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ON THE GENERAL CASE OF BRANNAN CONJECTURE

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Abstract. We will prove the Brannan conjecture provided that the parameters α and β verify the conditions $\frac{3}{4} \leqslant \alpha \leqslant \beta \leqslant 1$. The basic tool of the study is a new integral representation deduced in this paper.

1. Introduction

We denote $\mathbb{N}^* = \{1,2,3,\dots\}$ and $U = \{z \in \mathbb{C} : |z| < 1\}$.
For $\alpha > 0$, $\beta > 0$, $x = e^{i\theta}$, $\theta \in [-\pi,\pi]$, and $z \in U$ the development holds

$$\frac{(1+xz)^{\alpha}}{(1-z)^{\beta}} = \sum_{n=0}^{\infty} A_n(\alpha,\beta,x) z^n.$$

The inequality

$$\left|A_n(\alpha,\beta,x)\right| \leqslant A_n(\alpha,\beta,1),\ x \in \mathbb{C},\ |x| = 1,$$

has been proved in [1] for $\alpha > 1$ and $\beta > 1$. In [5] D. A. Brannan stattes the following conjecture.

CONJECTURE 1. If $\alpha,\beta \in (0,1]$, and $x \in \mathbb{C}$, $|x| = 1$, then the inequality

$$|A_{2n+1}(\alpha,\beta,x)| \leqslant A_{2n+1}(\alpha,\beta,1),\ x \in \mathbb{C},\ |x| = 1 \tag{1}$$

holds for every natural number n.

The particular case $\beta = 1$ and $\alpha \in (0,1)$ was intensively studied. The inequality

$$|A_{2n+1}(\alpha,1,x)| \leqslant A_{2n+1}(\alpha,1,1), \quad |x| = 1$$

has been proved in [5] for $n = 1$, in [7] for $n = 2$, in [4] for $n = 3$ and in [6] for $n \leqslant 25$. In case $\beta = 1$ the proof for every natural number n is given in [6] and [4]. The second particular case $\alpha = \beta$, that is

$$|A_{2n+1}(\alpha,\alpha,x)| \leqslant A_{2n+1}(\alpha,\alpha,1),\ \alpha \in (0,1)$$

is proved in [9]. We will prove a partial result of the general case, which suggests that Brannan's conjecture holds in the general case too.

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2. The main result

THEOREM 1. *If* $\frac{3}{4} \leqslant \alpha \leqslant \beta \leqslant 1$ *and* $x \in \mathbb{C}$ *with* $|x| = 1$, *then inequality* (1) *holds, that is*

$$|A_{2n+1}(\alpha,\beta,x)| \leqslant A_{2n+1}(\alpha,\beta,1), \quad (\forall)\ n \in \mathbb{N}. \tag{$*$}$$

3. Preliminaries

In order to prove the main result, we need the following lemmas. A simple calculation gives:

$$(1+xz)^{\alpha} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(0-\alpha)(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{n!} x^n z^n$$

and

$$(1-z)^{-\beta} = 1 + \sum_{n=1}^{\infty} \frac{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}{n!} z^n.$$

Thus it follows that

$$\begin{aligned} A_n(\alpha,\beta,x) = & \frac{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)}{n!} + \frac{\beta(\beta+1)(\beta+2)\dots(\beta+n-2)}{(n-1)!} \cdot \frac{\alpha}{1!} x \\ & - \frac{\beta(\beta+1)(\beta+2)\dots(\beta+n-3)}{(n-2)!} \cdot \frac{\alpha(1-\alpha)}{2!} x^2 + \dots \\ & + (-1)^{n-2} \frac{\beta}{1!} \cdot \frac{\alpha(1-\alpha)(2-\alpha)\cdots(n-2-\alpha)}{(n-1)!} x^{n-1} \\ & + (-1)^{n-1} \frac{\alpha(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{n!} x^n. \end{aligned}$$

