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GLOBAL REGULARITY IN PARABOLIC WEIGHTED ORLICZ–MORREY SPACES OF SOLUTIONS TO PARABOLIC EQUATIONS WITH VMO COEFFICIENTS

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Abstract. We show continuity in parabolic generalized weighted Orlicz-Morrey spaces $M_w^{\Phi,\varphi}$ of sublinear integral operators generated by parabolic singular and nonsingular operators and their commutators with BMO functions. The obtained estimates are used to study global regularity of the solution of the Cauchy-Dirichlet problem for linear uniformly parabolic operators of second order with discontinuous data.

1. Introduction

There has been tremendous work on the Calderón-Zygmund theory to weak solutions of various elliptic and parabolic equations in recent decades. As we know, many elliptic and parabolic equations with discontinuous coefficients are often proposed in models of deformations in composite materials as fiberreinforced media, in the mechanics of membranes and films of simple non-homogeneous materials which form a linear laminated medium. In particular, a highly twinned elastic or ferroelectric crystal is a typical situation where the laminates appear.

As a starting point of the Calderón-Zygmund theory to partial differential equations involving discontinuous coefficients, both interior and boundary $W^{2,p}$ estimates were first established by Chiarenza et al. [6] for nondivergence linear elliptic equations when each $a_{ij}(x)$ belongs to VMO spaces for every $i,j = 1,\ldots,n$, and later attained by Bramanti and Cerutti [2] in the case of parabolic problems. Since then, there was a great deal of literature concerning the topic of Calderón-Zygmund theory to various elliptic and parabolic problems with discontinuous coefficients, for details see [4, 8, 11, 12, 22, 25, 26, 28, 30, 34].

Weighted Orlicz spaces are the natural generalizations of weighted Sobolev spaces, and the estimates in weighted Orlicz spaces to partial differential equations have become an extremely popular research nowadays. Areas of its applications include the study of geometric, probability, stochastic, Fourier analysis and so on, also see [45].

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Motivated by the extension of parabolic Calderon-Zygmund theory to the weighted Orlicz context, we study the boundedness of parabolic singular and nonsingular integral operators and their commutators with BMO functions, on *parabolic generalized weighted Orlicz-Morrey spaces*. Also we show some applications to strong solutions to non-divergence parabolic equations of second order with VMO coefficients. In the present work we study the global regularity of the solutions of a class of parabolic partial differential equations (PDEs) in parabolic generalized weighted Orlicz-Morrey spaces. In connection with elliptic partial differential equations, C. Morrey proposed a weak condition for the solution to be continuous enough in [39]. Later on, his condition became a family of normed spaces and they are called Morrey spaces $L^{p,\lambda}$. Although the notion is originally from the partial differential equations, the space turned out to be important in many branches of mathematics.

Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [17, 38, 40] introduced generalized Morrey spaces $M^{p,\varphi}$ (see, also [18, 47]). Komori and Shirai [33] defined weighted Morrey spaces $L^{p,\kappa}(w)$. Guliyev [20] gave a concept of the generalized weighted Morrey spaces $M_w^{p,\varphi}$ which could be viewed as extension of both $M^{p,\varphi}$ and $L^{p,\kappa}(w)$. The boundedness of the classical operators and their commutators in spaces $M_w^{p,\varphi}$ was studied in [1, 10, 19, 20, 23]. In [22, 25, 27, 28] we apply these estimates to study the regularity of the solution of Dirichlet problem for linear elliptic and parabolic partial differential equation with discontinuous coefficients. The presented result is a generalization of previous works [2, 25, 28].

The reason to study continuity properties of these integrals in various functional spaces is that they permit to investigate the regularity of solutions to linear elliptic and parabolic partial differential equations and systems in terms of the data of the corresponding problems. The method, associated to the names of A. Calderón and A. Zygmund (see [5]) uses explicit representation formula for the highest-order derivatives of the solution in terms of singular integrals acting on the known right-hand side plus another one acting on the very same derivatives. This last term appears in a commutator which norm can be made small enough if the coefficients have small oscillation over small balls. This way, suitable ”integral continuity” of the principal coefficients ensure boundedness of the commutator and therefore validity of the corresponding a priori estimate. The Sarason class of functions with vanishing mean oscillation verifies this requirement although they could be discontinuous. Their good behavior on small balls allows to extend the classical theory of elliptic and parabolic equations and systems with continuous coefficients (see [14, 34, 35, 37, 44]) to operators with discontinuous coefficients (cf. [2, 6, 37]). A vast number of works are dedicated to boundary value problems for linear elliptic and parabolic operators with VMO coefficients in the framework of Sobolev and Sobolev-Morrey spaces (see [12, 22, 25, 26, 28]).

The main goal of the present paper is to extend the global parabolic weighted Morrey regularity results from [27, 28], regarding linear parabolic equations with VMO principal coefficients, to the settings of parabolic generalized weighted Orlicz-Morrey spaces $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})$ (see Definition 2.2). The approach adopted is that of [22, 26] and relies on proving boundedness of suitable integral operators and their commutators, that appear at the representation formula for the second order derivatives of the solution. Even if standard in some sense, that method requires precise analysis due to the specifics of the considered parabolic generalized weighted Orlicz-Morrey spaces, and we employ our results from [25] to get the desired $M_w^{\Phi,\varphi}$-boundedness of the parabolic singular and nonsingular integrals and their commutators.

The article is organized as follows. In Section 2 we introduce the problem and give some basic notions. In this section we recall also continuity results regarding the parabolic Calderón-Zygmund integrals that appear in the interior representation formula of the derivatives D_{ij} of the solution. The corresponding nonsingular integrals are studied in Section 3. These results permit to obtain $M_w^{\Phi,\varphi}$-estimate of $D_t u$, $D_{ij}u$, $i,j=1,\dots,n$ near the boundary. The a priori estimate is established in the last section.

Throughout this paper the following notations will be used:

$x=(x',t), y=(y',\tau)\in\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}$, $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times\mathbb{R}_+$;

$x=(x'',x_n,t)\in\mathbb{D}^{n+1}_+=\mathbb{R}^{n-1}\times\mathbb{R}_+\times\mathbb{R}_+$, $\mathbb{D}^{n+1}_-=\mathbb{R}^{n-1}\times\mathbb{R}_-\times\mathbb{R}_+$;

$|\cdot|$ is the Euclidean metric, $|x|=\left(\sum_{i=1}^n x_i^2+t^2\right)^{1/2}$;

$D_i u=\partial u/\partial x_i$, $Du=(D_1u,\dots,D_nu)$, $D_t u=u_t=\partial u/\partial t$;

$D_{ij}u=\partial^2 u/\partial x_i\partial x_j$, $D^2u=\{D_{ij}u\}_{ij=1}^n$ means the Hessian matrix of u;

$\mathscr{B}_r(x')=\{y'\in\mathbb{R}^n:\ |x'-y'|<r\}$, $|\mathscr{B}_r|=Cr^n$;

$\mathscr{C}_r(x)=\{y\in\mathbb{R}^{n+1}:\ |x'-y'|<r, t-\tau<r^2\}$ is a parabolic cylinder;

$\mathscr{E}_r(x)=\{y\in\mathbb{R}^{n+1}:\ \frac{(x_1-y_1)^2}{r^2}+\dots+\frac{(x_n-y_n)^2}{r^2}+\frac{(t-\tau)^2}{r^4}<1\}$;

$\mathscr{E}_r^c(x)=\mathbb{R}^{n+1}\setminus\mathscr{E}_r(x)$, $|\mathscr{E}_r(x)|=Cr^{n+2}$, $2\mathscr{E}_r(x)=\mathscr{E}_{2r}(x)$;

$\mathbb{S}^n$ is a unit sphere in $\mathbb{R}^{n+1}$;

for any bounded domain Ω and cylinder $Q=\Omega\times(0,T)$ define
$\Omega_r=\Omega\cap\mathscr{B}_r(x'), x'\in\Omega,\quad Q_r=Q\cap\mathscr{C}_r(x), x\in Q$.

The standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various positive constants and may change from one occurrence to another. In this paper, we shall use the symbol $A\lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A\leqslant CB$. $A\approx B$ means that $A\lesssim B$ and $B\lesssim A$.

2. Some preliminaries on weighted Orlicz and parabolic generalized weighted Orlicz-Morrey spaces

We recall the definition of Young functions.

DEFINITION 2.1. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0}\Phi(r)=\Phi(0)=0$ and $\lim\limits_{r\to\infty}\Phi(r)=\infty$.

From the convexity and $\Phi(0)=0$ it follows that any Young function is increasing. If there exists $s\in(0,\infty)$ such that $\Phi(s)=\infty$, then $\Phi(r)=\infty$ for $r\geqslant s$. The set of Young functions such that

$$0<\Phi(r)<\infty \qquad \text{for} \qquad 0<r<\infty$$

will be denoted by $\mathscr{Y}$. If $\Phi\in\mathscr{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0\leqslant s\leqslant\infty$, let

$$\Phi^{-1}(s)=\inf\{r\geqslant 0:\Phi(r)>s\}.$$

If $\Phi\in\mathscr{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r))\leqslant r\leqslant \Phi^{-1}(\Phi(r)) \quad \text{for} \quad 0\leqslant r<\infty.$$

It is well known that

$$r\leqslant \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r)\leqslant 2r \qquad \text{for} \quad r\geqslant 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r)=\begin{cases}\sup\{rs-\Phi(s):s\in[0,\infty)\}, & r\in[0,\infty)\\ \infty, & r=\infty.\end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi\in\Delta_2$, if

$$\Phi(2r)\leqslant k\Phi(r) \quad \text{for} \quad r>0$$

for some $k>1$. If $\Phi\in\Delta_2$, then $\Phi\in\mathscr{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi\in\nabla_2$, if

$$\Phi(r)\leqslant\frac{1}{2k}\Phi(kr), \qquad r\geqslant 0$$

for some $k>1$.

We recall an important pair of indices used for Young functions. For any Young function Φ, write

$$h_\Phi(t)=\sup_{s>0}\frac{\Phi(st)}{\Phi(s)}, \quad t>0.$$

The lower and upper dilation indices of Φ are defined by

$$i_\Phi = \lim_{t\to 0^+} \frac{\log h_\Phi(t)}{\log t} \text{ and } I_\Phi = \lim_{t\to\infty} \frac{\log h_\Phi(t)}{\log t},$$

respectively.

We have $i_\Phi > 1$ as consequence of $\Phi \in \nabla_2$ (see [13]). On the other hand, $\Phi \in \Delta_2$ implies that there exist two exponents $p_1, p_2 \in (1,\infty)$, $p_1 \leqslant p_2$, such that

$$c^{-1} \min\{\lambda^{p_1}, \lambda^{p_2}\} \Phi(t) \leqslant \Phi(\lambda t) \leqslant c \max\{\lambda^{p_1}, \lambda^{p_2}\} \Phi(t) \quad \text{for } \lambda, t > 0, \tag{2.2}$$

with a constant c is independent of λ and t (see [32]), from which one can easily check that

$$L^\infty \subset L^{p_2}_w \subset L^\Phi_w \subset L^{p_1}_w \subset L^1.$$

The supremum of those p_1 for which (2.2) holds true with $\lambda \geqslant 1$ being equal to i_Φ. If, for instance, $\Phi(t) = t^p$ with $p > 1$ then $i_\Phi = p$.

