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UPPER BOUNDS ON THE HARMONIC STATUS INDEX

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Abstract. The harmonic status index of a simple connected graph G is defined as the sum of the weights $\frac{2}{\sigma_G(u)+\sigma_G(v)}$ over all edges uv of G, where $\sigma_G(u)$ denotes the status of the vertex u in G which is the sum of distances between u and all other vertices of G. In this paper, we present upper bounds on the harmonic status index of some families of graph products in terms of certain structural invariants such as the order, size, maximum degree, inverse status and harmonic status index of their components. The graph products considered here are sum, disjunction, symmetric difference, Indu-Bala product, corona product, Cartesian product, lexicographic product, and strong product. Some applications of the obtained results are also presented as corollaries.

1. Introduction

Throughout this paper, all graphs are considered to be simple, connected, and finite. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. The open neighborhood $N_G(u)$ of a vertex $u \in V(G)$ is the set of vertices adjacent to u. The set $N_G(u) \cup \{u\}$ is called the closed neighborhood of the vertex u in G. The order of $N_G(u)$ is called the degree of u in G and denoted by $d_G(u)$. A graph in which every vertex has the same degree is called a regular graph. A regular graph with vertices of degree k is said to be k-regular. The distance $d_G(u,v)$ between the vertices $u, v \in V(G)$ is defined as the length of any shortest path in G connecting u and v. The diameter $d(G)$ is the largest distance between all pairs of vertices of G. The status (also called transmission) $\sigma_G(u)$ of a vertex $u \in V(G)$ is the sum of distance between u and all other vertices of G.

A *graph invariant* is a property of graphs that depends only on the abstract structure, not on graph representations. A *topological index* is a graph invariant associated to the molecular graph of a chemical compound which quantifies its topological characteristics. Two of the most famous categories of topological indices are distance-based and degree-based indices. Among them, several indices are recognized to be useful tools in chemical researches. One of the best-known and well-studied degree-based topological indices is the *harmonic index* which was introduced in 1987 by Fajtlowicz [10] within some conjectures, generated by the computer program Graffiti. It was defined for a graph G as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u)+d_G(v)}.$$

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It was showed that the harmonic index correlates well with the π-electronic energy of benzenoid hydrocarbons. Further results concerning mathematical properties and applications of the harmonic index can be found in the papers [1, 2, 21], the recent survey [3] and the references quoted therein.

Inspired by the definition of the harmonic index, Ramane, Basavanagoud and Yalnaik [19] introduced the *harmonic status index* of a graph G as

$$HS(G) = \sum_{uv \in E(G)} \frac{2}{\sigma_G(u) + \sigma_G(v)}.$$

Ramane *et al*. [19] showed that the correlation between the boiling point of the paraffin hydrocarbons and the harmonic status index of the corresponding molecular graphs is good. In addition, the authors computed the exact value of the harmonic status index for some specific graphs and gave a number of upper and lower bounds on this invariant. Jog and Patil [13] constructed new graphs of fixed diameter and computed the harmonic status index of those graphs. In this paper, we present upper bounds for the eccentric harmonic index of some families of graph products such as the sum, disjunction, symmetric difference, Indu-Bala product, corona product, Cartesian product, lexicographic product, and strong product in terms of the harmonic status indices of their components and/or some auxiliary invariants. These results lead us to compute the exact value of the harmonic status index for some other families of graphs. Our results follows the line of research of several recent papers [4, 11, 17, 20] dealing with computing the harmonic index of graph products.

2. Definitions and preliminaries

As usual, we denote the path, cycle, star, complete graph, and empty graph on n vertices by P_n, C_n, S_n, K_n, and $\overline{K}_n$, respectively. Also, we introduce the *inverse status* of a graph G as

$$\sigma^{-1}(G) = \sum_{u \in V(G)} \frac{1}{\sigma_G(u)}.$$

The status of a vertex and the harmonic status index of graphs with diameter at most 2 are given in the following lemma.

LEMMA 1. *Let G be a graph of order n and diameter at most 2. Then*

1. *For each $u \in V(G)$, $\sigma_G(u) = 2(n-1) - d_G(u)$,*
2. $HS(G) = \sum_{uv \in E(G)} \frac{2}{4(n-1)-(d_G(u)+d_G(v))}$.

See, for example [19], for the proof of the lemma.

A special case of the Jensen's inequality [16] presented in the following lemma is useful in the proof of our main theorems.

LEMMA 2. [16] *Let $x_1, x_2, \cdots, x_n$ be positive real numbers. Then*

$$\frac{n}{x_1 + x_2 + \cdots + x_n} \leqslant \frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right),$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

At this point, we express the definitions of some graph operations which will be studied in the next section. Throughout the paper, we denote the components of each operation by G_1 and G_2 which are considered to be nontrivial graphs. The order, size, maximum degree, and minimum degree of the graph G_i are denoted by n_i, m_i, Δ_i, and δ_i, respectively, where $i = 1,2$. If the number of components of a graph operation is more than 2, the values of subscripts will vary accordingly.

The *sum* $G_1 + G_2 + \cdots + G_k$ of graphs $G_1, G_2, \cdots, G_k$ is the graph union $G_1 \cup G_2 \cup \cdots \cup G_k$ together with all the edges joining $V(G_i)$ and $V(G_j)$ for all $1 \leqslant i < j \leqslant k$. It is obvious that $G_1 + G_2 + \cdots + G_k$ has diameter at most 2 and $d_{G_1+G_2+\cdots+G_k}(u) = n - n_i + d_{G_i}(u)$, where $u \in V(G_i)$ and $n = n_1 + n_2 + \cdots + n_k$. Hence by Lemma 1, we arrive at:

LEMMA 3. *The status of a vertex $u \in V(G_1 + G_2 + \cdots + G_k)$ is given by*

$$\sigma_{G_1+G_2+\cdots+G_k}(u) = n + n_i - (d_{G_i}(u) + 2),$$

where $n = n_1 + n_2 + \cdots + n_k$ and G_i is the component of $G_1 + G_2 + \cdots + G_k$ containing the vertex u.