This equality can be rewritten as follows:

$$\begin{aligned} A_n(\alpha,\beta,x) = & \frac{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)}{n!} \\ & \times \left(1 + \sum_{k=1}^{n} (-1)^k x^k C_n^k \frac{(0-\alpha)(1-\alpha)\dots(k-1-\alpha)}{(\beta+n-1)(\beta+n-2)\dots(\beta+n-k)}\right). \end{aligned}$$

Let $B_n(\alpha,\beta,x)$ be defined by

$$B_n(\alpha,\beta,x) = 1 + \sum_{k=1}^{n} (-1)^k x^k C_n^k \frac{(0-\alpha)(1-\alpha)(2-\alpha)\dots(k-1-\alpha)}{(n-1+\beta)(n-2+\beta)\dots(n-k+\beta)}. \tag{2}$$

The conjecture (1) is equivalent to

$$\left|B_{2n+1}(\alpha,\beta,x)\right| \leqslant B_{2n+1}(\alpha,\beta,1) \quad \text{where } x \in \mathbb{C},\ |x| = 1 \text{ and } \alpha\beta \in [0,1],\ n \in \mathbb{N}^*. \tag{3}$$

LEMMA 1. *The following equality holds:*

$$(1-\alpha)\sum_{p=0}^{\infty}\frac{(\alpha)_p}{(p+1)!}=1. \tag{4}$$

Proof. We have the development

$$(1-x)^{-\alpha}=\sum_{p=0}^{\infty}\frac{(\alpha)_p}{p!}x^p,\ \ |x|<1.$$

Integrating the equality, we get

$$\frac{1}{1-\alpha}=\int_0^1(1-t)^{-\alpha}dt=\sum_{p=0}^{\infty}\frac{(\alpha)_p}{(p+1)!}, \tag{5}$$

and the proof is done. □

LEMMA 2. *If* $\alpha,\beta\in(0,1),\ k,n\in\mathbb{N},\ n\geqslant k\geqslant 2$ *and*

$$I_{n,k}=\int_0^1 t^{k-1-\alpha}(1-t)^{n-k-1+\beta}dt,$$

then the following equalities hold:

$$I_{n,k}=\frac{k-1-\alpha}{n-k+\beta}\cdot\frac{k-2-\alpha}{n-k+1+\beta}\cdot\frac{k-3-\alpha}{n-k+2+\beta}\cdot\ldots\cdot\frac{1-\alpha}{n-2+\beta}I_{n,1} \tag{6}$$

and

$$B_n(\alpha,\beta,x)=1+\frac{\alpha}{(n-1+\beta)I_{n,1}}\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[(1-t)^n-(1-t-xt)^n\right]dt. \tag{7}$$

Proof. Integrating by parts, we get

$$\begin{aligned}I_{n,k}&=\int_0^1 t^{k-1-\alpha}(1-t)^{n-k-1+\beta}dt=-\int_0^1 t^{k-1-\alpha}\left(\frac{(1-t)^{n-k+\beta}}{n-k+\beta}\right)'dt\\&=-t^{k-1-\alpha}\left(\frac{(1-t)^{n-k+\beta}}{n-k+\beta}\right)\Bigg|_0^1+\frac{k-1-\alpha}{n-k+\beta}\int_0^1 t^{k-2-\alpha}(1-t)^{n-k+\beta}dt\\&=\frac{k-1-\alpha}{n-k+\beta}I_{n,k-1}.\end{aligned} \tag{8}$$

Equality (8) implies (6). Finally we have

$$\begin{aligned}B_n(\alpha,\beta,x)&=1+\frac{x\alpha}{(n-1+\beta)I_{n,1}}\sum_{k=1}^{n}C_n^k(-1)^{k-1}x^{k-1}I_{n,k}\\&=1+\frac{x\alpha}{(n-1+\beta)I_{n,1}}\int_0^1 t^{-\alpha}(1-t)^{-1+\beta}\sum_{k=1}^{n}C_n^k(-xt)^{k-1}(1-t)^{n-k}dt.\end{aligned}$$