DEFINITION 2.2. (Weighted Orlicz space). For a Young function Φ and $w \in A_\infty$, the set

$$L^\Phi_w(\mathbb{R}^{n+1}) = \left\{ f \in L^{1,\mathrm{loc}}_w(\mathbb{R}^{n+1}) : \int_{\mathbb{D}^{n+1}_+} \Phi(k|f(x)|)w(x)dx < \infty \text{ for some } k > 0 \right\}$$

is called weighted Orlicz space. The local weighted Orlicz space $L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^{n+1})$ is defined as the set of all functions f such that $f\chi_{\mathscr{E}} \in L^\Phi_w(\mathbb{R}^{n+1})$ for all parabolic balls $\mathscr{E} \subset \mathbb{D}^{n+1}_+$.

Note that $L^\Phi_w(\mathbb{R}^{n+1})$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi_w(\mathbb{R}^{n+1})} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^{n+1}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)dx \leqslant 1 \right\}.$$

For a weight w, a measurable function f and $t > 0$, let

$$m(w,f,t) = w(\{x \in \mathbb{R}^{n+1} : |f(x)| > t\}).$$

The weak weighted Orlicz space

$$WL^\Phi_w(\mathbb{R}^{n+1}) = \{f \in L^{1,\mathrm{loc}}_w(\mathbb{R}^{n+1}) : \|f\|_{WL^\Phi_w(\mathbb{R}^{n+1})} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL^\Phi_w(\mathbb{R}^{n+1})} = \inf\Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(w, \frac{f}{\lambda}, t\Big) \leqslant 1 \Big\}.$$

If $\Phi(t) = t^q$ with $1 < q < \infty$, it is clear that satisfies the $\Delta_2 \cap \nabla_2$-condition. In this case, the weighted Orlicz space $L^\Phi_w(\mathbb{R}^{n+1})$ coincides with the weighted Lebesgue space $L_w^q(\mathbb{R}^{n+1})$. In other words, the weighted Orlicz spaces are the generalized ones of the weighted Lebesgue spaces.

We can prove the following by a direct calculation:

$$\|\chi_{\mathscr{E}}\|_{WL_w^{\Phi}(\mathbb{R}^{n+1})} = \|\chi_{\mathscr{E}}\|_{L_w^{\Phi}(\mathbb{R}^{n+1})} = \frac{1}{\Phi^{-1}\left(w(\mathscr{E})^{-1}\right)}. \tag{2.3}$$

Let $\mathbb{E} = \{\mathscr{E}_r(x) : x \in \mathbb{R}^{n+1},\ r>0\}$. The parabolic maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|\mathscr{E}_r(x)|} \int_{\mathscr{E}_r(x)} |f(y)|dy, \qquad x \in \mathbb{R}^{n+1}$$

for a locally integrable function f on $\mathbb{R}^{n+1}$.

THEOREM 2.1. [15, Proposition 2.4] *Let Φ be a Young function. Assume in addition $w \in A_{i_\Phi}$. Then, there is a constant $C>1$ such that*

$$\Phi(t)m\Big(w, Mf,\ t\Big) \leqslant C \int_{\mathbb{R}^{n+1}} \Phi\left(C|f(x)|\right) w(x)dx$$

for every locally integrable f and every $t>0$.

REMARK 2.1. For a sublinear operator S, weak modular inequality

$$\Phi(t)m\Big(w, Sf,\ t\Big) \leqslant C \int_{\mathbb{R}^{n}} \Phi\left(C|f(x)|\right) w(x)dx \tag{2.4}$$

implies the corresponding norm inequality. Indeed, let (2.4) holds. Then, we have

$$\begin{aligned}
&\Phi(t)w\left(\{x \in \mathbb{R}^{n+1} : \frac{|Sf(x)|}{C^2\|f\|_{L_w^{\Phi}}} > t\}\right) \\
&= \Phi(t)w\left(\{x \in \mathbb{R}^{n+1} : \left|S\Big(\frac{f}{C^2\|f\|_{L_w^{\Phi}}}\Big)(x)\right| > t\}\right) \\
&\leqslant C \int_{\mathbb{R}^{n+1}} \Phi\left(\frac{|f(x)|}{C\|f\|_{L_w^{\Phi}}}\right) w(x)dx \leqslant 1,
\end{aligned}$$

which implies $\|Sf\|_{WL_w^{\Phi}} \lesssim \|f\|_{L_w^{\Phi}}$.

LEMMA 2.1. *Let Φ be a Young function and $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$. Assume in addition $w \in A_{i_\Phi}$. For a parabolic ball $\mathscr{E}$, the following inequality is valid:*

$$\|f\|_{L^1(\mathscr{E})} \lesssim |\mathscr{E}|\,\Phi^{-1}\left(w(\mathscr{E})^{-1}\right)\|f\|_{L_w^{\Phi}(\mathscr{E})}.$$

Proof. Let

$$\mathfrak{M}f(x) = \sup_{\mathscr{E}\in\mathbb{E}} \frac{\chi_{\mathscr{E}}(x)}{|\mathscr{E}|} \int_{\mathscr{E}} |f(y)|dy, \quad x \in \mathbb{R}^{n+1}$$

and $\tilde{f}$ denotes the extension of f from $\mathscr{E}$ to $\mathbb{R}^{n+1}$ by zero. It is well known that $\mathfrak{M}f(x) \leqslant 2^{n+2}Mf(x)$ for all $x \in \mathbb{R}^{n+1}$. Then taking into account Remark 2.1 and using Theorem 2.1, we have

$$\begin{aligned}\frac{\|f\|_{L^1(\mathscr{E})}}{|\mathscr{E}|}\|\chi_{\mathscr{E}}\|_{WL^\Phi_w(\mathscr{E})} &= \frac{\|\tilde{f}\|_{L^1(\mathscr{E})}}{|\mathscr{E}|}\|\chi_{\mathscr{E}}\|_{WL^\Phi_w(\mathscr{E})} \lesssim \|\mathfrak{M}\tilde{f}\|_{WL^\Phi_w(\mathscr{E})} \\ &\lesssim \|M\tilde{f}\|_{WL^\Phi_w(\mathscr{E})} \leqslant \|M\tilde{f}\|_{WL^\Phi_w(\mathbb{R}^{n+1})} \lesssim \|\tilde{f}\|_{L^\Phi_w(\mathbb{R}^{n+1})} = \|f\|_{L^\Phi_w(\mathscr{E})}.\end{aligned}$$

So, Lemma 2.1 is proved. □

Even though the A_p class is well known, for completeness, we offer the definition of A_p weight functions.

DEFINITION 2.3. For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^{n+1} \to [0,\infty)$ is said to be an A_p weight if

$$\sup_{\mathscr{E}\in\mathbb{E}} \left(\frac{1}{|\mathscr{E}|}\int_{\mathscr{E}} w(x)dx\right)\left(\frac{1}{|\mathscr{E}|}\int_{\mathscr{E}} w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

A locally integrable function $w : \mathbb{R}^{n+1} \to [0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|\mathscr{E}|}\int_{\mathscr{E}} w(y)dy \leqslant Cw(x), \qquad a.e. \ x \in \mathscr{E}$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\geqslant 1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.

DEFINITION 2.4. (parabolic generalized weighted Orlicz-Morrey space) Let φ be a positive measurable function on $\mathbb{D}^{n+1}_+ \times (0,\infty)$, let w be a non-negative measurable function on $\mathbb{R}^{n+1}$ and Φ any Young function. Denote by $M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})$ the generalized weighted Orlicz-Morrey space, the space of all functions $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^{n+1})$ such that

$$\begin{aligned}\|f\|_{M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})} \equiv \|f\|_{M^{\Phi,\varphi}_w} &= \sup_{x\in\mathbb{R}^{n+1}, r>0} \varphi(x,r)^{-1}\,\Phi^{-1}\big(w(\mathscr{E}_r(x))^{-1}\big)\,\|f\|_{L^\Phi_w(\mathscr{E}_r(x))} \\ &\equiv \sup_{\mathscr{E}\in\mathbb{E}} \varphi(\mathscr{E})^{-1}\,\Phi^{-1}\big(w(\mathscr{E})^{-1}\big)\,\|f\|_{L^\Phi_w(\mathscr{E})} < \infty.\end{aligned}$$

We denote by $WM^{\Phi,\varphi}_w(\mathbb{R}^{n+1})$ the weak generalized weighted Orlicz-Morrey space, the space of all functions $f \in WL^{\Phi,\mathrm{loc}}_w(\mathbb{R}^{n+1})$ such that

$$\|f\|_{WM^{\Phi,\varphi}_w} = \sup_{x\in\mathbb{R}^{n+1}, r>0} \varphi(x,r)^{-1}\,\Phi^{-1}\big(w(\mathscr{E}_r(x))^{-1}\big)\,\|f\|_{WL^\Phi_w(\mathscr{E}_r(x))} < \infty.$$

EXAMPLE 1. Let $1 \leqslant p < \infty$ and $0 < \kappa < 1$.

- If $\Phi(r) = r^p$ and $\varphi(x,r) = w(\mathscr{E}_r(x))^{-1/p}$, then $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1}) = L_w^p(\mathbb{R}^{n+1})$.
- If $\Phi(r) = r^p$ and $\varphi(x,r) = w(\mathscr{E}_r(x))^{\frac{\kappa-1}{p}}$, then $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1}) = L^{p,\kappa}(w)$.
- If $\Phi(r) = r^p$, then $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1}) = M_w^{p,\varphi}(\mathbb{R}^{n+1})$.
- If $\varphi(x,r) = \Phi^{-1}\big(w(\mathscr{E}_r(x))^{-1}\big)$, then $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1}) = L_w^{\Phi}(\mathbb{R}^{n+1})$.

3. Definitions and statement of the problem

In the present section we give the definitions of the functional spaces to which the coefficients and the data of the problem belong. The domain $\Omega \subset \mathbb{R}^n$ supposed to be bounded with $\partial\Omega \in C^{1,1}$.

DEFINITION 3.5. Let $\varphi(x,r)$ be a measurable function in $Q \times \mathbb{R}_+ \to \mathbb{R}_+$, w be a non-negative measurable function on Q and Φ any Young function. The parabolic generalized weighted Orlicz-Morrey space $M_w^{\Phi,\varphi}(Q)$ consists of all functions $f \in L_w^{\Phi}(Q)$ such that

$$\|f\|_{M_w^{\Phi,\varphi}(Q)} = \|f\|_{\Phi,\varphi,w;Q} := \sup_{x\in Q, r>0} \varphi(x,r)^{-1}\Phi^{-1}\big(w(Q_r(x))^{-1}\big)\,\|f\|_{L_w^{\Phi}(Q_r(x))} < \infty,$$

where $Q_r(x) = Q \cap \mathscr{E}_r(x)$.

The parabolic generalized weighted Sobolev-Orlicz-Morrey space $W^{2,1}M_w^{\Phi,\varphi}(Q)$ consists of all weighted Sobolev functions $u \in W^{2,1}L_w^{\Phi}(Q)$ with distributional derivatives $D_t^l D_{x'}^s u \in M_w^{\Phi,\varphi}(Q)$, endowed with the norm

$$\|u\|_{W^{2,1}M_w^{\Phi,\varphi}(Q)} = \|D_t u\|_{M_w^{\Phi,\varphi}(Q)} + \sum_{0\leqslant |s|\leqslant 2} \|D_{x'}^s u\|_{M_w^{\Phi,\varphi}(Q)},$$

$$\overset{\circ}{W}{}^{2,1}M_w^{\Phi,\varphi}(Q) = \big\{u \in W^{2,1}M_w^{\Phi,\varphi}(Q):\ u(x) = 0, x \in \partial Q\big\},$$
$$\|u\|_{\overset{\circ}{W}{}^{2,1}M_w^{\Phi,\varphi}(Q)} = \|u\|_{W^{2,1}M_w^{\Phi,\varphi}(Q)},$$

where ∂Q means the parabolic boundary $\Omega \cup (\partial\Omega \times (0,T))$.