The *disjunction* $G_1 \vee G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent, whenever $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$. From Lemma 1 of [14], $G_1 \vee G_2$ has diameter at most 2 and for each vertex $(u_1, u_2) \in V(G_1 \vee G_2)$, $d_{G_1 \vee G_2}((u_1, u_2)) = n_2 d_{G_1}(u_1) + n_1 d_{G_1}(u_2) - d_{G_1}(u_1) d_{G_1}(u_2)$. Now Lemma 1 implies:

LEMMA 4. *The status of a vertex $(u_1, u_2) \in V(G_1 \vee G_2)$ is given by*

$$\sigma_{G_1 \vee G_2}((u_1, u_2)) = 2(n_1 n_2 - 1) - \big(n_2 d_{G_1}(u_1) + n_1 d_{G_1}(u_2) - d_{G_1}(u_1) d_{G_1}(u_2)\big).$$

The *symmetric difference* $G_1 \oplus G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent, whenever $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$, but not both. One can easily see that, the diameter of $G_1 \oplus G_2$ is 2 and from Lemma 1 of [14], for each vertex $(u_1, u_2) \in V(G_1 \oplus G_2)$, $d_{G_1 \oplus G_2}((u_1, u_2)) = n_2 d_{G_1}(u_1) + n_1 d_{G_1}(u_2) - 2 d_{G_1}(u_1) d_{G_1}(u_2)$. Now Lemma 1 yields:

LEMMA 5. *The status of a vertex $(u_1, u_2) \in V(G_1 \oplus G_2)$ is given by*

$$\sigma_{G_1 \oplus G_2}(u) = 2(n_1 n_2 - 1) - \big(n_2 d_{G_1}(u_1) + n_1 d_{G_1}(u_2) - 2 d_{G_1}(u_1) d_{G_1}(u_2)\big).$$

The *Indu-Bala product* $G_1 \diamondsuit G_2$ of graphs G_1 and G_2 is obtained from two disjoint copies of $G_1 + G_2$ by joining the corresponding vertices in the two copies of G_2 (see [12]).

LEMMA 6. *The status of a vertex* $(u_1,u_2) \in V(G_1 \vee G_2)$ *is given by*

$$\sigma_{G_1 \diamondsuit G_2}(u) = \begin{cases} 5n_1 + 3n_2 - d_{G_1}(u) - 2 & \text{if } u \in V(G_1), \\ 3n_1 + 5n_2 - 2d_{G_2}(u) - 4 & \text{if } u \in V(G_2). \end{cases}$$

Proof. If $u \in V(G_1)$, then

$$\begin{aligned} \sigma_{G_1 \diamondsuit G_2}(u) = & \sum_{v \in N_{G_1}(u)} 1 + \sum_{v \in V(G_1) \setminus N_{G_1}[u]} 2 + \sum_{v \in V(G_2)} (1+2) + \sum_{v \in V(G_1)} 3 \\ = & d_{G_1}(u) + 2\big(n_1 - 1 - d_{G_1}(u)\big) + 3n_2 + 3n_1 = 5n_1 + 3n_2 - d_{G_1}(u) - 2, \end{aligned}$$

and if $u \in V(G_2)$, then

$$\begin{aligned} \sigma_{G_1 \diamondsuit G_2}(u) = & \sum_{v \in N_{G_2}(u)} (1+2) + \sum_{v \in V(G_2) \setminus N_{G_2}[u]} (2+3) + 1 + \sum_{v \in V(G_1)} (1+2) \\ = & 3d_{G_2}(u) + 5\big(n_2 - 1 - d_{G_2}(u)\big) + 1 + 3n_1 = 3n_1 + 5n_2 - 2d_{G_2}(u) - 4, \end{aligned}$$

from which the result follows. □

The *corona product* $G_1 \circ G_2$ of graphs G_1 and G_2 is obtained by taking one copy of G_1 and n_1 copies of G_2, and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1, for $i = 1, 2, \cdots, n_1$.

LEMMA 7. [5] *The status of a vertex* $u \in V(G_1 \circ G_2)$ *is given by*

$$\sigma_{G_1 \circ G_2}(u) = \begin{cases} (n_2+1)\sigma_{G_1}(u) + n_1 n_2 & \text{if } u \in V(G_1), \\ (n_2+1)\sigma_{G_1}(x) - d_{G_2}(u) + n_1 + 2(n_1 n_2 - 1) & \text{if } u \in V(G_{2x}), \end{cases}$$

where G_{2x} *denotes the copy of* G_2 *attached to the vertex* $x \in V(G_1)$.

The *Cartesian product* $G_1 \square G_2 \square \cdots \square G_k$ of graphs $G_1, G_2, \cdots, G_k$ has the vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ and vertices $(u_1, u_2, \cdots, u_k)$ and $(v_1, v_2, \cdots, v_k)$ are adjacent whenever they differ in exactly one position, say in i-th, and $u_i v_i \in E(G_i)$. Using the fact that for vertices $u = (u_1, u_2, \cdots, u_k)$, $v = (v_1, v_2, \cdots, v_k) \in V(G_1 \square G_2 \square \cdots \square G_k)$, $d_{G_1 \square G_2 \square \cdots \square G_k}(u,v) = \sum_{i=1}^{k} d_{G_i}(u_i, v_i)$ from [5], we arrive at:

LEMMA 8. *The status of a vertex* $(u_1, u_2, \cdots, u_k) \in V(G_1 \square G_2 \square \cdots \square G_k)$ *is given by*

$$\sigma_{G_1 \square G_2 \square \cdots \square G_k}((u_1, u_2, \cdots, u_k)) = (n_1 n_2 \cdots n_k) \sum_{i=1}^{k} \frac{\sigma_{G_i}(u_i)}{n_i}.$$

The *lexicographic product* $G_1[G_2]$ of graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_1 v_1 \in E(G_1)$ or $[u_1 = v_1$ and $u_2 v_2 \in E(G_2)]$.