Using the equality

$$\sum_{k=1}^{n} C_n^k(-v)^{k-1}u^{n-k} = \frac{u^n-(u-v)^n}{v},$$

we get

$$\begin{aligned}B_n(\alpha,\beta,x) &= 1+\frac{x\alpha}{(n-1+\beta)I_{n,1}}\int_0^1 t^{-\alpha}(1-t)^{-1+\beta}\frac{\left[(1-t)^n-(1-t-xt)^n\right]}{xt}dt\\ &= 1+\frac{\alpha}{(n-1+\beta)I_{n,1}}\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[(1-t)^n-(1-t-xt)^n\right]dt,\end{aligned}$$

and the proof is done. □

LEMMA 3. *If* $\alpha,\beta\in(0,1)$, $\alpha\leqslant\beta$ *and* $n\in\mathbb{N}$, *then the following inequality holds:*

$$1-\frac{\alpha}{(n-1+\beta)I_{n,1}}\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}[1-(1-t)^n]dt>0. \tag{9}$$

Proof. Inequality (9) is equivalent to

$$(n-1+\beta)\int_0^1 t^{-\alpha}(1-t)^{n-2+\beta}dt>\alpha\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}[1-(1-t)^n]dt.$$

We introduce the notations $m_1=\int_0^1 t^{-\alpha}(1-t)^{n-2+\beta}dt$ and $m_2=\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}[1-(1-t)^n]dt$.

A change of variable leads to $m_1=\int_0^1 t^{-\alpha}(1-t)^{n-2+\beta}dt=\int_0^1(1-t)^{-\alpha}t^{n-2+\beta}dt$, and $m_2=\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}[1-(1-t)^n]dt=\int_0^1(1-t)^{-1-\alpha}t^{-1+\beta}[1-t^n]dt$.

The developments

$$(1-t)^{-\alpha}=\sum_{p=0}^{\infty}\frac{(\alpha)_p}{p!}t^p \text{ and } (1-t)^{-1-\alpha}=\sum_{p=0}^{\infty}\frac{(1+\alpha)_p}{p!}t^p$$

hold for every $t\in(-1,1)$, and we infer

$$\int_0^s(1-t)^{-\alpha}t^{n-2+\beta}dt=\sum_{p=0}^{\infty}\frac{(\alpha)_p}{p!}\int_0^s t^{n+p-2+\beta}dt=\sum_{p=0}^{\infty}\frac{(\alpha)_p}{p!(n+p-1+\beta)}s^p$$

and

$$\begin{aligned}&\int_0^s(1-t)^{-1-\alpha}t^{-1+\beta}[1-t^n]dt\\ &=\sum_{p=0}^{\infty}\frac{(1+\alpha)_p}{p!}\int_0^s t^{p-1+\beta}dt-\sum_{p=0}^{\infty}\frac{(1+\alpha)_p}{p!}\int_0^s t^{p+n-1+\beta}dt\\ &=\sum_{p=0}^{\infty}\frac{(1+\alpha)_p}{p!(p+\beta)}s^p-\sum_{p=0}^{\infty}\frac{(1+\alpha)_p}{p!(p+n+\beta)}s^p\\ &=\sum_{p=0}^{\infty}\frac{n(1+\alpha)_p}{p!(p+\beta)(p+n+\beta)}s^p,\end{aligned}$$

for every $s \in (-1,1)$.