DEFINITION 3.6. Let $\varphi : Q \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function, w be a non-negative measurable function on Q, the parabolic generalized weak weighted Orlicz-Morrey space $WM_w^{\Phi,\varphi}(Q)$ consists of all measurable functions such that

$$\|f\|_{WM_w^{\Phi,\varphi}(Q)} = \sup_{x\in Q, r>0} \varphi(x,r)^{-1}\Phi^{-1}\big(w(Q_r(x))^{-1}\big)\,\|f\|_{WL_w^{\Phi}(Q_r(x))},$$

where $WL_w^{\Phi}(Q_r(x))$ denotes the weak weighted L^{Φ}-space of measurable functions f for which

$$\|f\|_{WL_w^{\Phi}(Q_r(x))} \equiv \|f\chi_{Q_r(x)}\|_{WL_w^{\Phi}(\mathbb{R}^{n+1})}.$$

For a bounded domain Q we define the space $WM_w^{\Phi,\varphi}(Q)$ taking $f \in WL_w^{\Phi}(Q)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain and $Q = \Omega \times (0,T)$, $T > 0$ be a cylinder in $\mathbb{R}^{n+1}_+$. We give the definitions of the functional spaces which we are going to use.

DEFINITION 3.7. Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $a_{\mathscr{E}_r} = |\mathscr{E}_r|^{-1}\int_{\mathscr{E}_r} a(y)dy$ be the mean integral of a. Denote

$$\eta_a(R) = \sup_{r \leqslant R} \frac{1}{|\mathscr{E}_r|}\int_{\mathscr{E}_r} |f(y) - f_{\mathscr{E}_r}|dy \quad \text{for every } R > 0.$$

We say that

- $a \in BMO(\mathbb{R}^{n+1})$ (*bounded mean oscillation*, [31]) provided the following is finite
$$\|a\|_* = \sup_{R>0} \eta_a(R).$$
The quantity $\|\cdot\|_*$ is a norm in BMO modulo constant function under which $BMO(\mathbb{R}^{n+1})$ is a Banach space.
- $a \in VMO(\mathbb{R}^{n+1})$ (*vanishing mean oscillation*, [46]) if $a \in BMO(\mathbb{R}^{n+1})$ and
$$\lim_{R \to 0} \eta_a(R) = 0.$$
The quantity $\eta_a(R)$ is called VMO-modulus of a.

For any bounded cylinder Q we define $BMO(Q)$ and $VMO(Q)$ taking $a \in L^1(Q)$ and Q_r instead of $\mathscr{E}_r$ in the definition above.

In the Sections 4 and 5 we study continuity in the spaces $M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})$ of sublinear integral operators generated by parabolic singular and nonsingular operators and their commutators with $BMO(\mathbb{R}^{n+1})$ functions. These results unified with known estimates in $L^\Phi_w(\mathbb{R}^{n+1})$ permit to obtain continuity of the parabolic singular and nonsingular operators in $M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})$ that is shown in Section 6. The last section is dedicated to the Cauchy-Dirichlet problem for linear parabolic equation of second order

$$u_t - a^{ij}(x)D_{ij}u(x) = f(x) \;\text{ a.a. } x \in Q, \quad u \in \overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}_w(Q). \tag{3.1}$$

where the coefficient matrix $\mathbf{a}(x) = \{a^{ij}(x)\}^n_{i,j=1}$ satisfies

$$\begin{cases} \exists\, \Lambda > 0 : \;\Lambda^{-1}|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2 \;\text{ for a.a. } x \in Q,\ \forall \xi \in \mathbb{R}^n \\ a^{ij}(x) = a^{ji}(x) \text{ that implies } a^{ij} \in L^\infty(Q). \end{cases} \tag{3.2}$$

The main theorem is stated as follows.

THEOREM 3.2. (Main result) *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $\mathbf{a} \in VMO(Q)$ satisfy (3.2) and $u \in \overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}_w(Q)$ be a strong solution of (3.1). If $f \in M^{\Phi,\varphi}_w(Q)$ with $\varphi(x,r)$ being measurable positive function satisfying*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(w(Q_s(x))^{-1}\big)}\Big)\Phi^{-1}\big(w(Q_t(x))^{-1}\big)\frac{dt}{t} \leqslant C\varphi(x,r), \tag{3.3}$$

then $u \in \overset{\circ}{W}{}^{2,1}M_w^{\Phi,\varphi}(Q)$ *and*

$$\|u\|_{\overset{\circ}{W}{}^{2,1}M_w^{\Phi,\varphi}(Q)} \leqslant C\|f\|_{M_w^{\Phi,\varphi}(Q)} \tag{3.4}$$

with $C = C(n,\Phi,w,\Lambda,\partial\Omega,T,\|\mathbf{a}\|_{\infty;Q},\eta_a)$.

4. Sublinear operators generated by parabolic singular integrals in parabolic generalized weighted Orlicz-Morrey spaces

Let $f \in L^1(\mathbb{R}^{n+1})$ be a function with a compact support and $a \in BMO(\mathbb{R}^{n+1})$. For $x \notin \mathrm{supp} f$ define the sublinear operators T and T_a such that

$$|Tf(x)| \leqslant C\int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho(x-y)^{n+2}}\,dy, \tag{4.1}$$

$$|T_a f(x)| \leqslant C\int_{\mathbb{R}^{n+1}} |a(x)-a(y)|\frac{|f(y)|}{\rho(x-y)^{n+2}}\,dy \tag{4.2}$$

with constants independent of a and f.

Suppose in addition that the both operators are bounded in $L_w^{\Phi}(\mathbb{R}^{n+1})$ satisfying the estimates

$$\|Tf\|_{L_w^{\Phi}(\mathbb{R}^{n+1})} \leqslant C\|f\|_{L_w^{\Phi}(\mathbb{R}^{n+1})}, \quad \|T_a f\|_{L_w^{\Phi}(\mathbb{R}^{n+1})} \leqslant C\|a\|_*\|f\|_{L_w^{\Phi}(\mathbb{R}^{n+1})} \tag{4.3}$$

with constants independent of a and f.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^{\infty} g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem were proved in [21] and in the case $w = 1$ in [3].

THEOREM 4.3. *Let* v_1, v_2 *and* w *be weights on* $(0,\infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)H_w^* g(t) \leqslant C\sup_{t>0} v_1(t)g(t) \tag{4.4}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t)\int_t^{\infty} \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{4.5}$$

Moreover, the value $C = B$ *is the best constant for* (4.4).

REMARK 4.2. In (4.4) and (4.5) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

LEMMA 4.2. *Let* Φ *be a Young function and* $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$, *be such that for each* $(x_0,r) \in \mathbb{R}^{n+1} \times \mathbb{R}_+$

$$\int_r^\infty \|f\|_{L_w^\Phi(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s} < \infty \tag{4.6}$$

and T *be a sublinear operator satisfying* (4.1).

(i) *If* T *bounded on* $L_w^\Phi(\mathbb{R}^{n+1})$, *then*

$$\|Tf\|_{L_w^\Phi(\mathscr{E}_r(x_0))} \leqslant C\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big) \int_{2r}^\infty \|f\|_{L_w^\Phi(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}. \tag{4.7}$$

(ii) *If* T *bounded from* $L_w^\Phi(\mathbb{R}^{n+1})$ *on* $WL_w^\Phi(\mathbb{R}^{n+1})$, *then*

$$\|Tf\|_{WL^\Phi(\mathscr{E}_r(x_0))} \leqslant C\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big) \int_{2r}^\infty \|f\|_{L_w^\Phi(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}, \tag{4.8}$$

where the constants are independent of r, x_0 *and* f.

Proof. (i) Consider the decomposition of f with respect to the ellipsoid $\mathscr{E}_r(x_0)$

$$f = f\chi_{2\mathscr{E}_r(x_0)} + f\chi_{2\mathscr{E}_r^c(x_0)} = f_1 + f_2.$$

We remark that due to the lack of density of smooth functions in the parabolic generalized weighted Orlicz-Morrey spaces the parabolic singular integral operators need to be defined in a convenient way. For the moment, in the case when the sublinear operator is the parabolic singular integral operator, we denote the operator T on $L_w^\Phi(\mathbb{R}^{n+1})$ by T_0 to avoid confusion.

For all $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$ in the case when the sublinear operator is the parabolic singular integral operator we define

$$Tf(x) := T_0 f_1(x) + \int_{\mathbb{R}^{n+1}} K(x,y) f_2(y)dy. \tag{4.9}$$

First we show that $Tf(x)$ is well defined for almost all x and independent of the choice $\mathscr{E}$ containing x. As T_0 is bounded on $L_w^\Phi(\mathbb{R}^{n+1})$ and $f_1 \in L_w^\Phi(\mathbb{R}^n)$, $T_0 f_1$ is well defined. Next, we show that the second-term of the right-hand side defining $Tf(x)$ converges absolutely for any $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$ and almost every $x \in \mathbb{R}^{n+1}$.

Finally it remains to show that the definition is independent of the choice of $\mathscr{E}$. Let $\mathbb{E} = \{\mathscr{E}_r(x) : x \in \mathbb{R}^{n+1},\ r > 0\}$. That is, if $\mathscr{E}_1, \mathscr{E}_2 \in \mathbb{E}$ and $x \in \mathscr{E}_1 \cap \mathscr{E}_2$, then

$$T_0(f\chi_{2\mathscr{E}_1})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_1} K(x,y)f(y)dy = T_0(f\chi_{2\mathscr{E}_2})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_2} K(x,y)f(y)dy. \tag{4.10}$$

Actually, let $\mathscr{E}_3 \in \mathbb{E}$ be selected so that $2\mathscr{E}_1 \cup \mathscr{E}_2 \subset \mathscr{E}_3$.

Since $f\chi_{2\mathscr{E}_1}, f\chi_{\mathscr{E}_3\setminus 2\mathscr{E}_1} \in L_w^{\Phi}(\mathbb{R}^{n+1})$, the linearity of T_0 on $L_w^{\Phi}(\mathbb{R}^{n+1})$ yields

$$\begin{aligned}
&T_0(f\chi_{2\mathscr{E}_1})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_1} K(x,y)f(y)dy \\
&= T_0(f\chi_{2\mathscr{E}_1})(x) + \int_{\mathscr{E}_3\setminus 2\mathscr{E}_1} K(x,y)f(y)dy + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)f(y)dy \\
&= T_0(f\chi_{2\mathscr{E}_1})(x) + T_0(f\chi_{\mathscr{E}_3\setminus 2\mathscr{E}_1})(x) + \int_{\mathbb{R}^{n}\setminus \mathscr{E}_3} K(x,y)f(y)dy \\
&= T_0(f\chi_{\mathscr{E}_3})(x) + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)f(y)dy. \qquad (4.11)
\end{aligned}$$

Similarly, we also have

$$T_0(f\chi_{2\mathscr{E}_2})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_2} K(x,y)f(y)dy = T_0(f\chi_{\mathscr{E}_3})(x) + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)f(y)dy. \qquad (4.12)$$

Thus, combining (4.11) and (4.12) we obtain (4.10).