LEMMA 9. [5, 18] *The status of a vertex* $(u_1, u_2) \in V(G_1[G_2])$ *is given by*

$$\sigma_{G_1[G_2]}((u_1, u_2)) = n_2 \sigma_{G_1}(u_1) + 2(n_2 - 1) - d_{G_2}(u_2).$$

The *strong product* $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(G_1)]$ or $[u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)]$.

LEMMA 10. *The status of a vertex $(u_1, u_2) \in V(G_1 \boxtimes G_2)$ is given by*

$$\begin{aligned}\sigma_{G_1 \boxtimes G_2}((u_1,u_2)) \geqslant &(d_{G_2}(u_2)+1)\sigma_{G_1}(u_1) + (d_{G_1}(u_1)+1)\sigma_{G_2}(u_2) + 2(n_1-1)(n_2-1)\\ &-2(n_2-1)d_{G_1}(u_1) - 2(n_1-1)d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2),\end{aligned}$$

with equality if and only if for each $v_1 \notin N_{G_1}[u_1]$ and $v_2 \notin N_{G_2}[u_2]$, we have

$$\max\{d_{G_1}(u_1,v_1), d_{G_2}(u_2,v_2)\} = 2.$$

Proof. Using the fact that for each $(u_1,u_2),(v_1,v_2) \in V(G_1 \boxtimes G_2)$,

$$d_{G_1 \boxtimes G_2}((u_1,u_2),(v_1,v_2)) = \max\{d_{G_1}(u_1,v_1), d_{G_1}(u_2,v_2)\}$$

from [22], we have

$$\begin{aligned}\sigma_{G_1 \boxtimes G_2}((u_1,u_2)) = &\sum_{v_1 \in V(G_1)\setminus\{u_1\}} \sum_{v_2 \in N_{G_2}[u_2]} d_{G_1}(u_1,v_1)\\ &+ \sum_{v_1 \in N_{G_1}[u_1]} \sum_{v_2 \in V(G_2)\setminus\{u_2\}} d_{G_2}(u_2,v_2)\\ &+ \sum_{v_1 \notin N_{G_1}[u_1]} \sum_{v_2 \notin N_{G_2}[u_2]} \max\{d_{G_1}(u_1,v_1), d_{G_1}(u_2,v_2)\}\\ &- \sum_{v_1 \in N_{G_1}(u_1)} \sum_{v_2 \in N_{G_2}(u_2)} 1.\end{aligned}$$

Note that for each $v_1 \notin N_{G_1}[u_1]$ and $v_2 \notin N_{G_2}[u_2]$, $\max\{d_{G_1}(u_1,v_1), d_{G_2}(u_2,v_2)\} \geqslant 2$. Hence

$$\begin{aligned}\sigma_{G_1 \boxtimes G_2}((u_1,u_2)) \geqslant &(d_{G_2}(u_2)+1)\sigma_{G_1}(u_1) + (d_{G_1}(u_1)+1)\sigma_{G_2}(u_2) - d_{G_1}(u_1)d_{G_2}(u_2)\\ &+2(n_1-1-d_{G_1}(u_1))(n_2-1-d_{G_2}(u_2)),\end{aligned}$$

from which the result follows immediately. □

We refer the readers to [5–9, 15, 18] for more information on topological indices of graph products.

3. Main results

In this section, we present upper bounds on the harmonic status index of the product graphs introduced in Section 2. We start with the sum of graphs.

THEOREM 1. *Let* $G = G_1 + G_2 + \cdots + G_k$ *and* $n = n_1 + n_2 + \cdots + n_k$. *The harmonic status index of* G *satisfies the following inequality:*

$$HS(G) \leqslant \sum_{i=1}^{k} \frac{m_i}{n + n_i - (\Delta_i + 2)} + \sum_{1 \leqslant i < j \leqslant k} \frac{2n_i n_j}{2n + n_i + n_j - (\Delta_i + \Delta_j + 4)}, \qquad (1)$$

with equality if and only if G_i *is regular for each* $1 \leqslant i \leqslant k$.

Proof. By Lemma 3 and using the fact that for each $u \in V(G_i)$, $1 \leqslant i \leqslant k$, $d_{G_i}(u) \leqslant \Delta_i$, we have

$$\sigma_G(u) = n + n_i - (d_{G_i}(u) + 2) \geqslant n + n_i - (\Delta_i + 2),$$

with equality if and only if $d_{G_i}(u) = \Delta_i$. Now from the definition of the harmonic status index, we get

$$\begin{aligned} HS(G) &\leqslant \sum_{i=1}^{k} \sum_{uv \in E(G_i)} \frac{2}{2n + 2n_i - 2(\Delta_i + 2)} \\ &\quad + \sum_{1 \leqslant i < j \leqslant k} \sum_{u \in V(G_i)} \sum_{v \in V(G_j)} \frac{2}{2n + n_i + n_j - (\Delta_i + \Delta_j + 4)} \\ &= \sum_{i=1}^{k} \frac{m_i}{n + n_i - (\Delta_i + 2)} + \sum_{1 \leqslant i < j \leqslant k} \frac{2n_i n_j}{2n + n_i + n_j - (\Delta_i + \Delta_j + 4)}, \end{aligned}$$

from which Eq. (1) follows. The equality holds in (1) if and only if for each $u \in V(G_i)$, $1 \leqslant i \leqslant k$, $d_{G_i}(u) = \Delta_i$, which implies that G_i is regular for each $1 \leqslant i \leqslant k$. $\square$

The complete r-partite graph $K_{n_1, n_2, \cdots, n_r}$ with classes of partitions of sizes $n_1, n_2, \cdots, n_r$ is the sum of r empty graphs $\overline{K}_{n_1}, \overline{K}_{n_2}, \cdots, \overline{K}_{n_r}$ and by Theorem 1, we easily arrive at:

COROLLARY 1. *Let* $n = n_1 + n_2 + \cdots + n_r$ *and* $n_1, n_2, \cdots, n_r \geqslant 2$. *Then*

$$HS(K_{n_1, n_2, \cdots, n_r}) = \sum_{1 \leqslant i < j \leqslant k} \frac{2n_i n_j}{2n + n_i + n_j - 4}.$$

Using Theorem 1, we can get the following upper bound for the harmonic status index of the suspension $K_1 + G$ of graph G.