For $s \nearrow 1$, we get

$$m_1 = \int_0^1 (1-t)^{-\alpha} t^{n-2+\beta} dt = \sum_{p=0}^{\infty} \frac{(\alpha)_p}{p!(n+p-1+\beta)} \tag{10}$$

and

$$m_2 = \int_0^1 (1-t)^{-1-\alpha} t^{-1+\beta} [1-t^n] dt = \sum_{p=0}^{\infty} \frac{n(1+\alpha)_p}{p!(p+\beta)(p+n+\beta)}. \tag{11}$$

In order to prove inequality (4) we have to prove

$$(n-1+\beta)m_1 > \alpha m_2.$$

According to (10) and (11), this inequality is equivalent to

$$1 + \sum_{p=1}^{\infty} \frac{(\alpha)_p (n-1+\beta)}{p!(n+p-1+\beta)} - \alpha \sum_{p=0}^{\infty} \frac{n(1+\alpha)_p}{p!(p+\beta)(p+n+\beta)} > 0. \tag{12}$$

We have the following equivalences:

$$\begin{aligned}
1 + \sum_{p=1}^{\infty} \frac{(\alpha)_p (n-1+\beta)}{p!(n+p-1+\beta)} - \alpha \sum_{p=0}^{\infty} \frac{n(1+\alpha)_p}{p!(p+\beta)(p+n+\beta)} > 0 \Leftrightarrow \\
1 > \alpha \sum_{p=0}^{\infty} \frac{n(1+\alpha)_p}{p!(p+\beta)(p+n+\beta)} - \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}(n-1+\beta)}{(p+1)!(n+p+\beta)} \Leftrightarrow \\
1 > \sum_{p=0}^{\infty} \left[\frac{n(\alpha)_{p+1}}{p!(p+\beta)(p+n+\beta)} - \frac{(\alpha)_{p+1}(n-1+\beta)}{(p+1)!(n+p+\beta)} \right] \Leftrightarrow \\
1 > \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}(1-\beta)(n+p+\beta)}{(p+1)!(p+\beta)(p+n+\beta)} \Leftrightarrow \\
1 > (1-\beta) \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}}{p!(p+\beta)(p+1)}
\end{aligned} \tag{13}$$

The last inequality holds because the condition $0 < \alpha \leqslant \beta \leqslant 1$ implies

$$(1-\alpha) \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}}{p!(p+\alpha)(p+1)} \geqslant (1-\beta) \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}}{p!(p+\beta)(p+1)},$$

and taking into account (4), we have

$$(1-\alpha) \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}}{p!(p+\alpha)(p+1)} = (1-\alpha) \sum_{p=0}^{\infty} \frac{(\alpha)_p}{p!(p+1)} = 1.$$

Consequently, inequality (12) holds and this implies (9). $\square$

The next lemma concerns the function Υ_n, which is defined by:

$$\Upsilon_n : (0,\infty) \to \mathbb{R}, \text{ by the equality} \Upsilon_n(v) = v^{\alpha}\frac{1-u^{2n+1}}{1-u} + \frac{1}{v^{\beta}}\frac{1+u^{2n+1}}{1-u}. \tag{14}$$

LEMMA 4. *Let* $x \in \mathbb{C}$ *be a complex number with* $|x| = 1$. *If* $0 \leqslant \alpha \leqslant \beta \leqslant 1$, *then the condition*

$$\int_0^1 \left(\frac{2}{1+u}\right)^{1+\alpha-\beta} \Upsilon_n\left(\frac{|x+u|}{1-u}\right) du \leqslant \int_0^1 \left(\frac{2}{1+u}\right)^{1+\alpha-\beta} \Upsilon_n\left(\frac{1+u}{1-u}\right) du, \tag{15}$$

implies the inequality:

$$\left|B_{2n+1}(\alpha,\beta,x)\right| \leqslant B_{2n+1}(\alpha,\beta,1), \tag{16}$$

where $B_{2n+1}(\alpha,\beta,x)$ *is defined by (2).*

Proof. We use equality (7) and we get:

$$\begin{aligned} &B_{2n+1}(\alpha,\beta,x) \\ &= 1 + \frac{\alpha}{(n-1+\beta)I_{2n+1,1}} \int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[(1-t)^{2n+1} - (1-t-xt)^{2n+1}\right] dt. \end{aligned}$$