It is easy to see that for arbitrary points $x \in \mathscr{E}_r(x_0)$ and $y \in 2\mathscr{E}_r^c(x_0)$ it holds

$$\frac{1}{2}\rho(x_0-y) \leqslant \rho(x-y) \leqslant \frac{3}{2}\rho(x_0-y). \qquad (4.13)$$

Applying (4.1), (4.13), the Fubini theorem and the Hölder inequality to Tf_2 we get

$$\begin{aligned}
|Tf_2(x)| &\lesssim \int_{2\mathscr{E}_r^c(x_0)} \frac{|f(y)|}{\rho(x_0-y)^{n+2}}dy \lesssim \int_{2\mathscr{E}_r^c(x_0)} |f(y)| \left(\int_{\rho(x_0-y)}^{\infty} \frac{ds}{s^{n+3}}\right)dy \\
&\leqslant \int_{2r}^{\infty} \left(\int_{2r\leqslant\rho(x_0-y)<s} |f(y)|dy\right)\frac{ds}{s^{n+3}} \leqslant \int_{2r}^{\infty} \left(\int_{\mathscr{E}_s(x_0)} |f(y)|dy\right)\frac{ds}{s^{n+3}}.
\end{aligned}$$

Applying Lemma 2.1, we get

$$|Tf_2(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \Phi^{-1}\left(w(\mathscr{E}_s(x_0))^{-1}\right)\frac{ds}{s}. \qquad (4.14)$$

Therefore, from (4.14) we get second-term of the right-hand side $\int_{\mathbb{R}^{n+1}} K(x,y)f_2(y)dy$ converges absolutely for any $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$ and almost every $x \in \mathbb{R}^n$, and therefore we get the right-hand side of (4.9) is finite.

Therefore, in the case when the sublinear operator is the parabolic singular integral operator $Tf(x)$ is well defined for almost all x and independent of the choice $\mathscr{E}$ containing x.

Because of the L_w^{Φ} boundedness of the operator T and $f_1 \in L_w^{\Phi}(\mathbb{R}^{n+1})$ we have

$$\|Tf_1\|_{L_w^{\Phi}(\mathscr{E}_r(x_0))} \leqslant \|Tf_1\|_{L_w^{\Phi}(\mathbb{R}^{n+1})} \lesssim \|f_1\|_{L_w^{\Phi}(\mathbb{R}^{n+1})} = \|f\|_{L_w^{\Phi}(2\mathscr{E}_r(x_0))}.$$

Direct calculations give

$$\|Tf_2\|_{L_w^{\Phi}(\mathscr{E}_r(x_0))} \lesssim \Phi^{-1}\left(w(\mathscr{E}_r(x_0))^{-1}\right) \int_{2r}^{\infty} \|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \Phi^{-1}\left(w(\mathscr{E}_s(x_0))^{-1}\right)\frac{ds}{s} \qquad (4.15)$$

Thus

$$\|Tf\|_{L_w^\Phi(\mathscr{E}_r(x_0))} \lesssim \|f\|_{L_w^\Phi(2\mathscr{E}_r(x_0))} + \Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big) \\ \times \int_{2r}^{\infty} \|f\|_{L_w^\Phi(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}. \tag{4.16}$$

On the other hand

$$\|f\|_{L_w^\Phi(2\mathscr{E}_r(x_0))} \leqslant C\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big) \int_{2r}^{\infty} \|f\|_{L_w^\Phi(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s} \tag{4.17}$$

which unified with (4.16) gives (4.7).

(ii) Let $f \in L_w^\Phi(\mathbb{R}^{n+1})$, the weak L_w^Φ boundedness of T implies

$$\|Tf_1\|_{WL_w^\Phi(\mathscr{E}_r(x_0))} \leqslant \|Tf_1\|_{WL_w^\Phi(\mathbb{R}^{n+1})} \leqslant C\|f_1\|_{L_w^\Phi(\mathbb{R}^{n+1})} = C\|f\|_{L_w^\Phi(2\mathscr{E}_r(x_0))} \\ \leqslant C\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big) \int_{2r}^{+\infty} \|f\|_{L_w^\Phi(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}$$

unified with (4.15) gives (4.8). □

THEOREM 4.4. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $\varphi(x,r)$: $\mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function satisfying*

$$\int_r^{\infty} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big)} \Big) \Phi^{-1}\big(w(\mathscr{E}_t(x_0))^{-1}\big) \frac{dt}{t} \leqslant C\varphi(x,r) \tag{4.18}$$

and T be sublinear operator satisfying (4.1).

(i) *If T bounded on $L_w^\Phi(\mathbb{R}^{n+1})$ then T is bounded on $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})$ and*

$$\|Tf\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \leqslant C\|f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \tag{4.19}$$

with constants independent on f.

(ii) *If T bounded from $L_w^\Phi(\mathbb{R}^{n+1})$ on $WL_w^\Phi(\mathbb{R}^{n+1})$ then it is bounded from $M_w^{1,\varphi}(\mathbb{R}^{n+1})$ to $WM_w^{1,\varphi}(\mathbb{R}^{n+1})$ and*

$$\|Tf\|_{WM_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \leqslant C\|f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \tag{4.20}$$

with constants independent on f.

Proof. (i) By Lemma 4.2 we have

$$\|Tf\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \lesssim \sup_{(x,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x,r)^{-1} \int_r^{\infty} \|f\|_{\Phi;\mathscr{E}_s(x)} \Phi^{-1}\big(w(\mathscr{E}_s(x))^{-1}\big) \frac{ds}{s}.$$

Applying the Theorem 4.3 with

$$w(r) = \Phi^{-1}\big(w(\mathscr{E}_r(x))^{-1}\big), \quad v(r) = \varphi(x,r)^{-1}, \quad g(r) = \|f\|_{L_w^\Phi(\mathscr{E}_r(x))}$$

$$H_w^* g(r) = \int_r^{\infty} \|f\|_{L_w^\Phi(\mathscr{E}_s(x))} w(s)ds$$

where the condition (4.5) is equivalent to (4.18), we get (4.19).

(ii) Making use of (4.8) we get

$$\|Tf\|_{WM_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \lesssim \sup_{(x_0,r)\in\mathbb{R}^{n+1}\times\mathbb{R}_+} \varphi(x_0,r)^{-1}\,\|f\|_{L_w^{\Phi}(\mathscr{E}_r(x_0))} = C\|f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})}. \quad \square$$

Our next step is to show boundedness of T_a in $M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})$. For this goal we recall some properties of the BMO functions.

LEMMA 4.3. (John-Nirenberg type lemma, [2, Lemma 2.8]) *Let $a\in BMO(\mathbb{R}^{n+1})$ and $p\in[1,\infty)$. Then for any $\mathscr{E}_r$ there holds*

$$\left(\frac{1}{|\mathscr{E}_r|}\int_{\mathscr{E}_r}|a(y)-a_{\mathscr{E}_r}|^p dy\right)^{\frac{1}{p}} \leqslant C(p)\|a\|_*. \tag{4.21}$$

DEFINITION 4.8. A Young function Φ is said to be of upper type p (resp. lower type p) for some $p\in[0,\infty)$, if there exists a positive constant C such that, for all $t\in[1,\infty)$ (resp. $t\in[0,1]$) and $s\in[0,\infty)$,

$$\Phi(st)\leqslant Ct^p\Phi(s).$$

REMARK 4.3. We know that if Φ is lower type p_0 and upper type p_1 with $1<p_0\leqslant p_1<\infty$, then $\Phi\in\Delta_2\cap\nabla_2$. Conversely if $\Phi\in\Delta_2\cap\nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1<p_0\leqslant p_1<\infty$ (see [32]).

As an immediate consequence of Lemma 4.3 we get the following property.

COROLLARY 4.1. *Let $a\in BMO(\mathbb{R}^{n+1})$ then for all $0<2r<s$ it holds*

$$|a_{\mathscr{E}_r}-a_{\mathscr{E}_s}|\leqslant C(n)\left(1+\ln\frac{s}{r}\right)\|a\|_*. \tag{4.22}$$

Proof. Since $s>2r$ there exists $k\in\mathbb{N},\ k\geqslant 1$ such that $2^k r<s\leqslant 2^{k+1}r$ and hence $k\ln 2<\ln\frac{s}{r}\leqslant(k+1)\ln 2$. By [2, Lemma 2.9] we have

$$\begin{aligned}
|a_{\mathscr{E}_s}-a_{\mathscr{E}_r}| &\leqslant |a_{2^k\mathscr{E}_r}-a_{\mathscr{E}_r}|+|a_{2^k\mathscr{E}_r}-a_{\mathscr{E}_r}|\\
&\leqslant C(n)k\,\|a\|_*+\frac{1}{|2^k\mathscr{E}_r|}\int_{2^k\mathscr{E}_r}|a(y)-a_{\mathscr{E}_s}|dy\\
&\leqslant C(n)k\,\|a\|_*+\frac{2^{n+2}}{|\mathscr{E}_s|}\int_{\mathscr{E}_s}|a(y)-a_{\mathscr{E}_s}|dy\\
&< C(n)\left(\ln\frac{s}{r}+1\right)\|a\|_*. \quad \square
\end{aligned}$$

In the following lemma which was proved in [24, 29] we provide a generalization of the property (4.21) from L^p-norms to weight Orlicz norms.

LEMMA 4.4. *Let $a \in BMO(\mathbb{R}^{n+1})$ and Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$. Let Φ is lower type p_0 and upper type p_1 with $1 \leqslant p_0 \leqslant p_1 < \infty$, then*

$$\|a\|_* \approx \sup_{x \in \mathbb{R}^{n+1}, r>0} \Phi^{-1}\big(w(Q_r(x))^{-1}\big) \big\|a(\cdot) - a_{\mathscr{E}_r(x)}\big\|_{L^\Phi_w(\mathscr{E}_r(x))}.$$

Additionally, we need the following lemma. For the proof of Lemma 4.5, see [16] for example.

LEMMA 4.5. *Let $0 < p < \infty$, $w \in A_\infty$ and $a \in BMO$. Then for any parabolic ball $\mathscr{E}$, we have that*

$$\left(\frac{1}{w(\mathscr{E})} \int_{\mathscr{E}} |a(y) - a_{\mathscr{E}}|^p w(y) dy\right)^{\frac{1}{p}} \leqslant C\|a\|_*.$$

DEFINITION 4.9. Let Φ be a Young function. Let

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

REMARK 4.4. It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \leqslant b_\Phi < \infty$.

REMARK 4.5. Remark 4.4 and Remark 4.3 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leqslant p_1 < \infty$ if and only if $1 < a_\Phi \leqslant b_\Phi < \infty$.

To estimate the norm of T_a we shall employ the same idea which we have used in the proof of Lemma 4.2.

LEMMA 4.6. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $a \in BMO(\mathbb{R}^{n+1})$ and T_a be a bounded operator in $L^\Phi_w(\mathbb{R}^{n+1})$ satisfying (4.2) and (4.3). Suppose that for any $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^{n+1})$ and $(x_0, r) \in \mathbb{R}^{n+1} \times \mathbb{R}_+$*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) \|f\|_{L^\Phi_w(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s} < \infty. \tag{4.23}$$

Then

$$\|T_a f\|_{L^\Phi_w(\mathscr{E}_r(x_0))} \lesssim \|a\|_* \Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big) \times \int_{2r}^\infty \left(1 + \ln\frac{s}{r}\right) \|f\|_{L^\Phi_w(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}. \tag{4.24}$$

Proof. Consider the decomposition $f = f\chi_{2\mathscr{E}_r(x_0)} + f\chi_{2\mathscr{E}^c_r(x_0)} = f_1 + f_2$.

