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SOME SINGULAR VALUE INEQUALITIES FOR MATRICES

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Abstract. In this paper, we prove some singular value inequalities for sums and products of matrices. Some of our inequalities will give several generalizations of recent known inequalities. Among other inequalities, we prove that if A,B,C,D,X,Y are $n\times n$ complex matrices such that X and Y are positive semidefinite, then

$$s_j\left(AXB^*+CYD^*\right)\leqslant\sqrt{\left\||A^*|^2+|C^*|^2\right\|\left\||B^*|^2+|D^*|^2\right\|}\,s_j(X\oplus Y),$$

for $j=1,2,\dots,n,$ which is a generalization of an inequality in [12]. Here, s_j and $\|\cdot\|$ denote the singular value and the spectral norm of matrices, respectively.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the $\mathcal{C}^*$-algebra of all $n\times n$ complex matrices. The matrix $A\in\mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathbb{C}^n$, where $\langle\cdot,\cdot\rangle$ is the inner product defined on $\mathbb{C}^n$. The absolute of $A\in\mathbb{M}_n(\mathbb{C})$, written as $|A|$, is defined by $|A|=(A^*A)^{1/2}$, where A^* denotes the adjoint (conjugate transpose) of the matrix A.

The singular values of $A\in\mathbb{M}_n(\mathbb{C})$, written as $s_1(A)\geqslant s_2(A)\geqslant\ldots\geqslant s_n(A)$ are the eigenvalues of $|A|$, i.e., $s_j(A)=\lambda_j(|A|)$ for $j=1,2,\dots,n$. In fact, it can be seen that $s_j(A)=s_j(|A|)=s_j(A^*)$ for $j=1,2,\dots,n$.

A norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$, is said to be unitarily invariant if $|||UAV|||=|||A|||$ for all $A\in\mathbb{M}_n(\mathbb{C})$ and for all unitary matrices $U,V\in\mathbb{M}_n(\mathbb{C})$. The spectral norm, written as $\|\cdot\|$, defined on $\mathbb{M}_n(\mathbb{C})$ by $\|A\|=\max\limits_{\|x\|=1}\|Ax\|$ for $A\in\mathbb{M}_n(\mathbb{C})$ and $x\in\mathbb{C}^n$. It can be seen that $\|A\|=s_1(A)$ for $A\in\mathbb{M}_n(\mathbb{C})$. The direct sum of $A,B\in\mathbb{M}_n(\mathbb{C})$ is denoted by $A\oplus B$ and is defined on $\mathbb{M}_{2n}(\mathbb{C})$ by $A\oplus B=\begin{bmatrix}A&0\\0&B\end{bmatrix}$. Note that $s_j(A\oplus 0)=s_j(A)$ for $j=1,\dots,n,$ and $s_j(A\oplus 0)=0$ for $j=n+1,\dots,2n.$

In [12], the authors proved that if $A,B\in\mathbb{M}_n(\mathbb{C}),$ then

$$s_j(A+B)\leqslant 2s_j(A\oplus B)\tag{1}$$

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for $j=1,2,\ldots,n$. In [15], the author proved that if $A,B\in\mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$s_j(A-B)\leqslant s_j(A\oplus B) \tag{2}$$

for $j=1,2,\ldots,n$.

In this paper, we give singular value inequalities for matrices. Some of our results represents generalizations of the inequalities (1) and (2). Other singular value inequalities will also be given. It is known that a unitarily invariant norm of a matrix is a symmetric gauge function of singular values of this matrix, and so, our results extend to every unitarily invariant norm. For recent results concerning singular value inequalities we refer the reader to [5], [7], [9], [11], [13] and [14]. Also, for recent results concerning unitarily invariant norm inequalities we refer the reader to [2], [3], [4], and [8].

2. Main results

We start this section with some singular value inequalities. For $A,B,X\in\mathbb{M}_n(\mathbb{C})$ where X is positive semidefinite and for $j=1,2,\ldots,n$, we have the following list of inequalities:

$$s_j(AXB^*)\leqslant\frac{1}{2}s_j\left(\left(|A|^2+|B|^2\right)^{1/2}X\left(|A|^2+|B|^2\right)^{1/2}\right). \tag{3}$$

The inequality (3) can be found in [16].

Applying the useful inequality (see e.g., [10, p. 75]),

$$s_j(AXB)\leqslant\|A\|\,\|B\|\,s_j(X), \tag{4}$$

for the right hand side of the inequality (3), we get

$$s_j(AXB^*)\leqslant\frac{1}{2}\left\||A|^2+|B|^2\right\|s_j(X), \tag{5}$$

which was given in [6]. Also, the authors in [6] gave a refinement of the inequality (4). This refinement asserts that

$$s_j(AXB^*)\leqslant\frac{1}{2}\left\|\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right\|\|A\|\,\|B\|\,s_j(X). \tag{6}$$

Clearly, inequality (6) can be obtained from inequality (5) by replacing A and B by $\sqrt{\frac{\|B\|}{\|A\|}}A$ and $\sqrt{\frac{\|A\|}{\|A\|}}B$, respectively.