COROLLARY 2. *Let* G *be a graph on* n *vertices,* m *edges and maximum degree* Δ. *The harmonic status index of the suspension of* G *satisfies the following inequality:*

$$HS(K_1 + G) \leqslant \frac{m}{2n - 1 - \Delta} + \frac{2n}{3n - 1 - \Delta},$$

with equality if and only if G *is regular.*

We can apply Corollary 2 to obtain the harmonic status index of the star and windmill graphs.

COROLLARY 3. *The following hold:*

1. $HS(S_{n+1}) = HS(K_1 + \overline{K}_n) = \frac{2n}{3n-1}$;
2. $HS(K_1 + mK_{n-1}) = \frac{m(n-1)(n-2)}{2(2mn-2m-n+1)} + \frac{2m(n-1)}{3mn-3m-n+1}$,

where mK_{n-1} is the union of m copies of K_{n-1}.

THEOREM 2. *The harmonic status index of $G_1 \vee G_2$ satisfies the following inequality:*

$$HS(G_1 \vee G_2) \leqslant \frac{m_1 n_2^2 + m_2 n_1^2 - 2m_1 m_2}{2(n_1 n_2 - 1) - \big(n_2 \Delta_1 + n_1 \Delta_2 - \delta_1 \delta_2\big)}, \tag{2}$$

with equality if and only if G_1 and G_2 are regular.

Proof. By Lemma 4 and using the fact that for each $u_i \in V(G_i)$, $1 \leqslant i \leqslant 2$, $\delta_i \leqslant d_{G_i}(u) \leqslant \Delta_i$, for each $(u_1, u_2) \in V(G_1 \vee G_2)$, we have

$$\begin{aligned}\sigma_{G_1 \vee G_2}((u_1, u_2)) &= 2(n_1 n_2 - 1) - (n_2 d_{G_1}(u_1) + n_1 d_{G_1}(u_2) - d_{G_1}(u_1) d_{G_1}(u_2)) \\ &\geqslant 2(n_1 n_2 - 1) - (n_2 \Delta_1 + n_1 \Delta_2 - \delta_1 \delta_2),\end{aligned}$$

with equality if and only if G_1 and G_2 are regular. Now from the definition of the harmonic status index, we have

$$\begin{aligned}HS(G_1 \vee G_2) &\leqslant \sum_{(u_1,u_2)(v_1,v_2) \in E(G_1 \vee G_2)} \frac{2}{4(n_1 n_2 - 1) - 2\big(n_2 \Delta_1 + n_1 \Delta_2 - \delta_1 \delta_2\big)} \\ &= \frac{m_1 n_2^2 + m_2 n_1^2 - 2m_1 m_2}{2(n_1 n_2 - 1) - \big(n_2 \Delta_1 + n_1 \Delta_2 - \delta_1 \delta_2\big)},\end{aligned}$$

from which Eq. (2) holds. The equality holds in (2) if and only if G_1 and G_2 are regular. □

Application of Theorem 2 yields:

COROLLARY 4.

$$HS(C_n \vee C_m) = \frac{nm(n+m-2)}{2(nm-n-m+1)}.$$

Using Lemma 5 and the same argument as in the proof of Theorem 2, we arrive at:

THEOREM 3. *The harmonic status index of $G_1 \oplus G_2$ satisfies the following inequality:*

$$HS(G_1 \oplus G_2) \leqslant \frac{m_1 n_2^2 + m_2 n_1^2 - 4m_1 m_2}{2(n_1 n_2 - 1) - \big(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2\big)},$$

with equality if and only if G_1 and G_2 are regular.

Theorem 3 has the following consequence.

COROLLARY 5.

$$HS(C_n \oplus C_m) = \frac{nm(n+m-4)}{2(nm-n-m+3)}.$$

THEOREM 4. *The harmonic status index of $G_1 \diamondsuit G_2$ satisfies the following inequality:*

$$\begin{aligned} HS(G_1 \diamondsuit G_2) \leqslant & \frac{2m_1}{5n_1+3n_2-\Delta_1-2} + \frac{n_2+2m_2}{3n_1+5n_2-2\Delta_2-4} \\ & + \frac{4n_1n_2}{8n_1+8n_2-\Delta_1-2\Delta_2-6}, \end{aligned} \tag{3}$$

with equality if and only if G_1 and G_2 are regular.

Proof. From the definition of the harmonic status index and Lemma 6, we have

$$\begin{aligned} HS(G_1 \diamondsuit G_2) = & \sum_{uv \in E(G_1 \diamondsuit G_2)} \frac{2}{\sigma_{G_1 \diamondsuit G_2}(u) + \sigma_{G_1 \diamondsuit G_2}(v)} \\ = & 2 \sum_{uv \in E(G_1)} \frac{2}{10n_1+6n_2-d_{G_1}(u)-d_{G_1}(v)-4} \\ & + 2 \sum_{uv \in E(G_2)} \frac{2}{6n_1+10n_2-2d_{G_2}(u)-2d_{G_2}(v)-8} \\ & + 2 \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{8n_1+8n_2-d_{G_1}(u)-2d_{G_2}(u)-6} \\ & + \sum_{u \in V(G_2)} \frac{2}{2\big(3n_1+5n_2-2d_{G_2}(u)-4\big)}. \end{aligned}$$