For all $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^{n+1})$ in the case when the sublinear operator T_a is the commutator of parabolic singular integral operator we define

$$T_a f(x) := T_{a,0} f_1(x) + \int_{\mathbb{R}^{n+1}} (a(x) - a(y)) K(x,y) f_2(y) dy. \tag{4.25}$$

First we show that $T_a f(x)$ is well defined for almost all x and independent of the choice $\mathscr{E} = \mathscr{E}_r(x_0)$ containing x.

As $T_{a,0}$ is bounded on $L_w^\Phi(\mathbb{R}^{n+1})$ and $f_1 \in L_w^\Phi(\mathbb{R}^{n+1})$, $T_{a,0} f_1$ is well defined.

Next, we show that the second term of the right-hand side defining $T_a f(x)$ converges absolutely for any $f \in M_w^{\Phi,\varphi_1}(\mathbb{R}^{n+1})$ and almost every $x \in \mathbb{R}^{n+1}$.

Due to the inequality (4.13) for all $x \in \mathscr{E}$ we have

$$
\begin{aligned}
|T_a f_2(x)| &\lesssim \int_{\complement(2\mathscr{E})} \frac{|a(y)-a(x)|}{\rho(x-y)^{n+2}} |f(y)| dy \\
&\lesssim \int_{\complement(2\mathscr{E})} \frac{|a(y)-a(x)|}{\rho(x_0-y)^{n+2}} |f(y)| dy \\
&\lesssim \int_{\complement(2\mathscr{E})} \frac{|a(y)-a_B|}{\rho(x_0-y)^{n+2}} |f(y)| dy + \int_{\complement(2\mathscr{E})} \frac{|a(x)-a_B|}{\rho(x_0-y)^{n+2}} |f(y)| dy \\
&= J_1 + J_2.
\end{aligned}
$$

By an argument similar to that used in the estimate (2.25) in [36], we have

$$
\big\| |a(\cdot) - a_{\mathscr{E}}| w^{-1}(\cdot) \big\|_{L_w^{\widetilde{\Phi}}(\mathscr{E})} \lesssim \Phi^{-1}\big(w(\mathscr{E})^{-1}\big) |\mathscr{E}|. \tag{4.26}
$$

For the sake of completeness, we prove estimate (4.26). Taking into account (2.1) and Remark 4.3, we conclude that

$$
\begin{aligned}
&\int_{\mathscr{E}} \widetilde{\Phi}\Big(\frac{|a(x) - a_{\mathscr{E}}| w^{-1}(x)}{\Phi^{-1}\big(w(\mathscr{E})^{-1}\big) |\mathscr{E}|} \Big) w(x) dx \\
&\lesssim \int_{\mathscr{E}} \widetilde{\Phi}\Big(\frac{|a(x) - a_{\mathscr{E}}| \widetilde{\Phi}^{-1}\big(w(\mathscr{E})^{-1}\big) w(\mathscr{E})}{w(x) |\mathscr{E}|} \Big) w(x) dx \\
&\lesssim \frac{1}{w(\mathscr{E})} \int_{\mathscr{E}} \Big\{ \sum_{i=0}^{1} \Big[\frac{|a(x) - a_{\mathscr{E}}|}{w(x)} \Big]^{p_i'} \Big[\frac{w(\mathscr{E})}{|\mathscr{E}|} \Big]^{p_i'} \Big\} w(x) dx.
\end{aligned}
$$

Since $w \in A_{p_0} \subset A_{p_1}$, we know that $w^{1-p_i'} \in A_{p_i'}$ for $i \in \{0,1\}$ (see, for example, [9, p. 136]). By this, the Hölder inequality and Lemma 4.5, we conclude that, for $i \in \{0,1\}$,

$$
\begin{aligned}
&\frac{1}{w(\mathscr{E})} \int_{\mathscr{E}} |a(x) - a_{\mathscr{E}}|^{p_i'} \Big[\frac{w(\mathscr{E})}{|\mathscr{E}|} \Big]^{p_i'} \frac{1}{w^{p_i'}(x)} w(x) dx \\
&\approx \Big[\frac{1}{|\mathscr{E}|} \int_{\mathscr{E}} w(x) dx \Big]^{p_i'-1} \Big[\frac{1}{|\mathscr{E}|} \int_{\mathscr{E}} w^{1-p_i'}(x) dx \Big] \\
&\quad \times \Big\{ \frac{1}{[w(\mathscr{E})]^{1-p_i'}} \int_{\mathscr{E}} |a(x) - a_{\mathscr{E}}|^{p_i'} w^{1-p_i'}(x) dx \Big\} \lesssim 1,
\end{aligned}
$$

which yields to (4.26).

Now, let us estimate I_1.

$$\begin{aligned}
I_1 &\approx \int_{\complement_{(2\mathscr{E})}} |a(y)-a_{\mathscr{E}}||f(y)| \int_{\rho(x_0-y)}^{\infty} \frac{dt}{t^{n+3}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r \leqslant \rho(x_0-y) \leqslant t} |a(y)-a_{\mathscr{E}}||f(y)| dy \frac{dt}{t^{n+3}} \\
&\lesssim \int_{2r}^{\infty} \int_{\mathscr{E}_t(x_0)} |a(y)-a_{\mathscr{E}}||f(y)| dy \frac{dt}{t^{n+3}}.
\end{aligned}$$

Applying Hölder's inequality, by (4.26), (4.22), (4.23) and Lemma 2.1 we get

$$\begin{aligned}
I_1 &\lesssim \int_{2r}^{\infty} \int_{\mathscr{E}_t(x_0)} |a(y)-a_{\mathscr{E}_t(x_0)}||f(y)| dy \frac{dt}{t^{n+3}} \\
&\quad + \int_{2r}^{\infty} |a_{\mathscr{E}_r(x_0)} - a_{\mathscr{E}_t(x_0)}| \int_{\mathscr{E}_t(x_0)} |f(y)| dy \frac{dt}{t^{n+3}} \\
&\lesssim \int_{2r}^{\infty} \left\| |a(\cdot)-a_{\mathscr{E}_t(x_0)}| w^{-1}(\cdot) \right\|_{L_w^{\widetilde{\Phi}}(\mathscr{E}_t(x_0))} \|f\|_{L_w^{\Phi}(\mathscr{E}_t(x_0))} \frac{dt}{t^{n+3}} \\
&\quad + \int_{2r}^{\infty} |a_{\mathscr{E}_r(x_0)} - a_{\mathscr{E}_t(x_0)}| \|f\|_{L_w^{\Phi}(\mathscr{E}_t(x_0))} \Phi^{-1}\left(w(\mathscr{E}_t(x_0))^{-1}\right) \frac{dt}{t} \\
&\lesssim \|a\|_* \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_w^{\Phi}(\mathscr{E}_t(x_0))} \Phi^{-1}\left(w(\mathscr{E}_t(x_0))^{-1}\right) \frac{dt}{t} \\
&\lesssim \|a\|_* \|f\|_{M_w^{\Phi,\varphi_1}} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \varphi_1(x_0,t) \frac{dt}{t} \\
&\lesssim \|a\|_* \|f\|_{M_w^{\Phi,\varphi_1}} \varphi_2(x_0,r) < \infty.
\end{aligned} \tag{4.27}$$

In order to estimate I_2 note that $a \in BMO$ implies that $a(\cdot)-a_{\mathscr{E}}$ is integrable on $\mathscr{E}$, so $a(\cdot)-a_{\mathscr{E}}$ is finite almost everywhere on $\mathscr{E}$. From this fact, (4.14) and (4.23), we get

$$\begin{aligned}
I_2 &\lesssim |a(x)-a_{\mathscr{E}}| \int_{2r}^{\infty} \|f\|_{L_w^{\Phi}(\mathscr{E}_t(x_0))} \Phi^{-1}\left(w(\mathscr{E}_t(x_0))^{-1}\right) \frac{dt}{t} \\
&\lesssim \|f\|_{M_w^{\Phi,\varphi_1}} |a(x)-a_{\mathscr{E}}| \int_r^{\infty} \varphi_1(x_0,t) \frac{dt}{t} \\
&\lesssim \|f\|_{M_w^{\Phi,\varphi_1}} |a(x)-a_{\mathscr{E}}| \varphi_2(x_0,r) < \infty.
\end{aligned} \tag{4.28}$$

Therefore, from (4.27) and (4.28) we get second-term of the right-hand side of (4.25) $\int_{\mathbb{R}^{n+1}} (a(x)-a(y)) K(x,y) f_2(y) dy$ converges absolutely for any $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$ and almost every $x \in \mathbb{R}^{n+1}$, and therefore we get the right-hand side of (4.25) is finite.

Therefore, in the case when the sublinear operator is a commutator of the parabolic singular integral operator $T_a f(x)$ is well defined for almost all x and does not depend on the choice $\mathscr{E}$ containing x.

Finally it remains to show that the definition is independent of the choice of B.

That is, if $\mathscr{E}_1, \mathscr{E}_2 \in \mathbb{E}$ and $x \in \mathscr{E}_1 \cap \mathscr{E}_2$, then

$$\begin{aligned} &T_{a,0}(f\chi_{2\mathscr{E}_1})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_1} K(x,y)(a(y)-a(x))f(y)dy \\ &= T_{a,0}(f\chi_{2\mathscr{E}_2})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_2} K(x,y)(a(y)-a(x))f(y)dy. \end{aligned} \tag{4.29}$$

Actually, let $\mathscr{E}_3 \in \mathbb{E}$ be selected so that $2\mathscr{E}_1 \cup 2\mathscr{E}_2 \subset \mathscr{E}_3$. Since $f\chi_{2\mathscr{E}_1}, f\chi_{\mathscr{E}_3\setminus 2\mathscr{E}_1} \in L^{\Phi}(\mathbb{R}^{n+1})$, the linearity of $T_{a,0}$ on $L^{\Phi}(\mathbb{R}^{n+1})$ yields

$$\begin{aligned} &T_{a,0}(f\chi_{2\mathscr{E}_1})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_1} K(x,y)(a(y)-a(x))f(y)dy \\ &= T_{a,0}(f\chi_{2\mathscr{E}_1})(x) + \int_{\mathscr{E}_3\setminus 2\mathscr{E}_1} K(x,y)(a(y)-a(x))f(y)dy \\ &\quad + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)(a(y)-a(x))f(y)dy \\ &= T_{a,0}(f\chi_{2\mathscr{E}_1})(x) + T_{a,0}(f\chi_{\mathscr{E}_3\setminus 2\mathscr{E}_1})(x) + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)(a(y)-a(x))f(y)dy \\ &= T_{a,0}(f\chi_{\mathscr{E}_3})(x) + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)(a(y)-a(x))f(y)dy. \end{aligned} \tag{4.30}$$

Similarly, we also have

$$\begin{aligned} &T_{a,0}(f\chi_{2\mathscr{E}_2})(x) + \int_{\mathbb{R}^{n+1}\setminus 2\mathscr{E}_2} K(x,y)(a(y)-a(x))f(y)dy \\ &= T_{a,0}(f\chi_{\mathscr{E}_3})(x) + \int_{\mathbb{R}^{n+1}\setminus \mathscr{E}_3} K(x,y)(a(y)-a(x))f(y)dy. \end{aligned} \tag{4.31}$$

Therefore, combining (4.30) and (4.31) we obtain (4.29).

Now, we show the boundedness. Hence

$$\|T_a f\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} \leqslant \|T_a f_1\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} + \|T_a f_2\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))}$$

and by (4.3) as in Lemma 4.2 we have

$$\|T_a f_1\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} \leqslant C\|a\|_* \|f\|_{L^{\Phi}_w(2\mathscr{E}_r(x_0))} \tag{4.32}$$

with constants independent on f.