This equality is equivalent to:

$$\begin{aligned} &B_{2n+1}(\alpha,\beta,x) \\ &= 1 - \frac{\alpha}{(2n+\beta)I_{2n+1,1}} \int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t)^{2n+1}\right] dt \\ &\quad + \frac{\alpha}{(2n+\beta)I_{2n+1,1}} \int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t-xt)^{2n+1}\right] dt. \end{aligned} \tag{17}$$

Since according to Lemma 3 we have

$$1 - \frac{\alpha}{(2n+\beta)I_{2n+1,1}} \int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t)^{2n+1}\right] dt \geqslant 0,$$

it follows that in order to prove inequality (16), it is enough to show that the following inequality holds:

$$|J_{2n+1}(\alpha,\beta,x)| \leqslant J_{2n+1}(\alpha,\beta,1), \text{ provided that } x \in \mathbb{C}, |x| = 1, \tag{18}$$

where

$$J_{2n+1}(\alpha,\beta,x) = \int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t(1+x))^{2n+1}\right] dt. \tag{19}$$

The change of variable $u = 1-(1+x)t$ leads to

$$\begin{aligned}
J_{2n+1}(\alpha,\beta,x) &= \int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t(1+x))^{2n+1}\right]dt \\
&= \frac{1}{1+x}\int_{-x}^1 \left(\frac{1-u}{1+x}\right)^{-1-\alpha}\left(\frac{x+u}{1+x}\right)^{-1+\beta}(1-u^{2n+1})du \\
&= \frac{1}{1+x}\int_0^1 \left(\frac{1-u}{1+x}\right)^{-1-\alpha}\left(\frac{x+u}{1+x}\right)^{-1+\beta}(1-u^{2n+1})du \\
&\quad -\frac{1}{1+x}\int_0^{-x} \left(\frac{1-u}{1+x}\right)^{-1-\alpha}\left(\frac{x+u}{1+x}\right)^{-1+\beta}(1-u^{2n+1})du \\
&= \frac{1}{1+x}\int_0^1 \left(\frac{1-u}{1+x}\right)^{-1-\alpha}\left(\frac{x+u}{1+x}\right)^{-1+\beta}(1-u^{2n+1})du \\
&\quad +\frac{x}{1+x}\int_0^1 \left(\frac{1+xu}{1+x}\right)^{-1-\alpha}\left(\frac{x-xu}{1+x}\right)^{-1+\beta}(1+x^{2n+1}u^{2n+1})du.
\end{aligned} \tag{20}$$

From (19), (20) and $|1+xu| = |x+u|$ we infer that

$$\begin{aligned}
|J_{2n+1}(\alpha,\beta,x)| \leqslant{}& \int_0^1 \left|\frac{1+x}{1-u}\right|^{\alpha}\left|\frac{1+x}{x+u}\right|^{1-\beta}\frac{1-u^{2n+1}}{1-u}du \\
&+\int_0^1 \frac{|1+x|^{\alpha}}{|x+u|^{1+\alpha}}\left|\frac{1+x}{1-u}\right|^{1-\beta}(1+u^{2n+1})du.
\end{aligned}$$

This inequality can be rewritten as follows:

$$\begin{aligned}
|J_{2n+1}(\alpha,\beta,x)| &\leqslant \int_0^1 \left|\frac{1+x}{x+u}\right|^{1+\alpha-\beta}\left(\left|\frac{x+u}{1-u}\right|^{\alpha}\frac{1-u^{2n+1}}{1-u}+\left|\frac{1-u}{x+u}\right|^{\beta}\frac{1+u^{2n+1}}{1-u}\right)du \\
&= \int_0^1 \left|\frac{1+x}{x+u}\right|^{1+\alpha-\beta}\Upsilon_n\left(\left|\frac{x+u}{1-u}\right|\right)du.
\end{aligned} \tag{21}$$

A simple calculation shows that $\left|\frac{1+x}{x+u}\right| \leqslant \frac{2}{1+u}$, for every $x \in \mathbb{C}$, with $|x| = 1$, and $u \in (0,1)$.