Related to the inequality (5), we have the inequality

$$s_j(AXB^*)\leqslant\frac{1}{2}\|X\|\,s_j\left(|A|^2+|B|^2\right), \tag{7}$$

which was given in [1]. The inequality (7) can also be obtained by applying inequality (4) on the right hand side of (3) as follows:

$$\begin{aligned} s_j(AXB^*) &\leqslant \frac{1}{2}s_j\left(\left(|A|^2+|B|^2\right)^{1/2}X\left(|A|^2+|B|^2\right)^{1/2}\right) \\ &= \frac{1}{2}\lambda_j\left(X^{1/2}\left(|A|^2+|B|^2\right)X^{1/2}\right) \\ &= \frac{1}{2}s_j\left(X^{1/2}\left(|A|^2+|B|^2\right)X^{1/2}\right) \\ &\leqslant \frac{1}{2}\|X^{1/2}\|\,\|X^{1/2}\|s_j\left(|A|^2+|B|^2\right) \\ &= \frac{1}{2}\|X\|s_j\left(|A|^2+|B|^2\right). \end{aligned}$$

Combining the inequalities (5) and (7) together, we obtain

$$s_j(AXB^*) \leqslant \frac{1}{2}\min\left\{\left\||A|^2+|B|^2\right\|s_j(X), \|X\|\,s_j\left(|A|^2+|B|^2\right)\right\}. \tag{8}$$

Based on the inequality (8), we can have a refinement of the inequality (6). This refinement can be seen in the following theorem.

THEOREM 1. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ be such that X is positive semidefinite. Then*

$$s_j(AXB^*) \leqslant \frac{\|A\|\,\|B\|}{2}\min\left\{\left\|\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right\|s_j(X), \|X\|\,s_j\left(\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right)\right\}$$

for $j = 1,2,\ldots,n$.

Proof. Replacing A by $\frac{A}{\|A\|}$ and B by $\frac{B}{\|B\|}$ in the inequality (8), we get

$$s_j\left(\frac{A}{\|A\|}X\frac{B^*}{\|B\|}\right) \leqslant \frac{1}{2}\min\left\{\left\|\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right\|s_j(X), \|X\|\,s_j\left(\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right)\right\}$$

and so

$$s_j(AXB^*) \leqslant \frac{\|A\|\,\|B\|}{2}\min\left\{\left\|\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right\|s_j(X), \|X\|\,s_j\left(\frac{|A|^2}{\|A\|^2}+\frac{|B|^2}{\|B\|^2}\right)\right\},$$

as required. □

Letting $X = (|A|^2+|B|^2)^m$, $m = 1,2,\ldots$ in the inequality (8), we get the following corollary.

COROLLARY 2. *Let $A,B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\begin{aligned} &s_j\left(A(|A|^2+|B|^2)^m B^*\right) \\ &\quad\leqslant \frac{1}{2}\min\left\{\left\||A|^2+|B|^2\right\|s_j((|A|^2+|B|^2)^m), \left\|(|A|^2+|B|^2)^m\right\|s_j\left(|A|^2+|B|^2\right)\right\}, \end{aligned}$$

for $j = 1,2,\ldots,n$.

To state our next result, we invoke the well known fact which asserts that for any $T \in \mathbb{M}_n(\mathbb{C})$, we have

$$s_j(T^*T) = s_j(TT^*) \tag{9}$$

for $j = 1,2,\ldots,n$. In particular, if $j = 1$, we have

$$\|T^*T\| = \|TT^*\|. \tag{10}$$

THEOREM 3. *Let $A,B,C,D,X,Y \in \mathbb{M}_n(\mathbb{C})$ be such that X and Y are positive semidefinite. Then*

$$s_j(AXB^* + CYD^*) \leqslant \sqrt{\left\||A^*|^2 + |C^*|^2\right\| \left\||B^*|^2 + |D^*|^2\right\|}\, s_j(X \oplus Y)$$

for $j = 1,2,\ldots,n$.