Using the fact that for each $u \in V(G_i)$, $1 \leqslant i \leqslant 2$, $d_{G_i}(u) \leqslant \Delta_i$, we get

$$\begin{aligned} HS(G_1 \diamondsuit G_2) \leqslant & 2 \sum_{uv \in E(G_1)} \frac{2}{10n_1+6n_2-2\Delta_1-4} + 2 \sum_{uv \in E(G_2)} \frac{2}{6n_1+10n_2-4\Delta_2-8} \\ & + 2 \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{8n_1+8n_2-\Delta_1-2\Delta_2-6} \\ & + \sum_{u \in V(G_2)} \frac{1}{3n_1+5n_2-2\Delta_2-4} \\ = & \frac{2m_1}{5n_1+3n_2-\Delta_1-2} + \frac{n_2+2m_2}{3n_1+5n_2-2\Delta_2-4} + \frac{4n_1n_2}{8n_1+8n_2-\Delta_1-2\Delta_2-6}, \end{aligned}$$

from which the inequality in (3) holds. The equality holds in (3) if and only if G_1 and G_2 are regular graphs. □

Application of Theorem 4 yields:

COROLLARY 6.

$$HS(C_n \Diamond C_m) = \frac{2n}{5n+3m-4} + \frac{3m}{3n+5m-8} + \frac{nm}{2n+2m-3}.$$

THEOREM 5. *The harmonic status index of $G_1 \circ G_2$ satisfies the following inequality:*

$$HS(G_1 \circ G_2) < \frac{1}{4}\Big(\frac{1}{n_2+1}HS(G_1) + \frac{n_2+m_2}{n_2+1}\sigma^{-1}(G_1) + \frac{m_1}{n_1 n_2} + \frac{n_1 m_2}{n_1+2(n_1n_2-1)-\Delta_2} + \frac{2n_1n_2}{n_1+3n_1n_2-2-\Delta_2}\Big). \quad (4)$$

Proof. From the definition of the harmonic status index and Lemma 7, we have

$$\begin{aligned}
&HS(G_1 \circ G_2)\\
&= \sum_{uv\in E(G_1\circ G_2)} \frac{2}{\sigma_{G_1\circ G_2}(u)+\sigma_{G_1\circ G_2}(v)}\\
&= \sum_{uv\in E(G_1)} \frac{2}{(n_2+1)\big(\sigma_{G_1}(u)+\sigma_{G_1}(v)\big)+2n_1n_2}\\
&\quad + \sum_{x\in V(G_1)} \sum_{uv\in E(G_{2x})} \frac{2}{2(n_2+1)\sigma_{G_1}(x)+2n_1+4(n_1n_2-1)-d_{G_2}(u)-d_{G_2}(v)}\\
&\quad + \sum_{u\in V(G_1)} \sum_{v\in V(G_{2u})} \frac{2}{2(n_2+1)\sigma_{G_1}(u)+n_1+3n_1n_2-2-d_{G_2}(v)}\\
&:= S_1+S_2+S_3.
\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}
S_1 &\leqslant \frac{1}{4}\sum_{uv\in E(G_1)}\Big(\frac{2}{(n_2+1)\big(\sigma_{G_1}(u)+\sigma_{G_1}(v)\big)} + \frac{2}{2n_1n_2}\Big)\\
&= \frac{1}{4}\Big(\frac{1}{n_2+1}HS(G_1) + \frac{m_1}{n_1n_2}\Big),
\end{aligned}$$

with equality if and only if for each $uv \in E(G_1)$, $(n_2+1)\big(\sigma_{G_1}(u)+\sigma_{G_1}(v)\big) = 2n_1n_2$.

Using the fact that $d_{G_2}(u) \leqslant \Delta_2$, for each $u \in V(G_2)$ and by Lemma 2, we have

$$\begin{aligned}
S_2 &\leqslant \frac{m_2}{4}\sum_{x\in V(G_1)}\Big(\frac{2}{2(n_2+1)\sigma_{G_1}(x)} + \frac{2}{2n_1+4(n_1n_2-1)-2\Delta_2}\Big)\\
&= \frac{m_2}{4}\Big(\frac{1}{n_2+1}\sigma^{-1}(G_1) + \frac{n_1}{n_1+2(n_1n_2-1)-\Delta_2}\Big),
\end{aligned}$$

with equality if and only if G_2 is regular and for each $x \in V(G_1)$, $2(n_2+1)\sigma_{G_1}(x) = 2n_1+4(n_1n_2-1)-2\Delta_2$, and

$$S_3 \leqslant \frac{n_2}{4}\sum_{u\in V(G_1)}\Big(\frac{2}{2(n_2+1)\sigma_{G_1}(u)}+\frac{2}{n_1+3n_1n_2-2-\Delta_2}\Big)$$
$$=\frac{n_2}{4}\Big(\frac{1}{n_2+1}\sigma^{-1}(G_1)+\frac{2n_1}{n_1+3n_1n_2-2-\Delta_2}\Big),$$

with equality if and only if G_2 is regular and for each $u \in V(G_1)$, $2(n_2+1)\sigma_{G_1}(u) = n_1+3n_1n_2-2-\Delta_2$.

Eq. (4) is obtained by adding S_1, S_2 and S_3 and simplifying the resulting expression. The equality in (4) holds if and only if G_2 is regular, for each $u_1v_1 \in E(G_1)$, $(n_2+1)\big(\sigma_{G_1}(u)+\sigma_{G_1}(v)\big) = 2n_1n_2$ and for each $x \in V(G_1)$, $2(n_2+1)\sigma_{G_1}(x) = 2n_1 + 4(n_1n_2-1)-2\Delta_2$, $2(n_2+1)\sigma_{G_1}(x) = n_1+3n_1n_2-2-\Delta_2$. From the last two relations, we get $\Delta_2 = n_1+n_1n_2-2 \geqslant 2+2n_2-2 = 2n_2$, which is a contradiction. Hence (4) is a strict inequality. $\square$

THEOREM 6. *The harmonic status index of $G_1\square G_2$ satisfies the following inequality:*

$$HS(G_1\square G_2) \leqslant \frac{1}{4}\Big(HS(G_1)+HS(G_2)+\frac{m_2}{n_2}\sigma^{-1}(G_1)+\frac{m_1}{n_1}\sigma^{-1}(G_2)\Big), \tag{5}$$

with equality if and only if for each $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, $n_2\sigma_{G_1}(u_1) = n_1\sigma_{G_2}(u_2)$.

