDec})_p$, if $t \to t^{-p}\varphi(x,t)$ is almost decreasing for almost in $x \in \mathbb{R}^n$.

The operator T is bounded from $L^{\varphi(\cdot)}(\mathbb{R}^n)$ to $L^{\psi(\cdot)}(\mathbb{R}^n)$ if $\|Tf\|_{L^{\psi(\cdot)}(\mathbb{R}^n)} \leqslant \|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}$ for all $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$. Provided that φ satisfies (A0)–(A2) and $(\mathrm{aInc})_p$ for some $p \in (1,\infty)$, the maximal operator is bounded on $L^{\varphi(\cdot)}(\mathbb{R}^n)$, which was proved in [25, Theorem 4.7].

LEMMA 3.2. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy (A0), (A1), (A2) and (aInc). Then the Hardy–Littlewood maximal operator M is bounded on $L^{\varphi(\cdot)}(\mathbb{R}^n)$.*

REMARK 3.3. A wealth of examples shown in [25, 14, 21] seems that in many situations the conditions on φ are optimal or near optimal except the weighted Lebesgue spaces $L^p_\omega(\mathbb{R}^n) := \{f : \ \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx < \infty\}$ with $\omega \in A_p(\mathbb{R}^n)$.

COROLLARY 3.4. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy (A0), (A1), (A2) and $(\mathrm{aInc})_p$ with $1 < p < \infty$. For any $s \in (0,p)$, then M^s is also bounded on $L^{\varphi(\cdot)}(\mathbb{R}^n)$, where $M^s(f) = [M(|f|^s)]^{1/s}$.*

Proof. For any given $s \in (0,p)$, $\varphi_s(x,t) := \varphi(x,t^{1/s})$. From [14, Proposition 3.5], we know that φ_s also satisfies (A0), (A1), (A2) and $(\mathrm{aInc})_{p/s}$. Thus, Lemma 3.2 yields that M is bounded on $L^{\varphi_s(\cdot)}(\mathbb{R}^n)$, which further implies that

$$\|M^s f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} = \|M(|f|^s)\|_{L^{\varphi_s(\cdot)}}^{1/s} \lesssim \||f|^s\|_{L^{\varphi_s(\cdot)}}^{1/s} = \|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}.$$

We get the desired result. $\square$

The following result is essentially proved in [14]. For the completeness, we give the details.

LEMMA 3.5. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy (A0), (A1), (A2) and $(\mathrm{aDec})_p$ with $1 < p < \infty$. Then the Hardy–Littlewood maximal operator M is bounded on $L^{\varphi^*(\cdot)}(\mathbb{R}^n)$.*

Proof. By Lemma 3.2, it suffices to check that φ^* satisfies (A0), (A1), (A2) and $(\mathrm{aInc})_{p'}$ with $1 < p < \infty$. From [21, Lemmas 3.7.6, 4.1.7 and 4.2.4], it follows that φ^* satisfies (A0), (A1), (A2). By [21, Proposition 2.4.9], we conclude that φ^* satisfies $(\mathrm{aInc})_{p'}$, where p' is the conjugate index of p, that is, $1/p + 1/p' = 1$. Thus, we finish the proof of Lemma 3.5. $\square$

Multilinear maximal function. Lerner et al. [32] introduced the Hardy–Littlewood maximal function in the multilinear setting. For an m-tuple $\vec{f} = (f_1, \cdots, f_m)$ of locally integrable functions, the *multilinear Hardy–Littlewood maximal function* $\mathscr{M}$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathscr{M}(\vec{f})(x) := \sup_{Q\,\text{cube}: Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|\, dy_j.$$

Obviously, for any $x \in \mathbb{R}^n$, $\mathscr{M}(\vec{f})(x) \leqslant \prod_{j=1}^m M(f_j)(x)$. It immediately yields the boundedness of the multilinear maximal function over generalized Orlicz spaces from Theorem 2.12 and Lemma 3.2.