On the other hand, because of (4.13) we can write

$$\begin{aligned} \|T_a f_2\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} &\lesssim \left\| \int_{2\mathscr{E}_r^c(x_0)} \frac{|a(x)-a(y)||f(y)|}{\rho(x_0-y)^{n+2}} dy \right\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} \\ &\leqslant \left\| \int_{2\mathscr{E}_r^c(x_0)} \frac{|a(y)-a_{\mathscr{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}} dy \right\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} \\ &\quad + \left\| \int_{2\mathscr{E}_r^c(x_0)} \frac{|a(\cdot)-a_{\mathscr{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}} dy \right\|_{L^{\Phi}_w(\mathscr{E}_r(x_0))} \\ &= I_1 + I_2. \end{aligned}$$

Applying (4.2), the Fubini theorem and the Hölder inequality as in Lemmate 4.2 and 4.3 we get

$$
\begin{aligned}
I_1 &\lesssim \frac{1}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2\mathscr{E}_r^c(x_0)} \frac{|a(y)-a_{\mathscr{E}_r(x_0)}||f(y)|}{\rho(x_0-y)^{n+2}}dy \\
&\lesssim \frac{1}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2\mathscr{E}_r^c(x_0)} |a(y)-a_{\mathscr{E}_r(x_0)}||f(y)|dy \int_{\rho(x_0-y)}^{\infty} \frac{ds}{s^{n+3}} \\
&\leqslant \frac{1}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2r}^{\infty} \int_{\mathscr{E}_s(x_0)} |a(y)-a_{\mathscr{E}_s(x_0)}||f(y)|dy \frac{ds}{s^{n+3}} \\
&\quad + \frac{1}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2r}^{\infty} |a_{\mathscr{E}_r(x_0)}-a_{\mathscr{E}_s(x_0)}| \left(\int_{\mathscr{E}_s(x_0)} |f(y)|dy\right) \frac{ds}{s^{n+3}} \\
&\lesssim \frac{1}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2r}^{\infty} \left\|a(y)-a_{\mathscr{E}_s(x_0)}\right\|_{\widetilde{\Phi},w;\mathscr{E}_s(x_0)} \|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \frac{ds}{s^{n+3}} \\
&\quad + \frac{1}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2r}^{\infty} |a_{\mathscr{E}_r(x_0)}-a_{\mathscr{E}_s(x_0)}|\|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s} \\
&\lesssim \frac{\|a\|_*}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2r}^{\infty} \left(1+\ln\frac{s}{r}\right)\|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}.
\end{aligned}
$$

In order to estimate I_2 we note that

$$
I_2 = \left\|a(\cdot)-a_{\mathscr{E}_r(x_0)}\right\|_{L_w^{\Phi}(\mathscr{E}_r(x_0))} \int_{2\mathscr{E}_r^c(x_0)} \frac{|f(y)|}{\rho(x_0-y)^{n+2}}dy.
$$

By Lemma 4.3 and (4.15) we obtain

$$
\begin{aligned}
I_2 &\lesssim \frac{\|a\|_*}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2\mathscr{E}_r^c(x_0)} \frac{|f(y)|}{\rho(x_0-y)^{n+2}}dy \\
&\lesssim \frac{\|a\|_*}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}.
\end{aligned}
$$

Summing up (4.32), I_1 and I_2 we get

$$
\begin{aligned}
\|T_a f\|_{L_w^{\Phi}(\mathscr{E}_r(x_0))} &\lesssim \|a\|_*\|f\|_{L_w^{\Phi}(2\mathscr{E}_r(x_0))} + \frac{\|a\|_*}{\Phi^{-1}\big(w(\mathscr{E}_r(x_0))^{-1}\big)} \\
&\quad \times \int_{2r}^{\infty} \left(1+\ln\frac{s}{r}\right)\|f\|_{L_w^{\Phi}(\mathscr{E}_s(x_0))} \Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big) \frac{ds}{s}
\end{aligned}
$$

and the statement follows after applying (4.17). □

THEOREM 4.5. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$ and $\varphi(x,r) : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable function such that*

$$
\int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(w(\mathscr{E}_s(x_0))^{-1}\big)}\right) \Phi^{-1}\big(w(\mathscr{E}_t(x_0))^{-1}\big) \frac{dt}{t} \leqslant C\varphi(x,r). \tag{4.33}
$$

Suppose $a \in BMO(\mathbb{R}^{n+1})$ and T_a be sublinear operator satisfying (4.2). If T_a is bounded in $L^{\Phi}_w(\mathbb{R}^{n+1})$, then it is bounded in $M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})$ and

$$\|T_a f\|_{M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})} \leqslant C\|a\|_* \|f\|_{M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})} \tag{4.34}$$

with a constant independent of a and f.

The statement of the theorem follows by Lemma 4.6 and Theorem 4.3 in the same manner as the Theorem 4.4.

EXAMPLE 2. The functions $\varphi(x,r) = r^{\beta}\,\Phi^{-1}\big(w(Q_r(x))^{-1}\big)$ with $0 < \beta < n+2$ are Morrey functions satisfying the condition (4.33).

EXAMPLE 3. The functions $\varphi(x,r) = r^{\beta}\,\Phi^{-1}\big(w(Q_r(x))^{-1}\big)\log^m(e+r)$ with $0 < \beta < n+2$ and $m \geqslant 1$ are Morrey functions satisfying the condition (4.33) and the space $M^{\Phi,\varphi}_w(\mathbb{R}^{n+1})$ does not coincide with any Morrey space.

5. Sublinear operators generated by parabolic nonsingular integrals in parabolic generalized weighted Orlicz-Morrey spaces

For any $x \in \mathbb{D}^{n+1}_+$ define $\widetilde{x} = (x'', -x_n, t) \in \mathbb{D}^{n+1}_-$ and call $x^0 = (x'', 0, 0) \in \mathbb{R}^{n-1}$. Consider the semi-ellipsoids $\mathscr{E}^+_r(x^0) = \mathscr{E}_r(x^0) \cap \mathbb{D}^{n+1}_+$. Let $f \in L^1(\mathbb{D}^{n+1}_+)$, $a \in BMO(\mathbb{D}^{n+1}_+)$ and $\widetilde{T}$ and $\widetilde{T}_a$ be sublinear operators such that

$$|\widetilde{T} f(x)| \leqslant C \int_{\mathbb{D}^{n+1}_+} \frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}}\, dy, \tag{5.1}$$

$$|\widetilde{T}_a f(x)| \leqslant C \int_{\mathbb{D}^{n+1}_+} |a(x)-a(y)| \frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}} dy \tag{5.2}$$

Suppose in addition that the both operators are bounded in $L^{\Phi}_w(\mathbb{D}^{n+1}_+)$ satisfying the estimates

$$\|\widetilde{T} f\|_{L^{\Phi}_w(\mathbb{D}^{n+1}_+)} \leqslant C\|f\|_{L^{\Phi}_w(\mathbb{D}^{n+1}_+)}, \quad \|\widetilde{T}_a f\|_{L^{\Phi}_w(\mathbb{D}^{n+1}_+)} \leqslant C\|a\|_*\|f\|_{L^{\Phi}_w(\mathbb{D}^{n+1}_+)} \tag{5.3}$$

with constants independent of a and f. The following assertions can be proved in the same manner as in § 4.

LEMMA 5.7. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{D}^{n+1}_+)$, and for all $(x^0, r) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$*

$$\int_r^{\infty} \|f\|_{L^{\Phi}_w(\mathscr{E}^+_s(x^0))} \Phi^{-1}\big(w(\mathscr{E}^+_s(x^0))^{-1}\big) \frac{ds}{s} < \infty. \tag{5.4}$$

If $\widetilde{T}$ is bounded on $L^{\Phi}_w(\mathbb{D}^{n+1}_+)$ then

$$\|\widetilde{T} f\|_{L^{\Phi}_w(\mathscr{E}^+_r(x^0))} \leqslant \frac{C}{\Phi^{-1}\big(w(\mathscr{E}^+_r(x^0))^{-1}\big)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}_w(\mathscr{E}^+_s(x^0))} \Phi^{-1}\big(w(\mathscr{E}^+_s(x^0))^{-1}\big) \frac{ds}{s} \tag{5.5}$$

where the constant is independent of r, x^0, and f.

THEOREM 5.6. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, φ be a weight function satisfying* (4.18) *and $\widetilde{T}$ be a sublinear operator satisfying* (5.1) *and* (5.3). *Then $\widetilde{T}$ is bounded in $M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$, and*

$$\|\widetilde{T} f\|_{M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} \leqslant C\|f\|_{M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} \tag{5.6}$$

with a constant independent of f.

LEMMA 5.8. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $a \in BMO(\mathbb{D}_+^{n+1})$ and $\widetilde{T}_a$ satisfy* (5.2) *and* (5.3). *Suppose that for all $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{D}_+^{n+1})$, $(x^0,r) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$*

$$\int_r^\infty \Big(1+\ln\frac{s}{r}\Big)\|f\|_{L_w^\Phi(\mathscr{E}_s^+(x^0))}\Phi^{-1}\big(w(\mathscr{E}_s^+(x^0))^{-1}\big)\frac{ds}{s} < \infty. \tag{5.7}$$

Then

$$\begin{aligned}\|\widetilde{T}_a f\|_{L_w^\Phi(\mathscr{E}_r^+(x^0))} \leqslant\ & \frac{C\,\|a\|_*}{\Phi^{-1}\big(w(\mathscr{E}_r^+(x^0))^{-1}\big)} \\ & \times \int_{2r}^\infty \Big(1+\ln\frac{s}{r}\Big)\|f\|_{L_w^\Phi(\mathscr{E}_s^+(x^0))}\,\Phi^{-1}\big(w(\mathscr{E}_s^+(x^0))^{-1}\big)\frac{ds}{s}\end{aligned}$$

with a constant independent of a and f.

THEOREM 5.7. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $a \in BMO(\mathbb{D}_+^{n+1})$, φ be measurable function satisfying* (4.33) *and $\widetilde{T}_a$ be a sublinear operator satisfying* (4.2) *and* (4.3). *Then $\widetilde{T}_a$ is bounded in $M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$, and*

$$\|\widetilde{T}_a f\|_{M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} \leqslant C\|a\|_*\,\|f\|_{M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} \tag{5.8}$$

with a constant independent of a and f.

6. Singular and nonsingular parabolic integral operators in generalized parabolic weighted Orlicz-Morrey spaces

In the present section we apply the above results to Calderón-Zygmund type operators with parabolic kernel. Since these operators are sublinear and bounded in $L_w^\Phi(\mathbb{R}^{n+1})$ their continuity in $M_w^{\Phi,\varphi}$ follows immediately.

DEFINITION 6.10. A measurable function $\mathscr{K}(x,\xi) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is called variable parabolic Calderón-Zygmund kernel if:

i) $\mathscr{K}(x,\cdot)$ is a parabolic Calderón-Zygmund kernel for a.a. $x \in \mathbb{R}^{n+1}$:

a) $\mathscr{K}(x,\cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$,

b) $\mathscr{K}(x,(\mu\xi',\mu^2 s)) = \mu^{-n-2}\mathscr{K}(x,\xi), \quad \forall \mu > 0,\ \xi = (\xi',s)$,

$$c)\quad \int_{\mathbb{S}^n} \mathscr{K}(x,\xi)d\sigma_\xi = 0, \qquad \int_{\mathbb{S}^n} |\mathscr{K}(x,\xi)|d\sigma_\xi < +\infty,$$

$ii)$ $\left\|D_\xi^\beta \mathscr{K}\right\|_{L^\infty(\mathbb{R}^{n+1}\times\mathbb{S}^n)} \leqslant M(\beta) < \infty$ for every multi-index β.