Proof. From the definition of the harmonic status index and Lemma 8, we have

$$HS(G_1\square G_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1\square G_2)} \frac{2}{\sigma_{G_1\square G_2}((u_1,u_2))+\sigma_{G_1\square G_2}((v_1,v_2))}$$
$$= \sum_{u_1\in V(G_1)}\sum_{u_2v_2\in E(G_2)} \frac{2}{2n_2\sigma_{G_1}(u_1)+n_1\big(\sigma_{G_2}(u_2)+\sigma_{G_2}(v_2)\big)}$$
$$+\sum_{u_2\in V(G_2)}\sum_{u_1v_1\in E(G_1)} \frac{2}{n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big)+2n_1\sigma_{G_2}(u_2)}.$$

Using Lemma 2, we obtain

$$HS(G_1\square G_2) \leqslant \frac{1}{4}\sum_{u_1\in V(G_1)}\sum_{u_2v_2\in E(G_2)}\Big(\frac{2}{2n_2\sigma_{G_1}(u_1)}+\frac{2}{n_1\big(\sigma_{G_2}(u_2)+\sigma_{G_2}(v_2)\big)}\Big)$$
$$+\frac{1}{4}\sum_{u_2\in V(G_2)}\sum_{u_1v_1\in E(G_1)}\Big(\frac{2}{n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big)}+\frac{2}{2n_1\sigma_{G_2}(u_2)}\Big)$$
$$=\frac{1}{4}\Big(\frac{m_2}{n_2}\sigma^{-1}(G_1)+HS(G_2)+HS(G_1)+\frac{m_1}{n_1}\sigma^{-1}(G_2)\Big),$$

from which the inequality (5) follows. By Lemma 2, the equality holds in (5) if and only if for each $u_1 \in V(G_1)$, $u_2v_2 \in E(G_2)$, $2n_2\sigma_{G_1}(u_1) = n_1\big(\sigma_{G_2}(u_2)+\sigma_{G_2}(v_2)\big)$ and for each $u_2 \in V(G_2)$, $u_1v_1 \in E(G_1)$, $n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big) = 2n_1\sigma_{G_2}(u_2)$. This holds if and only if for each $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, $n_2\sigma_{G_1}(u_1) = n_1\sigma_{G_2}(u_2)$. □

Using the expression for $HS(C_n)$ from Proposition 5 of [19] and Theorem 6, we can compute the status harmonic index of the C_4-nanotorus $C_n \Box C_n$.

COROLLARY 7.

$$HS(C_n \Box C_n) = \begin{cases} \frac{4n}{n^2-1} & \text{if } n \text{ is odd,} \\ \frac{4}{n} & \text{if } n \text{ is even.} \end{cases}$$

Using Theorem 6 and an inductive argument, we arrive at:

THEOREM 7. *The harmonic status index of $G_1 \Box G_2 \Box \cdots \Box G_k$ satisfies the following inequality:*

$$HS(G_1 \Box G_2 \Box \cdots \Box G_k) \leqslant \frac{1}{k^2}\Big(\sum_{i=1}^{k} HS(G_i) + \sum_{i=1}^{k} \frac{m_i}{n_i} \sum_{j=1, j\neq i}^{k} \sigma^{-1}(G_j)\Big),$$

with equality if and only if $n_j\sigma_{G_i}(u_i) = n_i\sigma_{G_j}(u_j)$, for each $1 \leqslant i \neq j \leqslant k$.

The k-dimensional hypercube Q_k is the Cartesian product $K_2 \Box K_2 \Box \cdots K_2$, ($k$ *times*) and from Theorem 7, we get

COROLLARY 8.

$$HS(Q_k) = 1.$$

The next operation which we consider is the lexicographic product. We first study the special case when the first component of the lexicographic product has diameter at most 2.

THEOREM 8. *If G_1 has diameter at most 2, then*

$$HS(G_1[G_2]) \leqslant \frac{m_1 n_2^2 + n_1 m_2}{2(n_1n_2 - 1) - \big(n_2\Delta_1 + \Delta_2\big)}, \tag{6}$$

with equality if and only if G_1 and G_2 are regular.

Proof. Note that $G_1[G_2]$ has diameter at most 2, as G_1 has diameter at most 2 and from Lemma 1 of [14], for each $(u_1,u_2) \in V(G_1[G_2])$, $d_{G_1[G_2]}((u_1,u_2)) = n_2 d_{G_1}(u_1) + d_{G_2}(u_2)$. Now by Lemma 1 and using the fact that for each $u_i \in V(G_i)$, $1 \leqslant i \leqslant 2$, $d_{G_i}(u_i) \leqslant \Delta_i$, we get

$$\begin{aligned}\sigma_{G_1[G_2]}((u_1,u_2)) &= 2(n_1n_2 - 1) - \big(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)\big) \\ &\geqslant 2(n_1n_2 - 1) - (n_2\Delta_1 + \Delta_2),\end{aligned}$$

with equality if and only if $d_{G_i}(u_i)=\Delta_i$, $1\leqslant i\leqslant 2$. Now from the definition of the harmonic status index, we have

$$\begin{aligned}HS(G_1[G_2]) &\leqslant \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1[G_2])}\frac{2}{4(n_1n_2-1)-2\big(n_2\Delta_1+\Delta_2\big)}\\ &=\frac{m_1n_2^2+n_1m_2}{2(n_1n_2-1)-\big(n_2\Delta_1+\Delta_2\big)},\end{aligned}$$

from which Eq. (6) holds. The equality holds in (6) if and only if G_1 and G_2 are regular. □

From Theorem 8 we get the following corollary.

COROLLARY 9.

$$HS(K_n[C_m])=\frac{nm(nm-m+2)}{2(nm+m-4)},$$

Now we tackle the harmonic status index of the lexicographic product in general case.