This result together with the inequality (21) lead to

$$|J_{2n+1}(\alpha,\beta,x)| \leqslant \int_0^1 \left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\Upsilon_n\left(\left|\frac{x+u}{1-u}\right|\right)du,$$

and

$$J_{2n+1}(\alpha,\beta,1) = \int_0^1 \left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\Upsilon_n\left(\frac{1+u}{1-u}\right)du \ \text{ for } |x| = 1,\ x \in \mathbb{C}. \tag{22}$$

and (15) imply (18). From (17) we get

$$|B_{2n+1}(\alpha,\beta,x)| \leqslant 1-\frac{\alpha}{(2n+\beta)I_{2n+1,1}}\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t)^{2n+1}\right]dt \\ +\frac{\alpha}{(2n+\beta)I_{2n+1,1}}|J_{2n+1}(\alpha,\beta,x)|.$$

and taking into the account (18) it follows

$$|B_{2n+1}(\alpha,\beta,x)| \leqslant 1-\frac{\alpha}{(2n+\beta)I_{2n+1,1}}\int_0^1 t^{-1-\alpha}(1-t)^{-1+\beta}\left[1-(1-t)^{2n+1}\right]dt \\ +\frac{\alpha}{(2n+\beta)I_{2n+1,1}}J_{2n+1}(\alpha,\beta,1)=B_{2n+1}(\alpha,\beta,1).$$

and consequently inequality (16) holds. □

We define the sequence $(D_n(\alpha,\beta,x))_{n\geqslant 1}$ by the equality

$$\begin{aligned} D_n(\alpha,\beta,x) &= \int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\Upsilon_n\left(\frac{1+u}{1-u}\right)du \\ &\quad -\int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\Upsilon_n\left(\frac{|u+x|}{1-u}\right)du \\ &= \int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\left\{\left[\left(\frac{1+u}{1-u}\right)^{\alpha}-\left(\frac{|u+x|}{1-u}\right)^{\alpha}\right]\frac{1-u^{2n+1}}{1-u}\right. \\ &\quad \left.+\left[\left(\frac{1-u}{1+u}\right)^{\beta}-\left(\frac{1-u}{|u+x|}\right)^{\beta}\right]\frac{1+u^{2n+1}}{1+u}\right\}du. \end{aligned}$$

LEMMA 5. *If* $0<\alpha\leqslant\beta\leqslant 1$ *and* $|x|=1$ *then we have*

$$D_{n+1}(\alpha,\beta,x)\geqslant D_n(\alpha,\beta,x). \tag{23}$$

Proof.

$$\begin{aligned} &D_{n+1}(\alpha,\beta,x)-D_n(\alpha,\beta,x) \\ &= \int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\left\{\left[\left(\frac{1+u}{1-u}\right)^{\alpha}-\left(\frac{|u+x|}{1-u}\right)^{\alpha}\right]u^{2n+1}(1+u)\right. \\ &\quad \left.+\left[\left(\frac{1-u}{|u+x|}\right)^{\beta}-\left(\frac{1-u}{1+u}\right)^{\beta}\right]u^{2n+1}(1-u)\right\}du. \end{aligned} \tag{24}$$

Since $\left(\frac{1+u}{1-u}\right)^{\alpha}\geqslant\left(\frac{|u+x|}{1-u}\right)^{\alpha}$ and $\left(\frac{1-u}{|u+x|}\right)^{\beta}\geqslant\left(\frac{1-u}{1+u}\right)^{\beta}$, for every $u\in[0,1]$, and $|x|=1$, equality (24) implies (23). □