Proof. Let $S^* = \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, $W = \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}$. Then for $j = 1,2,\ldots,n$, we have

$$\begin{aligned}
&s_j(AXB^* + CYD^*) \\
&\quad = s_j(S^*RW^*) \\
&\quad \leqslant \frac{1}{2}\left\||S^*|^2 + |W|^2\right\| s_j(R) \qquad \text{(by the inequality (5))} \\
&\quad = \frac{1}{2}\left\| \begin{bmatrix} A^* & 0 \\ C^* & 0 \end{bmatrix} \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B^* & 0 \\ D^* & 0 \end{bmatrix} \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix} \right\| s_j(X \oplus Y) \\
&\quad \leqslant \frac{1}{2}\left(\left\| \begin{bmatrix} A^* & 0 \\ C^* & 0 \end{bmatrix} \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} B^* & 0 \\ D^* & 0 \end{bmatrix} \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix} \right\| \right) s_j(X \oplus Y) \\
&\qquad \text{(by the triangle inequality)} \\
&\quad = \frac{1}{2}\left(\left\| \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ C^* & 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B^* & 0 \\ D^* & 0 \end{bmatrix} \right\| \right) s_j(X \oplus Y) \\
&\quad \leqslant \frac{1}{2}\left(\left\||A^*|^2 + |C^*|^2\right\| + \left\||B^*|^2 + |D^*|^2\right\| \right) s_j(X \oplus Y).
\end{aligned}$$

Now, for $t > 0$, replacing A by $\sqrt{t}A$, C by $\sqrt{t}C$, B by $\frac{1}{\sqrt{t}}B$, and D by $\frac{1}{\sqrt{t}}D$ and taking the minimum over $t > 0$, we have

$$s_j(AXB^* + CYD^*) \leqslant \sqrt{\left\||A^*|^2 + |C^*|^2\right\| \left\||B^*|^2 + |D^*|^2\right\|}\, s_j(X \oplus Y),$$

as required. $\square$

Note that Theorem 3 generalizes the inequality (1). In fact, letting $A = B = C = D = I$ in Theorem 3, we get the inequality (1).

Other generalizations of the inequality (1) can be seen in the following corollaries.

COROLLARY 4. *Let $A,B,C,D,X,Y \in \mathbb{M}_n(\mathbb{C})$ be such that X and Y are positive semidefinite. Then*

$$s_j(AXB^* + CYD^*) \leqslant \left\| \left(|A|^2 \oplus |C|^2\right) + \left(|B|^2 \oplus |D|^2\right) \right\| s_j(X \oplus Y)$$

for $j = 1,2,\dots,n$.

Proof. For $j = 1,2,\dots,n$, we have

$$\begin{aligned}
&s_j(AXB^* + CYD^*)\\
&\qquad \leqslant 2s_j\left((AXB^*) \oplus (CYD^*)\right) \qquad \text{(by the inequality (1))}\\
&\qquad = 2s_j\left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} B^* & 0 \\ 0 & D^* \end{bmatrix}\right)\\
&\qquad \leqslant \left\| \left(|A|^2 \oplus |C|^2\right) + \left(|B|^2 \oplus |D|^2\right) \right\| s_j(X \oplus Y)\\
&\qquad\qquad \text{(by the inequality (5))},
\end{aligned}$$

as required. $\square$

COROLLARY 5. *Let $A,B,X,Y,C,D \in \mathbb{M}_n(\mathbb{C})$ be such that X and Y are positive semidefinite. Then*

$$\begin{aligned}
&s_j(AXD + BYC)\\
&\leqslant \left\| \frac{\max(\|D\|,\|C\|)\left(|A|^2 \oplus |B|^2\right)}{\max(\|A\|,\|B\|)} + \frac{\max(\|A\|,\|B\|)\left(|D^*|^2 \oplus |C^*|^2\right)}{\max(\|D\|,\|C\|)} \right\| s_j\left(X \oplus Y\right),
\end{aligned}$$

for $j = 1,2,\dots,n$.

Proof. For $j = 1,2,\dots,n$, we have

$$\begin{aligned}
&s_j(AXD + BYC)\\
&\quad \leqslant 2s_j\left((AXD) \oplus (BYC)\right) \text{ (by the inequality (1))}\\
&\quad = 2s_j\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix}\right)\\
&\quad \leqslant \left\| \frac{|A|^2 \oplus |B|^2}{\left(\max(\|A\|,\|B\|)\right)^2} + \frac{|D^*|^2 \oplus |C^*|^2}{\left(\max(\|D\|,\|C\|)\right)^2} \right\|\\
&\qquad \times \max(\|A\|,\|B\|)\max(\|D\|,\|C\|)s_j\left(X \oplus Y\right) \qquad \text{(by the inequality (6))}\\
&\quad = \left\| \frac{\max(\|D\|,\|C\|)\left(|A|^2 \oplus |B|^2\right)}{\max(\|A\|,\|B\|)} + \frac{\max(\|A\|,\|B\|)\left(|D^*|^2 \oplus |C^*|^2\right)}{\max(\|D\|,\|C\|)} \right\|\\
&\qquad \times s_j\left(X \oplus Y\right),
\end{aligned}$$

as required. □

In our next work, we give generalizations of the inequality (2). We start with the following result which is an application of the inequality (6).