THEOREM 3.6. *Let $\varphi_1, \ldots, \varphi_m \in \Phi_w(\mathbb{R}^n)$ and define $\varphi = (\prod_{j=1}^m \varphi_j^{-1})^{-1}$. Suppose that for every $j \in \{1, \ldots, m\}$, φ_j satisfies (A0), (A1), (A2) and $(\mathrm{aInc})_{p_j}$ with $1 < p_j < \infty$. Then*

$$\left\|\mathscr{M}(\vec{f})\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant C \prod_{j=1}^{m} \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

Proof. By Theorem 2.12 and Lemma 3.2, we have

$$\left\|\mathscr{M}(\vec{f})\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant \left\|\prod_{j=1}^{m} M(f_j)\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|M(f_j)\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}. \quad \square$$

Sparse operators. Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z},\ j \in \mathbb{Z}^n.$$

Denote the standard grid by $\mathscr{D}$. By a *general dyadic grid* $\mathscr{D}$ we mean a collection of cubes with the following properties:

(i) for any $Q \in \mathscr{D}$ its sidelength $\ell(Q)$ is of the form 2^k, $k \in \mathbb{Z}$;

(ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q,\ R \in \mathscr{D}$;

(iii) the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

We say that $\{Q_j^k\}$ is a *sparse family* of cubes if: (i) the cubes Q_j^k are disjoint in j, with k fixed; (ii) if $\Omega_k := \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$; (iii) $|\Omega_{k+1} \cap Q_j^k| \leqslant \frac{1}{2}|Q_j^k|$.

With each sparse family $\{Q_j^k\}$ we have the associated sets $E_j^k = Q_j^k \setminus \Omega_{k+1}$. We find that the sets E_j^k are pairwise disjoint and $|Q_j^k| \leqslant 2|E_j^k|$.

Given a cube Q_0, denote by $\mathscr{D}(Q_0)$ the set of all dyadic cubes with respect to Q_0, that is, the cubes from $\mathscr{D}(Q_0)$ are formed by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes. It remarks here that if $Q_0 \in \mathscr{D}$, then each cube from $\mathscr{D}(Q_0)$ falls in $\mathscr{D}$.

Given a sparse family $\mathscr{S} := \{Q_j^k\}$ of cubes from a general dyadic grid $\mathscr{D}$, the sparse operator $\mathscr{A}_{\mathscr{D},\mathscr{S}}$ is defined by setting, for any $x \in \mathbb{R}^n$ and $\vec{f} := (f_1, \cdots, f_m)$,

$$\mathscr{A}_{\mathscr{D},\mathscr{S}}\left(\vec{f}\right)(x) := \sum_{j,k}\left[\prod_{i=1}^m \frac{1}{|Q_j^k|}\int_{Q_j^k} f_i(y_i)\,dy_i\right]\chi_{Q_j^k}(x).$$

THEOREM 3.7. *Let $\varphi_1, \ldots, \varphi_m \in \Phi_w(\mathbb{R}^n)$ and define $\varphi = (\prod_{j=1}^m \varphi_j^{-1})^{-1}$. Suppose that for every $j \in \{1, \ldots, m\}$, φ_j satisfies (A0), (A1), (A2), $(\mathrm{aInc})_{p_j}$ and $(\mathrm{aDec})_{q_j}$ with $1 < p_j, q_j < \infty$. Then $\mathscr{A}_{\mathscr{D},\mathscr{S}}$ is bounded from $L^{\varphi_1(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{\varphi_m(\cdot)}(\mathbb{R}^n)$ to $L^{\varphi(\cdot)}(\mathbb{R}^n)$, that is, there exists a positive constant C, independent of $\mathscr{S}$ and f_j, $j = 1, \ldots, m$, such that*

$$\left\|\mathscr{A}_{\mathscr{D},\mathscr{S}}(\vec{f})\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant C\prod_{j=1}^m \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

Proof. We first claim that there exists $p \in (0,\infty)$ such that φ_p satisfies (A0), (A1), (A2) and $(\mathrm{aInc})_{\bar{p}}$ for some $\bar{p} \in (1,\infty)$. To this end, let $1/\widetilde{p} = 1/p_1 + \cdots + 1/p_m$ and $1/\widetilde{q} = 1/q_1 + \cdots + 1/q_m$. By definitions of (A0), (A1), (A2) and Lemma 2.5, we know that φ also satisfies (A0), (A1) and (A2). From [21, Proposition 2.3.7], we can conclude that $\varphi(x,t)/t^{\widetilde{p}}$ is almost increasing for almost $x \in \mathbb{R}^n$. This shows that, for any $p \in (0,\widetilde{p})$, φ_p satisfies $(\mathrm{aInc})_{\widetilde{p}/p}$. Similarly, for any $q \in (0,\widetilde{q})$, φ_q satisfies $(\mathrm{aDec})_{\widetilde{q}/q}$. From Lemma 3.5, it follows that M is bounded on $L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)$.