LEMMA 6. *Let* $\kappa,\lambda : [a,b] \to \mathbb{R}$ *be two continuous functions. If* κ *is increasing,* $\kappa(a) \geqslant 0$ *and there is a point* $x_0 \in (a,b)$ *such that* $\lambda(x) \geqslant 0,\ x \in [x_0,b]$ *and* $\lambda(x) \leqslant 0,\ x \in [a,x_0]$ *then*

$$\int_a^b \kappa(x)\lambda(x)dx \geqslant \kappa(x_0)\int_a^b \lambda(x)dx.$$

Proof. We have

$$\begin{aligned}&\int_a^b \kappa(x)\lambda(x)dx - \kappa(x_0)\int_a^b \lambda(x)dx\\ =&\int_a^{x_0}\big[\kappa(x)-\kappa(x_0)\big]\lambda(x)dx + \int_{x_0}^b \big[(\kappa(x)-\kappa(x_0)\big]\lambda(x)dx \geqslant 0.\end{aligned}$$

The inequality holds because $\big[\kappa(x)-\kappa(x_0)\big]\lambda(x)\geqslant 0, x\in[a,x_0]$ and $\big[(\kappa(x)-\kappa(x_0)\big]\lambda(x) \geqslant 0,\ x\in[x_0,b]$. □

LEMMA 7. *The equation* $\lambda(u) = \left(\frac{8}{3}\right)^{\frac{1}{3}} - 1 - \left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}} + \frac{2^{\frac{2}{3}}}{3}\frac{1-u}{1+u}$ *has a unique root* $u_0 \in (0,1)$ *such that* $\lambda(u) \leqslant 0,\ u\in[0,u_0]$ *and* $\lambda(u)\geqslant 0,\ u \in [u_0,1]$.

Proof. We have

$$\lambda'(u) = \frac{1}{4}\left(\frac{1+u^2-\frac{1}{2}u}{1+u^2-2u}\right)^{\frac{5}{6}}\frac{1-u^2}{(1+u^2-\frac{1}{2}u)^2} - \frac{2^{\frac{5}{3}}}{3}\left(\frac{1}{1+u}\right)^2. \tag{25}$$

We will prove that the equation $\lambda'(u)=0$ has a unique root $u^* \in (0,1)$.

The equation $\lambda'(u)=0$ is equivalent to:

$$\frac{1}{(1-u)^{\frac{2}{3}}}(1+u)^{\frac{2}{3}}\left(\frac{1+u^2+2u}{1+u^2-\frac{1}{2}u}\right)^{\frac{7}{6}} = \frac{2^{\frac{11}{3}}}{3} \tag{26}$$

The mapping $\omega : (0,1)\to\mathbb{R},\ \ \omega(u) = \frac{1}{(1-u)^{\frac{2}{3}}}(1+u)^{\frac{2}{3}}\left(\frac{1+u^2+2u}{1+u^2-\frac{1}{2}u}\right)^{\frac{7}{6}}$ is strictly increasing since it is a product of positive, strictly increasing functions. Thus taking into the account that $\lim_{u\nearrow 1}\omega(u) = +\infty,\ \ \omega(0) < \frac{2^{\frac{11}{3}}}{3}$ the equation (26) has a unique root $u^*\in(0,1)$. Consequently the equation $\lambda'(u)=0$ has the unique root $u^*\in(0,1)$ too.

The equalities $\lim_{u\nearrow 1}\lambda'(u) = +\infty$ and $\lambda'(0) = \frac{1}{4} - \frac{2^{\frac{5}{3}}}{3} < 0$ imply that $\lambda'(u) < 0,\ u\in(0,u^*),\ \ \lambda'(u)>0,\ u\in(u^*,1)$. The inequalities $\lambda(0)<0$ and $\lambda(1)>0$ and the monotony of λ imply that the equation $\lambda(u)=0$ has a unique root $u_0\in(u^*,1)$. □