Moreover

$$|\mathscr{K}(x,x-y)| \leqslant \rho(x-y)^{-n-2}\left|\mathscr{K}\left(\frac{x'-y'}{\rho(x-y)}, \frac{t-\tau}{\rho^2(x-y)}\right)\right| \leqslant \frac{M}{\rho(x-y)^{n+2}}$$

which means that the singular integrals

$$\begin{cases} \mathfrak{K}f(x) = P.V.\displaystyle\int_{\mathbb{R}^{n+1}} \mathscr{K}(x,x-y)f(y)dy, \\ [8pt]\mathfrak{C}[a,f](x) = P.V.\displaystyle\int_{\mathbb{R}^{n+1}} \mathscr{K}(x,x-y)[a(x)-a(y)]f(y)dy \end{cases} \tag{6.1}$$

are sublinear and bounded in $L_w^\Phi(\mathbb{R}^{n+1})$ (see [41, 42, 43]). Let us note that any weight function φ satisfying (4.33) satisfies also (4.18) and hence the following holds as a simple application of the estimates proved in Section 4 (see Theorems 4.4 and 4.5).

THEOREM 6.8. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$ and $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable function satisfying* (4.33)*. Then for any $f \in M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})$ and $a \in BMO(\mathbb{R}^{n+1})$ there exist constants depending on n, Φ and the kernel such that*

$$\|\mathfrak{K}f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \lesssim \|f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})}, \quad \|\mathfrak{C}[a,f]\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \lesssim \|a\|_*\|f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})}. \tag{6.2}$$

COROLLARY 6.2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable function satisfying* (4.33)*, Q be a cylinder in $\mathbb{R}_+^{n+1}$, $\mathscr{K}(x,\xi) : Q \times \mathbb{R}_+^{n+1} \setminus \{0\} \to \mathbb{R}$, $a \in BMO(Q)$ and $f \in M_w^{\Phi,\varphi}(Q)$. Then the operators* (6.1) *are bounded in $M^{\Phi,\varphi}(Q)$ and*

$$\|\mathfrak{K}f\|_{M_w^{\Phi,\varphi}(Q)} \lesssim \|f\|_{M_w^{\Phi,\varphi}(Q)}, \quad \|\mathfrak{C}[a,f]\|_{M_w^{\Phi,\varphi}(Q)} \lesssim \|a\|_* \|f\|_{M_w^{\Phi,\varphi}(Q)}. \tag{6.3}$$

Proof. Define the extensions

$$\overline{\mathscr{K}}(x,\xi) = \begin{cases} \mathscr{K}(x,\xi) & (x,\xi) \in Q \times \mathbb{R}_+^{n+1} \setminus \{0\} \\ 0 & \text{elsewhere} \end{cases}, \quad \overline{f}(x) = \begin{cases} f(x) & x \in Q \\ 0 & x \notin Q. \end{cases}$$

Denote by $\overline{\mathfrak{K}}f$ the singular integral with a kernel $\overline{\mathscr{K}}$ and potential $\overline{f}$. Then

$$|\mathfrak{K}f(x)| \leqslant |\overline{\mathfrak{K}}f(x)| \leqslant C\int_{\mathbb{R}^{n+1}} \frac{|\overline{f}(y)|}{\rho(x-y)^{n+2}}dy$$

and

$$\|\mathfrak{K}f\|_{M_w^{\Phi,\varphi}(Q)} \leqslant \|\overline{\mathfrak{K}}f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} \leqslant C\|\overline{f}\|_{M_w^{\Phi,\varphi}(\mathbb{R}^{n+1})} = C\|f\|_{M_w^{\Phi,\varphi}(Q)}.$$

The estimate for the commutator follows in a similar way. $\square$

COROLLARY 6.3. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $a \in VMO(\mathbb{R}^{n+1})$ and φ be measurable function satisfying (4.33). Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_a)$ such that for any $\mathscr{E}_r(x)$ with a radius $r \in (0, r_0)$ and all $f \in M^{p,\varphi}(\mathscr{E}_r(x))$*

$$\|\mathfrak{C}[a,f]\|_{\Phi,\varphi,w;\mathscr{E}_r(x)} \leqslant C\varepsilon \|f\|_{\Phi,\varphi,w;\mathscr{E}_r(x)} \tag{6.4}$$

where C is independent of ε, f and r.

Proof. Since any VMO function can be approximated by BUC functions (see [46]) for each $\varepsilon > 0$ there exists $r_0(\varepsilon, \eta_a)$ and $g \in BUC$ with modulus of continuity $\omega_g(r_0) < \varepsilon/2$ such that $\|a-g\|_* < \varepsilon/2$. Fixing $\mathscr{E}_r(x_0)$ with $r \in (0, r_0)$ define the function

$$h(x) = \begin{cases} g(x), & x \in \mathscr{E}_r(x_0), \\ g\left(x_0 + r\dfrac{x'-x_0'}{\rho(x-x_0)}, t_0 + r^2\dfrac{t-t_0}{\rho^2(x-x_0)}\right), & x \in \mathscr{E}_r^c(x_0) \end{cases}$$

such that $h \in BUC(\mathbb{R}^{n+1})$ and $\omega_h(r_0) \leqslant \omega_g(r_0) < \varepsilon/2$. Hence

$$\begin{aligned} \|\mathfrak{C}[a,f]\|_{\Phi,\varphi,w;\mathscr{E}_r(x_0)} &\leqslant \|\mathfrak{C}[a-g,f]\|_{\Phi,\varphi,w;\mathscr{E}_r(x_0)} + \|\mathfrak{C}[g,f]\|_{\Phi,\varphi,w;\mathscr{E}_r(x_0)} \\ &\leqslant C\|a-g\|_*\|f\|_{\Phi,\varphi,w;\mathscr{E}_r(x_0)} + \|\mathfrak{C}[h,f]\|_{\Phi,\varphi,w;\mathscr{E}_r(x_0)} \\ &< C\varepsilon\|f\|_{\Phi,\varphi,w;\mathscr{E}_r(x_0)}. \quad \square \end{aligned}$$

For any $x' \in \mathbb{R}^n_+$ and any fixed $t > 0$ define the *generalized reflection*

$$\mathscr{T}(x) = (\mathscr{T}'(x), t), \qquad \mathscr{T}'(x) = x' - 2x_n \frac{\mathbf{a}^n(x',t)}{a^{nn}(x',t)}, \tag{6.5}$$

where $\mathbf{a}^n(x)$ is the last row of the coefficients matrix $\mathbf{a}(x)$ of (3.1). The function $\mathscr{T}'(x)$ maps $\mathbb{R}^n_+$ into $\mathbb{R}^n_-$ and the kernel $\mathscr{K}(x; \mathscr{T}(x)-y) = \mathscr{K}(x; \mathscr{T}'(x)-y', t-\tau)$ is nonsingular one for any $x, y \in \mathbb{D}^{n+1}_+$. Taking $\widetilde{x} \in \mathbb{D}^{n+1}_+$ there exist positive constants κ_1 and κ_2 such that

$$\kappa_1 \rho(\widetilde{x}-y) \leqslant \rho(\mathscr{T}(x)-y) \leqslant \kappa_2 \rho(\widetilde{x}-y). \tag{6.6}$$

For any $f \in M^{p,\varphi}_w(\mathbb{D}^{n+1}_+)$ and $a \in BMO(\mathbb{D}^{n+1}_+)$ define the nonsingular integral operators

$$\begin{cases} \widetilde{\mathfrak{K}} f(x) = \displaystyle\int_{\mathbb{D}^{n+1}_+} \mathscr{K}(x, \mathscr{T}(x)-y) f(y) dy, \\ \widetilde{\mathfrak{C}}[a,f](x) = \displaystyle\int_{\mathbb{D}^{n+1}_+} \mathscr{K}(x, \mathscr{T}(x)-y)[a(x)-a(y)] f(y) dy. \end{cases} \tag{6.7}$$

Since

$$|\mathscr{K}(x, \mathscr{T}(x)-y)| \leqslant \frac{M}{\rho(\mathscr{T}(x)-y))^{n+2}} \leqslant \frac{C}{\rho(\widetilde{x}-y)^{n+2}}$$

the operators (6.7) are sublinear and bounded in $L^\Phi_w(\mathbb{D}^{n+1}_+)$ (see [43]). The following estimates are simple consequence of the results in Section 5.

THEOREM 6.9. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $a \in BMO(\mathbb{D}_+^{n+1})$, $f \in M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$ and φ be measurable function satisfying (4.33). Then the operators $\widetilde{\mathfrak{K}}f$ and $\widetilde{\mathfrak{C}}[a,f]$ are continuous in $M_w^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$ and*

$$\|\widetilde{\mathfrak{K}}f\|_{\Phi,\varphi,w;\mathbb{D}_+^{n+1}} \leqslant C\|f\|_{\Phi,\varphi,w;\mathbb{D}_+^{n+1}}, \quad \|\widetilde{\mathfrak{C}}[a,f]\|_{\Phi,\varphi,w;\mathbb{D}_+^{n+1}} \leqslant C\|a\|_* \|f\|_{\Phi,\varphi,w;\mathbb{D}_+^{n+1}} \tag{6.8}$$

with a constant dependent on known quantities only.

COROLLARY 6.4. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_\Phi}$, $a \in VMO$ and p and φ be as above. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_a)$ such that for any $\mathscr{E}_r^+(x^0)$ with a radius $r \in (0, r_0)$ and all $f \in M_w^{\Phi,\varphi}(\mathscr{E}_r^+(x^0))$*

$$\|\mathfrak{C}[a,f]\|_{\Phi,\varphi,w;\mathscr{E}_r^+(x^0)} \leqslant C\varepsilon\|f\|_{\Phi,\varphi,w;\mathscr{E}_r^+(x^0)}, \tag{6.9}$$

where C is independent of ε, f, r and x^0.

7. Proof of the main result

Consider the problem (3.1) with $f \in M_w^{\Phi,\varphi}(Q)$ and φ satisfying (4.33). Since $M_w^{p,\varphi}(Q)$ is a proper subset of $L_w^{\Phi}(Q)$ than (3.1) is uniquely solvable and the solution u belongs to $\overset{\circ}{W}{}^{2,1}L_w^{\Phi}(Q)$. Our aim is to show that this solution belongs to $\overset{\circ}{W}{}^{2,1}M_w^{\Phi,\varphi}(Q)$. For this goal we need a priori estimate of u that we are going to prove in two steps.