Fix $\mathscr{S} \in \mathscr{D}$. Assume that $f_j \geqslant 0$. By Lemma 2.14

$$
\begin{aligned}
\left\|\mathscr{A}_{\mathscr{D},\mathscr{S}}(\vec{f})\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} &= \left\|\left[\mathscr{A}_{\mathscr{D},\mathscr{S}}(\vec{f})\right]^q\right\|_{L^{\varphi_q(\cdot)}(\mathbb{R}^n)}^{1/q} \\
&\leqslant C\left\{\sup_{\|g\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}\left[\sum_{j,k}\prod_{i=1}^m\frac{1}{|Q_j^k|}\int_{Q_j^k} f_i(y)\,dy\chi_{Q_j^k}(y)\right]^q|g(y)|\,dy\right\}^{\frac{1}{q}} \\
&\leqslant C\left\{\sup_{\|g\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)}\leqslant 1}\sum_{j,k}\prod_{i=1}^m\left[\frac{1}{|Q_j^k|}\int_{Q_j^k} f_i(y)\,dy\right]^q\int_{Q_j^k}|g(y)|\,dy\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Applying Theorem 2.12 and Lemma 3.2, we have

$$
\begin{aligned}
&\sum_{j,k}\prod_{i=1}^m\left[\frac{1}{|Q_j^k|}\int_{Q_j^k} f_i(y)\,dy\right]^q\int_{Q_j^k} g(y)\,dy \\
&\quad=\sum_{j,k}\prod_{i=1}^m\left[\frac{1}{|Q_j^k|}\int_{Q_j^k} f_i(y)\,dy\right]^q\frac{1}{|Q_j^k|}\int_{Q_j^k}|g(y)|\,dy|Q_j^k| \\
&\quad\leqslant\sum_{j,k}\int_{E_j^k}\left[\mathscr{M}(\vec{f})(x)\right]^q M(g)(x)\,dx\leqslant\int_{\mathbb{R}^n}\left[\mathscr{M}(\vec{f})(x)(x)\right]^q M(g)(x)\,dx \\
&\quad\leqslant\left\|\left[\mathscr{M}(\vec{f})\right]^q\right\|_{L^{\varphi_q(\cdot)}(\mathbb{R}^n)}\|M(g)\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}^q.
\end{aligned}
$$

whenever $\|g\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)}\leqslant 1$. We are done. □

Multilinear Calderón–Zygmund operators. Let T be a multilinear operator initially defined on the $m-$fold product of Schwarz spaces and take values into the space of tempered distributions,

$$T:\mathscr{S}(\mathbb{R}^n)\times\cdots\times\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n).$$

We say that T is an $m-$linear Calderón-Zygmund operator if for some $q_1,\cdots,q_m\in(1,\infty)$ and $q\in(0,\infty)$ with $1/q=1/q_1+\cdots+1/q_m$, it extends to a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q and satisfies the following conditions:

(1) If there exists a function K, defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$, such that

$$T(f_1,\ldots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x,y_1,\cdots,y_m)\prod_{j=1}^m f(y_j)d\vec{y}$$

for all $x\notin\cap_{j=1}^m\mathrm{supp}f_j$;

(2) There exists a constant C such that

$$|K(x,y_1,\ldots,y_m)| \leqslant \frac{C}{(|x-y_1|+\cdots+|x-y_m|)^{mn}};$$

(3) For some $\varepsilon > 0$, there exists a constant C such that

$$\left|K(x,y_1,\ldots,y_m)-K(x',y_1,\cdots,y_m)\right| \leqslant \frac{C|x-x'|^{\varepsilon}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon}},$$

whenever $|x-x'| \leqslant 1/2\max_{1\leqslant j\leqslant m}|x-y_j|$;

(4) For some $\varepsilon > 0$, there exists a constant C such that

$$\begin{aligned}&\left|K(x,y_1,\cdots,y_i,\cdots,y_m)-K(x,y_1,\cdots,y_i',\cdots,y_m)\right|\\ &\leqslant \frac{C|y_i-y_i'|^{\varepsilon}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon}},\end{aligned}$$

whenever $|y_i-y_i'| \leqslant 1/2\max_{1\leqslant j\leqslant m}|x-y_j|$ for $i\in\{1,2,\cdots,m\}$.