THEOREM 6. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$s_j(X^*AX - YBY^*) \leqslant \max(\|A\|, \|B\|) s_j(|X^*|^2 \oplus |Y|^2) \tag{11}$$

for $j = 1,2,\ldots,n$.

Proof. Since X^*AX and YBY^* are positive semidefinite, then for $j = 1,2,\ldots,n$, we have

$$\begin{aligned}
&s_j(X^*AX - YBY^*) \\
&\qquad \leqslant s_j(X^*AX \oplus YBY^*) \\
&\qquad = s_j\left(\begin{bmatrix} X^* & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y^* \end{bmatrix}\right) \\
&\qquad = s_j\left(\begin{bmatrix} X^* & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y^* \end{bmatrix}\right) \\
&\qquad = s_j\left(\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y^* \end{bmatrix}\begin{bmatrix} X^* & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\right) \\
&\qquad\qquad \text{(by the relation (9))} \\
&\qquad = s_j\left(\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} |X^*|^2 & 0 \\ 0 & |Y|^2 \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\right) \\
&\qquad \leqslant \frac{1}{2\max(\|A\|, \|B\|)} \left\| \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \right\| \left\| \begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix} \right\|^2 s_j(|X^*|^2 \oplus |Y|^2) \\
&\qquad\qquad \text{(by the inequality (6))} \\
&\qquad = \frac{1}{\max(\|A\|, \|B\|)} \max(\|A\|, \|B\|) \left\| \begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix} \right\|^2 s_j(|X^*|^2 \oplus |Y|^2) \\
&\qquad = \max(\|A\|, \|B\|) s_j(|X^*|^2 \oplus |Y|^2),
\end{aligned}$$

as required. □

In the inequality (11), letting $A = B = I$ and replace the matrices X and Y by the positive semidefinite matrices $X^{1/2}$ and $Y^{1/2}$, respectively, we get

$$s_j(X - Y) \leqslant s_j(X \oplus Y),$$

which is exactly the inequality (2).

COROLLARY 7. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ be such that X and Y are positive semidefinite. Then*

$$s_j(A^*XA - BYB^*) \leqslant (\max(\|A\|,\|B\|))^2 s_j(X \oplus Y)$$

for $j = 1,2,\ldots,n$.

Proof. For $j = 1,2,\ldots,n$, we have

$$\begin{aligned}
s_j(A^*XA - BYB^*) &\leqslant s_j\left(A^*XA \oplus BYB^*\right)\\
&= s_j\left(\begin{bmatrix} A^* & 0\\ 0 & B\end{bmatrix}\begin{bmatrix} X & 0\\ 0 & Y\end{bmatrix}\begin{bmatrix} A & 0\\ 0 & B^*\end{bmatrix}\right)\\
&\leqslant \frac{1}{2(\max(\|A\|,\|B\|))^2}\left\|\begin{bmatrix} AA^* & 0\\ 0 & B^*B\end{bmatrix} + \begin{bmatrix} AA^* & 0\\ 0 & B^*B\end{bmatrix}\right\|\\
&\quad \times (\max(\|A\|,\|B\|))^2 s_j(X \oplus Y)\\
&\qquad \text{(by the inequality (6))}\\
&= (\max(\|A\|,\|B\|))^2 s_j(X \oplus Y). \quad \square
\end{aligned}$$

We end this paper by the following corollary. This corollary deals with the largest singular value of a matrix which is, as we mentioned before, the spectral norm of this matrix.

THEOREM 8. *Let $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\|AX + XB\| \leqslant \sqrt{\|A + XBX^*\|\,\|XAX + B\|}.$$

Proof. Let $K_1 = \begin{bmatrix} A^{1/2} & XB^{1/2}\\ 0 & 0\end{bmatrix}$, $K_2^* = \begin{bmatrix} A^{1/2}X & 0\\ B^{1/2} & 0\end{bmatrix}$. Then

$$\begin{aligned}
\|AX + XB\| &= \|K_1K_2^*\|\\
&\leqslant \|K_1\|\,\|K_2^*\|\\
&= \|K_1\|\,\|K_2\|\\
&= \sqrt{\|K_1\|^2}\sqrt{\|K_2\|^2}\\
&= \sqrt{\|K_1K_1^*\|}\sqrt{\|K_2K_2^*\|}.
\end{aligned} \tag{12}$$

But,

$$K_1K_1^* = \begin{bmatrix} A + XBX^* & 0\\ 0 & 0\end{bmatrix} \tag{13}$$

and

$$K_2K_2^* = \begin{bmatrix} X^*A^{1/2}X + B & 0\\ 0 & 0\end{bmatrix}. \tag{14}$$

So, by the relations (12), (13), and (14), we have

$$\|AX+XB\| \leqslant \sqrt{\|A+XBX^*\| \, \|XAX+B\|},$$

as required. $\square$

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