Interior estimate. For any $x_0 \in \mathbb{R}_+^{n+1}$ consider the parabolic cylinder $\mathscr{C}_r(x_0) = \mathscr{B}_r(x_0') \times (t_0 - r^2, t_0)$. Let $v \in C_0^\infty(\mathscr{C}_r)$ with $v(x,t) = 0$ for $t \leqslant 0$. According to [2, Theorem 1.4] (see also [37]) for any $x \in \operatorname{supp} v$ the following representation formula for the second derivatives of v holds true

$$\begin{aligned} D_{ij}v(x) =& P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x-y)[a^{hk}(y) - a^{hk}(x)]D_{hk}v(y)dy \\ &+ P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x-y)\mathscr{P}v(y)dy + \mathscr{P}v(x)\int_{\mathbb{S}^n} \Gamma_j(x,y)\nu_i d\sigma_y, \end{aligned} \tag{7.1}$$

where $\nu(\nu_1, \ldots, \nu_{n+1})$ is the outward normal to $\mathbb{S}^n$. Here $\Gamma(x,\xi)$ is the fundamental solution of the operator $\mathscr{P}$ and $\Gamma_{ij}(x,\xi) = \partial^2\Gamma(x,\xi)/\partial\xi_i\partial\xi_j$. Since any function $v \in W^{2,1}L_w^{\Phi}$ can be approximated by C_0^∞ functions, the representation formula (7.1) still holds for any $v \in W^{2,1}L_w^{\Phi}(\mathscr{C}_r(x_0))$. The properties of the fundamental solution (cf. [2, 35, 37]) imply taht Γ_{ij} are variable Calderón-Zygmund kernels in the sense of Definition 6.10. Using the notations (6.1) we can write

$$\begin{aligned} D_{ij}v(x) =& \mathfrak{C}_{ij}[a^{hk}, D_{hk}v](x) \\ &+ \mathfrak{K}_{ij}(\mathscr{P}v)(x) + \mathscr{P}v(x)\int_{\mathbb{S}^n} \Gamma_j(x,y)\nu_i d\sigma_y, \end{aligned} \tag{7.2}$$

where $\mathfrak{K}_{ij}$ and $\mathfrak{C}_{ij}$ are the singular integrals defined in (6.1) with kernels $\mathscr{K}(x, x-y) = \Gamma_{ij}(x, x-y)$. Because of Corollaries 6.2 and 6.3 and the equivalence of the metrics we get

$$\|D^2v\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)} \leqslant C(\varepsilon\|D^2v\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)} + \|\mathscr{P}u\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)}) \tag{7.3}$$

for some r small enough. Moving the norm of D^2v on the left-hand side we get

$$\|D^2v\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)} \leqslant C(n,\Phi,\eta_a(r),\|D\Gamma\|_{\infty,Q})\|\mathscr{P}v\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)}.$$

Define a cut-off function $\phi(x)=\phi_1(x')\phi_2(t)$, with $\phi_1 \in C_0^\infty(\mathscr{B}_r(x_0'))$, $\phi_2 \in C_0^\infty(\mathbb{R})$ such that

$$\phi_1(x') = \begin{cases} 1 & x' \in \mathscr{B}_{\theta r}(x_0') \\ 0 & x' \notin \mathscr{B}_{\theta' r}(x_0') \end{cases}, \qquad \phi_2(t) = \begin{cases} 1 & t \in (t_0-(\theta r)^2, t_0] \\ 0 & t < t_0-(\theta' r)^2 \end{cases}$$

with $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and $|D^s\phi| \leqslant C[\theta(1-\theta)r]^{-s}$, $s=0,1,2$, $|\phi_t| \sim |D^2\phi|$. For any solution $u \in W^{2,1}L_w^\Phi(Q)$ of (3.1) define $v(x)=\phi(x)u(x) \in W^{2,1}L_w^\Phi(\mathscr{C}_r)$. Hence

$$\begin{aligned} \|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta r}(x_0)} &\leqslant \|D^2v\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)} \leqslant C\|\mathscr{P}v\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)} \\ &\leqslant C\left(\|f\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)} + \frac{\|Du\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)}}{\theta(1-\theta)r} + \frac{\|u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)}}{[\theta(1-\theta)r]^2}\right). \end{aligned}$$

Hence

$$\begin{aligned} &\big[\theta(1-\theta)r\big]^2\|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta r}(x_0)} \\ &\leqslant \Big([\theta(1-\theta)r]^2\|f\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)} + \theta(1-\theta)r\|Du\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)} + \|u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)}\Big) \\ &\quad \big(\text{by the definition of } \theta' \text{ it follows } \theta(1-\theta) \leqslant 2\theta'(1-\theta')\big) \\ &\leqslant C\left(r^2\|f\|_{M_w^{\Phi,\varphi}(Q)} + \theta'(1-\theta')r\|Du\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)} + \|u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta' r}(x_0)}\right). \end{aligned}$$

Introducing the semi-norms

$$\Theta_s = \sup_{0<\theta<1} \big[\theta(1-\theta)r\big]^s \|D^s u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta r}(x_0)}, \quad s=0,1,2,$$

the above inequality becomes

$$[\theta(1-\theta)r]^2\|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta r}(x_0)} \leqslant \Theta_2 \leqslant C\left(r^2\|f\|_{M_w^{\Phi,\varphi}(Q)} + \Theta_1 + \Theta_0\right). \tag{7.4}$$

The interpolation inequality [48, Lemma 4.2] gives that there exists a positive constant C independent of r such that

$$\Theta_1 \leqslant \varepsilon\Theta_2 + \frac{C}{\varepsilon}\Theta_0 \qquad \text{for any } \varepsilon \in (0,2).$$

Thus (7.4) becomes

$$[\theta(1-\theta)r]^2\|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_{\theta r}(x_0)} \leqslant \Theta_2 \leqslant C\left(r^2\|f\|_{M_w^{\Phi,\varphi}(Q)} + \Theta_0\right) \quad \forall\, \theta \in (0,1).$$

and taking $\theta = 1/2$ we get the Caccioppoli-type estimate

$$\|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_{r/2}(x_0)} \leqslant C\left(\|f\|_{M_w^{\Phi,\varphi}(Q)} + \frac{1}{r^2}\|u\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)}\right).$$

To estimate u_t we exploit the parabolic structure of the equation and the boundedness of the coefficients

$$\begin{aligned}\|u_t\|_{\Phi,\varphi,w;\mathscr{C}_{r/2}(x_0)} &\leqslant \|\mathbf{a}\|_{\infty;Q}\|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_{r/2}(x_0)} + \|f\|_{\Phi,\varphi,w;\mathscr{C}_{r/2}(x_0)}\\ &\leqslant C\big(\|f\|_{M_w^{\Phi,\varphi}(Q)} + \frac{1}{r^2}\|u\|_{\Phi,\varphi,w;\mathscr{C}_r(x_0)}\big).\end{aligned}$$

Consider cylinders $Q' = \Omega' \times (0,T)$ and $Q'' = \Omega'' \times (0,T)$ with $\Omega' \Subset \Omega'' \Subset \Omega$, by standard covering procedure and partition of the unity we get

$$\|u\|_{W^{2,1}M_w^{\Phi,\varphi}(Q')} \leqslant C\big(\|f\|_{M_w^{\Phi,\varphi}(Q)} + \|u\|_{\Phi,\varphi,w;Q''}\big), \tag{7.5}$$

where the constant depends on n, Φ, Λ, T, $\|D\Gamma\|_{\infty;Q}$, $\eta_{\mathbf{a}}(r)$, $\|\mathbf{a}\|_{\infty,Q}$ and $\mathrm{dist}(\Omega', \partial\Omega'')$.

Boundary estimates. For any fixed $r > 0$ and $x^0 = (x'', 0, 0)$ define the semi-cylinders

$$\mathscr{C}_r^+(x^0) = \mathscr{B}_r^+(x^{0'}) \times (0, r^2) = \{|x'| < r, 0 < x_n, 0 < t < r^2\}$$

with $\mathscr{S}_r^+ = \{(x'', 0, t) : |x''| < r, 0 < t < r^2\}$. For any solution $u \in W^{2,1}L_w^{\Phi}(\mathscr{C}_r^+(x^0))$ with $\mathrm{supp}\, u \in \mathscr{C}_r^+(x^0)$ the following boundary representation formula holds (cf. [2])

$$D_{ij}u = \mathfrak{C}_{ij}[a^{hk}, D_{hk}u](x) + \mathfrak{K}_{ij}(\mathscr{P}u)(x) + \mathscr{P}u(x)\int_{\mathbb{S}^n}\Gamma_j(x,y)\nu_i d\sigma_y - \mathfrak{I}_{ij}(x),$$

where

$$\begin{aligned}
&\mathfrak{I}_{ij}(x) = \widetilde{\mathfrak{C}}_{ij}[a^{hk}, D_{hk}u](x) + \widetilde{\mathfrak{K}}_{ij}(\mathscr{P}u)(x),\ i,j = 1,\ldots,n-1,\\
&\mathfrak{I}_{in}(x) = \mathfrak{I}_{ni}(x) = \sum_{l=1}^{n}\left(\frac{\partial\mathscr{T}(x)}{\partial x_n}\right)^l\left[\widetilde{\mathfrak{C}}_{il}[a^{hk}, D_{hk}u](x) + \widetilde{\mathfrak{K}}_{il}(\mathscr{P}u)(x)\right],\ i = 1,\ldots,n-1,\\
&\mathfrak{I}_{nn}(x) = \sum_{l,r=1}^{n}\left(\frac{\partial\mathscr{T}(x)}{\partial x_n}\right)^l\left(\frac{\partial\mathscr{T}(x)}{\partial x_n}\right)^r\left[\widetilde{\mathfrak{C}}_{il}[a^{hk}, D_{hk}u](x) + \widetilde{\mathfrak{K}}_{il}(\mathscr{P}u)(x)\right],\\
&\frac{\partial\mathscr{T}(x)}{\partial x_n} = \left(-2\frac{a^{n1}(x)}{a^{nn}(x)},\ldots,-2\frac{a^{nn-1}(x)}{a^{nn}(x)},-1,0\right).
\end{aligned}$$

Here $\widetilde{\mathfrak{K}}_{ij}$ and $\widetilde{\mathfrak{C}}_{ij}$ are the operators defined by (6.7) with kernels $\mathscr{K}(x, \mathscr{T}(x) - y) = \Gamma_{ij}(x, \mathscr{T}(x) - y)$. Applying the estimates (6.8) and (6.9) and having in mind that the components of the vector $\frac{\partial\mathscr{T}(x)}{\partial x_n}$ are bounded we get

$$\|D^2u\|_{\Phi,\varphi,w;\mathscr{C}_r^+(x^0)} \lesssim \|\mathscr{P}u\|_{\Phi,\varphi,w;\mathscr{C}_r^+(x^0)} + \|u\|_{\Phi,\varphi,w;\mathscr{C}_r^+(x^0)}.$$

The Jensen inequality applied to $u(x) = \int_0^t u_s(x', s)ds$ and the parabolic structure of the equation give

$$\|u\|_{\Phi,\varphi,w;\mathscr{C}_r^+(x^0)} \lesssim r^2\|u_t\|_{\Phi,\varphi,w;\mathscr{C}_r^+(x^0)} \lesssim \|f\|_{M_w^{\Phi,\varphi}(Q)} + r^2\|u\|_{\Phi,\varphi,w;\mathscr{C}_r^+(x^0)}.$$

Taking r small enough we can move the norm of u on the left-hand side obtaining

$$\|u\|_{\Phi,\varphi,w;\mathscr{C}_r^+} \leqslant C\|f\|_{M_w^{\Phi,\varphi}(Q)}$$

with a constant depending on $n, \Phi, \Lambda, T, \eta_{\mathbf{a}}, \|\mathbf{a}\|_{\infty,Q}$. By covering of the boundary with small cylinders, partition of the unit subordinated of that covering and local flattering we get that

$$\|u\|_{W^{2,1}M_w^{\Phi,\varphi}(Q\setminus Q')} \lesssim \|f\|_{M_w^{\Phi,\varphi}(Q)}. \tag{7.6}$$

Unifying (7.5) and (7.6) we get (3.4).

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