Let $\varphi\in\Phi_w(\mathbb{R}^n)$. From [15, Theorem 1.4], we know that

$$\left\|T\left(\vec{f}\right)\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant C\sup_{\mathscr{D},\mathscr{S}}\left\|\mathscr{A}_{\mathscr{D},\mathscr{S}}\left(\vec{f}\right)\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)},$$

which, together with Theorem 3.7, implies the boundedness of T over the product generalized Orlicz spaces immediately.

THEOREM 3.8. *Let $\varphi_1,\ldots,\varphi_m\in\Phi_w(\mathbb{R}^n)$ and define $\varphi=(\prod_{j=1}^m\varphi_j^{-1})^{-1}$. Let T be a multilinear Calderón–Zygmund operator. Suppose that for every $j\in\{1,\ldots,m\}$, φ_j satisfies (A0), (A1), (A2), (aInc)$_{p_j}$ and (aDec)$_{q_j}$ with $1<p_j,q_j<\infty$. Then T is bounded from $L^{\varphi_1(\cdot)}(\mathbb{R}^n)\times\cdots\times L^{\varphi_m(\cdot)}(\mathbb{R}^n)$ to $L^{\varphi(\cdot)}(\mathbb{R}^n)$, that is, there exists a positive constant C, independent of f_j, $j=1,\ldots,m$, such that*

$$\left\|T\left(\vec{f}\right)\right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leqslant C\prod_{j=1}^m\|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

REMARK 3.9. Our results could be extended to the multilinear singular integral operators with the L^r-Hörmander condition as in [33].

Multilinear fractional integral operators. Multilinear fractional integral operators were studied by many authors; see for instance [18, 30, 35, 38, 39, 44]. For $\vec{f}=(f_1,\ldots,f_m)$ and $0<\alpha<mn$, the multilinear fractional integral operator is defined by

$$\mathscr{I}_\alpha(\vec{f})(x) := \int_{\mathbb{R}^{mn}}\frac{|f_1(y_1)\cdots f_m(y_m)|}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\,dy_1\cdots dy_m,$$

and the associated multilinear fractional maximal operator $\mathscr{M}_\alpha$ is defined by

$$\mathscr{M}_\alpha(\vec{f})(x) := \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|^{1-\alpha/(mn)}} \int_Q |f_j(y_j)|\, dy_j.$$

When $m=1$, it goes back to the fractional maximal operator M_α. As in the linear case, $\mathscr{M}_\alpha(\vec{f}) \lesssim \mathscr{I}_\alpha(\vec{f})$ for $f_i \geqslant 0$, $i=1,\ldots,m$.

THEOREM 3.10. *Let $\varphi_1,\ldots,\varphi_m \in \Phi_w(\mathbb{R}^n)$. Suppose that for every $j \in \{1,\ldots,m\}$, φ_j satisfies (A0), (A1), (A2) and (aInc)$_{p_j}$, (aDec)$_{q_j}$ with $1 < p_j, q_j < \infty$. Let $\psi_j^{-1}(x,t) := t^{-\frac{\alpha}{mn}} \varphi_j^{-1}(x,t)$, $j=1,\ldots,m$ and $\psi = (\prod_{j=1}^m \psi_j^{-1}(x,t))^{-1}$. Then $\mathscr{I}_\alpha$ is bounded from $L^{\varphi_1(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{\varphi_m(\cdot)}(\mathbb{R}^n)$ to $L^{\psi(\cdot)}(\mathbb{R}^n)$, that is, there exists a positive constant C, independent of f_j, $j=1,\ldots,m$, such that*

$$\left\| \mathscr{I}_\alpha\left(\vec{f}\right) \right\|_{L^{\psi(\cdot)}(\mathbb{R}^n)} \leqslant C \prod_{j=1}^{m} \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

Proof. From [14, Propositions 3.5 and 3.6], we deduce that ψ_j, $j=1,\ldots,m$ satisfy (A0), (A1), (A2) and (aInc)$_{r_j}$ and (aDec)$_{\gamma_j}$, where $1/r_j = 1/p_j - \alpha/(mn)$ and $1/\gamma_j = 1/q_j - \alpha/(mn)$. Let $1/\widetilde{r} = 1/r_1 + \cdots + 1/r_m$ and $1/\widetilde{\gamma} = 1/\gamma_1 + \cdots + 1/\gamma_m$. By the same statement of the proof of Theorem 3.8, we know that for any $r \in (0,\widetilde{r})$, ψ_r satisfies (aInc)$_{\widetilde{r}/r}$. Similarly, for any $\gamma \in (0,\widetilde{\gamma})$, ψ_γ satisfies (aDec)$_{\widetilde{\gamma}/\gamma}$. From Lemma 3.5, it follows that M is bounded on $L^{\psi_\gamma^*(\cdot)}(\mathbb{R}^n)$.