THEOREM 9. *The harmonic status index of $G_1[G_2]$ satisfies the following inequality:*

$$HS(G_1[G_2])\leqslant\frac{1}{4}\Big(n_2HS(G_1)+\frac{m_2}{n_2}\sigma^{-1}(G_1)+\frac{m_1n_2^2+n_1m_2}{2(n_2-1)-\Delta_2}\Big), \tag{7}$$

with equality if and only if $G_1\cong P_2$ and G_2 is (n_2-2)-regular.

Proof. From the definition of the harmonic status and Lemma 9, we obtain

$$\begin{aligned}&HS(G_1[G_2])\\ &=\sum_{(u_1,u_2)(v_1,v_2)\in E(G_1[G_2])}\frac{2}{\sigma_{G_1[G_2]}((u_1,u_2))+\sigma_{G_1[G_2]}((v_1,v_2))}\\ &=\sum_{u_1v_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2)}\frac{2}{n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big)+4(n_2-1)-d_{G_2}(u_2)-d_{G_2}(v_2)}\\ &\quad+\sum_{u_1\in V(G_1)}\sum_{u_2v_2\in E(G_2)}\frac{2}{2n_2\sigma_{G_1}(u_1)+4(n_2-1)-d_{G_2}(u_2)-d_{G_2}(v_2)}\\ &:=S_1+S_2.\end{aligned}$$

Using the fact that for each $u_2\in V(G_2)$, $d_{G_2}(u_2)\leqslant\Delta_2$, and Lemma 2, we get

$$\begin{aligned}S_1&\leqslant\frac{n_2^2}{4}\sum_{u_1v_1\in E(G_1)}\Big(\frac{2}{n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big)}+\frac{2}{4(n_2-1)-2\Delta_2}\Big)\\ &=\frac{1}{4}\Big(n_2HS(G_1)+\frac{m_1n_2^2}{2(n_2-1)-\Delta_2}\Big),\end{aligned}$$

with equality if and only if G_2 is regular and for each $u_1v_1 \in E(G_1)$, $n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big) = 4(n_2-1)-2\Delta_2$. Similarly,

$$\begin{aligned} S_2 &\leqslant \frac{m_2}{4} \sum_{u_1 \in V(G_1)} \Big(\frac{2}{2n_2\sigma_{G_1}(u_1)} + \frac{2}{4(n_2-1)-2\Delta_2}\Big) \\ &= \frac{1}{4}\Big(\frac{m_2}{n_2}\sigma^{-1}(G_1) + \frac{n_1m_2}{2(n_2-1)-\Delta_2}\Big), \end{aligned}$$

with equality if and only if G_2 is regular and for each $u_1 \in V(G_1)$, $2n_2\sigma_{G_1}(u_1) = 4(n_2-1)-2\Delta_2$.

Eq. (7) is obtained by adding S_1 and S_2 and simplifying the resulting expression. The inequality in (7) holds if and only if G_2 is regular and for each $u_1v_1 \in E(G_1)$, $n_2\big(\sigma_{G_1}(u_1)+\sigma_{G_1}(v_1)\big) = 4(n_2-1)-2\Delta_2$ and for each $u_1 \in V(G_1)$, $2n_2\sigma_{G_1}(u_1) = 4(n_2-1)-2\Delta_2$. If the equality holds in (7) and there exists $u_1 \in V(G_1)$ with $\sigma_{G_1}(u_1) \geqslant 2$, then from the previous equation $4(n_2-1)-2\Delta_2 \geqslant 4n_2$, which implies that $\Delta_2 \leqslant -2$, a contradiction. Hence the equality holds in (7) if and only if for each $u_1 \in V(G_1)$, $\sigma_{G_1}(u_1) = 1$ and G_2 is regular with $2n_2 = 4(n_2-1)-2\Delta_2$. This holds if and only if $G_1 \cong P_2$ and G_2 is (n_2-2)-regular. □

The last operation which we consider is the strong product. We first consider the special case when both components of the strong product have diameter at most 2.

THEOREM 10. *If G_1 and G_2 are of diameter at most 2, then*

$$HS(G_1 \boxtimes G_2) \leqslant \frac{n_1m_2+n_2m_1+2m_1m_2}{2(n_1n_2-1)-\big(\Delta_1+\Delta_2+\Delta_1\Delta_2\big)}, \tag{8}$$

with equality if and only if G_1 and G_2 are regular.

Proof. Note that $G_1[G_2]$ has diameter at most 2, as G_1 and G_2 have diameter at most 2 and from [22], for each $(u_1,u_2) \in V(G_1[G_2])$, $d_{G_1\boxtimes G_2}((u_1,u_2)) = d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_1}(u_2)$. Now by Lemma 1 and using the fact that for each $u_i \in V(G_i)$, $1 \leqslant i \leqslant 2$, $d_{G_i}(u_i) \leqslant \Delta_i$, we get

$$\begin{aligned} \sigma_{G_1\boxtimes G_2}((u_1,u_2)) &= 2(n_1n_2-1) - \big(d_{G_1}(u_1)+d_{G_2}(u_2)+d_{G_1}(u_1)d_{G_1}(u_2)\big) \\ &\geqslant 2(n_1n_2-1)-(\Delta_1+\Delta_2+\Delta_1\Delta_2), \end{aligned}$$

with equality if and only if $d_{G_i}(u_i) = \Delta_i$, $1 \leqslant i \leqslant 2$. Now from the definition of the harmonic status index, we have

$$\begin{aligned} HS(G_1 \boxtimes G_2) &\leqslant \sum_{(u_1,u_2)(v_1,v_2) \in E(G_1\boxtimes G_2)} \frac{2}{4(n_1n_2-1)-2\big(\Delta_1+\Delta_2+\Delta_1\Delta_2\big)} \\ &= \frac{n_1m_2+n_2m_1+2m_1m_2}{2(n_1n_2-1)-\big(\Delta_1+\Delta_2+\Delta_1\Delta_2\big)}, \end{aligned}$$

from which Eq. (8) follows. The equality holds in (8) if and only if G_1 and G_2 are regular graphs. □

Theorem 10 yields:

COROLLARY 10.
$$HS(K_n \boxtimes K_m) = \frac{nm}{2}.$$

Now we tackle the case when at least one of the components in the strong product has diameter greater than 2.