From [35, (2.2)], we know that

$$\mathscr{I}_\alpha\left(\vec{f}\right)(x) \lesssim \sup_{\mathscr{D},\mathscr{S}} \left[\sum_{Q \in \mathscr{S}} \ell(Q)^\alpha \left\{ \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|\, dy_j \right\} \mathbf{1}_Q \right] =: \sup_{\mathscr{D},\mathscr{S}} \mathscr{I}_\alpha^{\mathscr{S}}\left(\vec{f}\right)(x).$$

Fix $\mathscr{S} \in \mathscr{D}$. Assume that $f_j \geqslant 0$. By Lemma 2.14 and a similar statement of the proof of Theorem 3.8, we have

$$\begin{aligned}
&\left\| \mathscr{I}_\alpha^{\mathscr{S}}\left(\vec{f}\right) \right\|_{L^{\psi(\cdot)}(\mathbb{R}^n)} \\
&= \left\{ \sup_{\|g\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)} \leqslant 1} \int_{\mathbb{R}^n} \sum_{Q \in \mathscr{S}} \ell(Q)^\alpha \left[\prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|\, dy_j \right]^\gamma \mathbf{1}_Q(x) g(x)\, dx \right\}^{\frac{1}{\gamma}} \\
&\lesssim \left\{ \sup_{\|g\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)} \leqslant 1} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \left[M_{\frac{\alpha}{m}}(f_j)(x) \right]^\gamma M(g)(x)\, dx \right\}^{\frac{1}{\gamma}} \\
&\lesssim \left\{ \sup_{\|g\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)} \leqslant 1} \left\| \prod_{j=1}^{m} \left[M_{\frac{\alpha}{m}}(f_j) \right]^\gamma \right\|_{L^{\psi_\gamma(\cdot)}(\mathbb{R}^n)} \|M(g)\|_{L^{\varphi_q^*(\cdot)}(\mathbb{R}^n)} \right\}^{\frac{1}{\gamma}}.
\end{aligned}$$

From [14, Corollary 6.8], we know that $M_{\frac{\alpha}{m}}$ is bounded from $L^{\varphi_j(\cdot)}(\mathbb{R}^n)$ to $L^{\psi_j(\cdot)}(\mathbb{R}^n)$ for all $j=1,\ldots,m$. This, together Theorem 2.12, yields that

$$\left\|\prod_{j=1}^m \left[M_{\frac{\alpha}{m}}(f_j)\right]^\gamma\right\|_{L^{\psi\gamma(\cdot)}(\mathbb{R}^n)} = \left\|\prod_{j=1}^m M_{\frac{\alpha}{m}}(f_j)\right\|^\gamma_{L^{\psi(\cdot)}(\mathbb{R}^n)} \lesssim \prod_{j=1}^m \left\|M_{\frac{\alpha}{m}}(f_j)\right\|^\gamma_{L^{\psi_j(\cdot)}(\mathbb{R}^n)}$$
$$\lesssim \prod_{j=1}^m \|f_j\|^\gamma_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

Combined with above estimates, we get $\|\mathscr{I}_\alpha(\vec{f})\|_{L^{\psi(\cdot)}(\mathbb{R}^n)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}$, which is the desired result. $\square$

COROLLARY 3.11. *Let $\varphi_1,\ldots,\varphi_m \in \Phi_w(\mathbb{R}^n)$. Suppose that for every $j\in\{1,\ldots,m\}$, φ_j satisfies (A0), (A1), (A2), (aInc)$_{p_j}$ and (aDec)$_{q_j}$ with $1<p_j,q_j<\infty$. Let $\psi_j^{-1}(x,t) := t^{-\frac{\alpha}{mn}}\varphi_j^{-1}(x,t)$, $j=1,\ldots,m$ and $\psi = (\prod_{j=1}^m \psi_j^{-1}(x,t))^{-1}$. Then $\mathscr{M}_\alpha$ is bounded from $L^{\varphi_1(\cdot)}(\mathbb{R}^n)\times\cdots\times L^{\varphi_m(\cdot)}(\mathbb{R}^n)$ to $L^{\psi(\cdot)}(\mathbb{R}^n)$, that is, there exists a positive constant C, independent of f_j, $j=1,\ldots,m$, such that*

$$\left\|\mathscr{M}_\alpha\left(\vec{f}\right)\right\|_{L^{\psi(\cdot)}(\mathbb{R}^n)} \leqslant C\prod_{j=1}^m \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)}.$$

REMARK 3.12. In the proofs of the boundedness of the multilinear singular integrals and multilinear fractional integrals on the generalized Orlicz spaces, we employ the sparse operators and the duality. However, it is interesting to prove them directly.

REMARK 3.13. The boundedness of commutators of Coifman–Meyer type generated by the multilinear singular integrals and the symbol $b\in BMO(\mathbb{R}^n)$ could be obtained by the Hölder inequality, the sparse operators and the duality. We leave it to the interested readers.

REFERENCES

[1] M. K. ALAOUI, T. NABIL, AND M. ALTANJI, *On some new non-linear diffusion models for the image filtering*, Appl. Anal. **93** (2) (2014), 269–280.
[2] T. ANH BUI, *Regularity estimates for nondivergence parabolic equations on generalized Orlicz spaces*, Inter. Math. Research Notices, vol. 00, no. 0 (2020), pp. 1–37.
[3] M. AVCI AND A. PANKOV, *Multivalued elliptic operators with nonstandard growth*, Advances in Nonlinear Analysis, March 2016, 1–14.
[4] P. BARONI, M. COLOMBO AND G. MINGIONE, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121** (2015), 206–222.
[5] P. BARONI, M. COLOMBO, AND G. MINGIONE, *Nonautonomous functionals, borderline cases and related function classes*, Algebra i Analiz **27** (2015), no. 3, 6–50; English transl., St. Petersburg Math. J. **27** (2016), no. 3, 347–379.
[6] A. BENEDEK AND R. PANZONE, *The space L^p, with mixed norm*, Duke Math. J. **28** (1961), 301–324.
[7] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, Pure Appl. Math. **129**, Academic Press, Boston, MA, 1988.

[8] Y. CHEN, S. LEVINE, AND M. RAO, *Variable exponent, linear growth functionals in image restoration*, SIAMJ. Appl. Math. **66** (2006), no. 4, 1383–1406.

[9] R. COIFMAN, Y. MEYER, *On commutators of singular integral and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[10] R. COIFMAN, Y. MEYER, *Au delà des opérateurs pseudo-différentiels, Astérisque*, **57** (1978).

[11] R. COIFMAN, Y. MEYER, *Ondelettes et opérateurs*, III, Hermann, Paris, 1990.

[12] M. CHRIST, J. L. JOURNÉ, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. **159** (1987), 51-80.

[13] M. COLOMBO AND G. MINGIONE, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218** (1) (2015), 219–273.

[14] D. CRUZ-URIBE, P. HÄSTÖ, *Extrapolation and interpolation in generalized Orlicz spaces*, Trans. Am. Math. Soc. **370** (6) (2018), 4323–4349.

[15] W. DAMIÁN, A. K. LERNER AND C. PÉREZ, *Sharp weighted bounds for multilinear maximal functions and Calderón–Zygmund operators*, J. Fourier Anal. Appl. **21** (2015), 161–181.

[16] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, M. RŮŽIČKA, *Lebesgue and Sobolev spaces with Variable Exponents*, vol. 2017 of Lecture Notes in Mathematics (Springer, Heidelberg, 2011).

[17] F. GIANNETTI AND A. PASSARELLI DI NAPOLI, *Regularity results for a new class of functionals with non-standard growth conditions*, J. Differ. Equ. **254** (2013), no. 3, 1280–1305.

[18] L. GRAFAKOS, *On multilinear fractional integrals*, Studia Math. **102** (1992), 49–56.

[19] L. GRAFAKOS, R. H. TORRES, *Multilinear Calderón–Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[20] L. GRAFAKOS, R. H. TORRES, *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J. **51** (2002), 1261–1276.

[21] P. HARJULEHTO AND P. HÄSTÖ, *Orlicz Spaces and Generalized Orlicz Spaces*, Springer Nature Switzerland AG 2019.