THEOREM 11. *If G_1 or G_2 are of diameter greater than 2, then*

$$\begin{aligned} HS(G_1 \boxtimes G_2) < \frac{1}{9}\Big(& \frac{n_2+2m_2}{\delta_2+1} HS(G_1) + \frac{n_1+2m_1}{\delta_1+1} HS(G_2) \\ & + \frac{m_2}{\delta_2+1}\sigma^{-1}(G_1) + \frac{m_1}{\delta_1+1}\sigma^{-1}(G_2) \\ & + \frac{n_1m_2+n_2m_1+2m_1m_2}{2(n_1-1)(n_2-1)-2(n_2-1)\Delta_1-2(n_1-1)\Delta_2+\delta_1\delta_2}\Big). \end{aligned} \tag{9}$$

Proof. By Lemma 10 and using the fact that for each $u_i \in V(G_i)$, $\delta_i \leqslant d_{G_i}(u_i) \leqslant \Delta_i$, $1 \leqslant i \leqslant 2$, we get

$$\begin{aligned} \sigma_{G_1\boxtimes G_2}((u_1,u_2)) > & (\delta_2+1)\sigma_{G_1}(u_1) + (\delta_1+1)\sigma_{G_2}(u_2) + 2(n_1-1)(n_2-1) \\ & -2(n_2-1)\Delta_1 - 2(n_1-1)\Delta_2 + \delta_1\delta_2. \end{aligned} \tag{10}$$

From the definition of the harmonic status index, we obtain

$$\begin{aligned} HS(G_1 \boxtimes G_2) &= \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1\boxtimes G_2)} \frac{2}{\sigma_{G_1\boxtimes G_2}((u_1,u_2)) + \sigma_{G_1\boxtimes G_2}((v_1,v_2))} \\ &= \sum_{u_1\in V(G_1)} \sum_{u_2v_2\in E(G_2)} \frac{2}{\sigma_{G_1\boxtimes G_2}((u_1,u_2)) + \sigma_{G_1\boxtimes G_2}((u_1,v_2))} \\ &\quad + \sum_{u_2\in V(G_2)} \sum_{u_1v_1\in E(G_1)} \frac{2}{\sigma_{G_1\boxtimes G_2}((u_1,u_2)) + \sigma_{G_1\boxtimes G_2}((v_1,u_2))} \\ &\quad + \sum_{u_1v_1\in E(G_1)} \sum_{u_2v_2\in E(G_2)} \frac{2}{\sigma_{G_1\boxtimes G_2}((u_1,u_2)) + \sigma_{G_1\boxtimes G_2}((v_1,v_2))} \\ &\quad + \sum_{u_1v_1\in E(G_1)} \sum_{u_2v_2\in E(G_2)} \frac{2}{\sigma_{G_1\boxtimes G_2}((v_1,u_2)) + \sigma_{G_1\boxtimes G_2}((u_1,v_2))} \\ &:= S_1+S_2+S_3+S_4. \end{aligned}$$

By Eq. (10), for each $u_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)$,

$$\begin{aligned} & \sigma_{G_1\boxtimes G_2}((u_1,u_2)) + \sigma_{G_1\boxtimes G_2}((u_1,v_2)) \\ > & 2(\delta_2+1)\sigma_{G_1}(u_1) + (\delta_1+1)(\sigma_{G_2}(u_2)+\sigma_{G_2}(v_2)) \\ & +4(n_1-1)(n_2-1) - 4(n_2-1)\Delta_1 - 4(n_1-1)\Delta_2 + 2\delta_1\delta_2, \end{aligned}$$

and by Lemma 2, we obtain

$$
\begin{aligned}
S_1 < \frac{1}{9} \sum_{u_1 \in V(G_1)} \sum_{u_2 v_2 \in E(G_2)} & \left(\frac{2}{2(\delta_2+1)\sigma_{G_1}(u_1)} + \frac{2}{(\delta_1+1)(\sigma_{G_2}(u_2)+\sigma_{G_2}(v_2))} \right. \\
& \left. + \frac{2}{4(n_1-1)(n_2-1) - 4(n_2-1)\Delta_1 - 4(n_1-1)\Delta_2 + 2\delta_1\delta_2} \right) \\
= \frac{1}{9} & \left(\frac{m_2}{\delta_2+1} \sigma^{-1}(G_1) + \frac{n_1}{\delta_1+1} HS(G_2) \right. \\
& \left. + \frac{n_1 m_2}{2(n_1-1)(n_2-1) - 2(n_2-1)\Delta_1 - 2(n_1-1)\Delta_2 + \delta_1\delta_2} \right).
\end{aligned}
$$

Similarly, we arrive at:

$$
\begin{aligned}
S_2 < \frac{1}{9} & \left(\frac{m_1}{\delta_1+1} \sigma^{-1}(G_2) + \frac{n_2}{\delta_2+1} HS(G_1) \right. \\
& \left. + \frac{n_2 m_1}{2(n_1-1)(n_2-1) - 2(n_2-1)\Delta_1 - 2(n_1-1)\Delta_2 + \delta_1\delta_2} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
S_3, S_4 < \frac{1}{9} & \left(\frac{m_2}{\delta_2+1} HS(G_1) + \frac{m_1}{\delta_1+1} HS(G_2) \right. \\
& \left. + \frac{m_1 m_2}{2(n_1-1)(n_2-1) - 2(n_2-1)\Delta_1 - 2(n_1-1)\Delta_2 + \delta_1\delta_2} \right).
\end{aligned}
$$

Eq. (9) is obtained by adding S_1, S_2, S_3, S_4 and simplifying the resulting expression. □

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