[22] P. HARJULEHTO AND P. HÄSTÖ, *The Riesz potential in generalized Orlicz spaces*, Forum Math. **29** (1) (2017), 229–244.

[23] P. HARJULEHTO, P. HÄSTÖ, R. KLÉN, *Generalized Orlicz spaces and related PDE*, Nonlinear Anal. **143** (2016) 155–173.

[24] P. HARJULEHTO, P. HÄSTÖ, V. LATVALA, AND O. TOIVANEN, *Critical variable exponent functionals in image restoration*, Appl. Math. Lett. **26** (1) (2013), 56–60.

[25] P. HÄSTÖ, *The maximal operator on generalized Orlicz spaces*, J. Funct. Anal. **269** (12) (2015), 4038–4048.

[26] P. HÄSTÖ, *Corrigendum to "The maximal operator on generalized Orlicz spaces"* [J. Funct. Anal. **269** (2015) 4038–4048], J. Funct. Anal. **271** (2016), no. 1, 240–243.

[27] P. HÄSTÖ AND J. OK, *Calderón–Zygmnund estimate in generalized Orlicz spaces*, J. Differ. Equ. **267** (2019), 2792–2823.

[28] A. W. HUANG AND J. S. XU, *Multilinear singular integrals and commutators in variable exponent Lebesgue spaces*, Appl. Math. J. Chinese Univ. Ser. B, **25** (1) (2010), 69–77.

[29] T. IIDA, E. SATO, Y. SAWANO, H. TANAKA, *Sharp bounds for multilinear fractional integral operators on morrey type spaces*, Positivity **16** (2) (2012), 339–358.

[30] C. E. KENIG AND E. M. STEIN, *Multilinear estimates and fractional integration*, Math. Res Lett. **6** (1999), 1–15.

[31] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czec. Math. J. **41** (4) (1991), 592–618.

[32] A. K. LERNER, S. OMBROSI, C. PÉREZ, R. H. TORRES, AND R. TRUJILLO-GONZÁLEZ, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), 1222–1264.

[33] K. LI, *Sparse domination theorem for multilinear singular integral operators with L^r-Hörmander condition*, Mich. Math. J. **67** (2) (2016), 253–265.

[34] K. LI, K. MOEN AND W. SUN, *The sharp weighted bound for multilinear maximal functions and Calderón–Zygmund operators*, J. Fourier Anal. Appl. **20** (2014), 751–765.

[35] K. LI, K. MOEN AND W. SUN, *Sharp weighted inequalities for multilinear fractional maximal operators and fractional integrals*, Math. Nachr. **288** (2015), 1–14.

[36] K. LI AND W. SUN, *Weak and strong type weighted estimates for multilinear Calderón–Zygmund operators*, Adv. Math. **254** (2014), 736–771.

[37] L. MALIGRANDA, *Why Hölder's inequality should be called Rogers' inequality*, Math. Inequal. Appl. **1** (1998), no. 1, 69–83.
[38] K. MOEN, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math. **60** (2009), 213–238.
[39] K. MOEN, *New weighted estimates for bilinear fractional integral operators*, Trans. Amer. Math. Soc. **366** (2014), 627–646.
[40] J. MUSIELAK, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, vol. 1034, Springer, Berlin, 1983.
[41] H. NAKANO, *Modulared Semi-ordered Linear Spaces*, Maruzen Co., Ltd., Tokyo, 1950.
[42] H. NAKANO, *Topology of Linear Topological Spaces*, Maruzen Co., Ltd., Tokyo, 1951.
[43] W. ORLICZ, *Über konjugierte Exponentenfolgen*, Stud. Math. **3** (1931) 200-211.
[44] G. PRADOLINI, *Weighted inequalities and pointwise estimates for the multilinear fractional integral and maximal operators*, J. Math. Anal. Appl. **367** (2010), 640–656.
[45] M. M. RAO, Z. D. REN, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker, Inc., New York, 1991.
[46] A. ŚWIERCZEWSKA–GWIAZDA, *Nonlinear parabolic problems in Musielak-Orlicz spaces*, Nonlinear Anal. **98** (2014), 48–65.
[47] H. WANG AND W. YI, *Multilinear singular and fractional integral operators on weighted Morrey spaces*, Journal of Function Spaces and Applications, vol. 2013 (2013), Article ID 735795, 11 pages.

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