LEMMA 8. *If* $\frac{2}{3}\leqslant\alpha\leqslant\beta\leqslant 1,\ \ |x|=1$ *and* $\arg(x)\in\big[\arccos(-\frac{1}{4}),\pi\big]$, *then the following inequality holds:*

$$J_{2n+1}(\alpha,\beta,1) \geqslant |J_{2n+1}(\alpha,\beta,x)|,\ n\in\mathbb{N}^*. \tag{27}$$

Proof. The equality (20) implies

$$\begin{aligned} &J_{2n+1}(\alpha,\beta,1)\\ &=2^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left(\frac{1}{1+u}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du\\ &\quad+2^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1+u}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1-u+u^2-u^3+\ldots+u^{2n})du, \end{aligned} \tag{28}$$

and since $|1+xu|=|x+u|,\ |1-xu+x^2u^2-x^3u^3+\ldots+x^{2n}u^{2n}|\leqslant 1+u+u^2+\ldots+u^{2n}$, we get

$$\begin{aligned} &|J_{2n+1}(\alpha,\beta,x)|\\ &\leqslant|1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left(\frac{1}{|x+u|}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du\\ &\quad+|1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{|x+u|}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du. \end{aligned} \tag{29}$$

Now (28) and (29) imply

$$\begin{aligned} &J_{2n+1}(\alpha,\beta,1)-|J_{2n+1}(\alpha,\beta,x)|\\ &\geqslant 2^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left(\frac{1}{1+u}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du\\ &\quad+2^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1+u}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1-u+u^2-u^3+\ldots+u^{2n})du\\ &\quad-|1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left(\frac{1}{|x+u|}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du\\ &\quad-|1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{|x+u|}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du. \end{aligned} \tag{30}$$

We use the notations:

$$\begin{aligned} M&=2^{\alpha}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left|\frac{1+x}{x+u}\right|^{1-\beta}(1+u+u^2+\ldots+u^{2n})du\\ &\quad-|1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left(\frac{1}{|x+u|}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du\\ &\quad-|1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{|x+u|}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1+u+u^2+\ldots+u^{2n})du,\\ m&=2^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1+u}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1-u+u^2-u^3+\ldots+u^{2n})du. \end{aligned} \tag{31}$$

Taking into the account (30), the inequality $\frac{2}{1+u}\geqslant\left|\frac{1+x}{u+x}\right|$ implies

$$J_{2n+1}(\alpha,\beta,1)-|J_{2n+1}(\alpha,\beta,x)|\geqslant M+m.$$

Thus, in order to prove (27), we have to show that $M+m \geqslant 0$.

Since $m \geqslant 0$, it follows that we have to prove the inequality $M \geqslant 0$ in order to finish the proof.

We have

$$\begin{aligned} M &= \int_0^1 \Big(\frac{1}{1-u}\Big)^{\alpha} \left|\frac{1+x}{x+u}\right|^{1-\beta} \left[2^{\alpha} - |1+x|^{\alpha} - |1+x|^{\alpha}\left|\frac{1-u}{x+u}\right|^{\alpha+\beta-1}\right] \\ &\quad \times (1+u+u^2+\ldots+u^{2n})du \\ &= |1+x|^{\alpha} \int_0^1 \Big(\frac{1}{1-u}\Big)^{\alpha} \left|\frac{1+x}{x+u}\right|^{1-\beta} \left[\Big(\frac{2}{|1+x|}\Big)^{\alpha} - 1 - \left|\frac{1-u}{x+u}\right|^{\alpha+\beta-1}\right] \\ &\quad \times (1+u+u^2+\ldots+u^{2n})du. \end{aligned} \tag{32}$$

Since $x=-t+i\sqrt{1-t^2}$, $t \in [\frac{1}{4},1)$ it follows

$$\begin{aligned} \Big(\frac{2}{|1+x|}\Big)^{\alpha} - 1 - \left|\frac{1-u}{x+u}\right|^{\alpha+\beta-1} &\geqslant +\Big(\frac{2}{|1+x|}\Big)^{\frac{2}{3}} - 1 - \left|\frac{1-u}{x+u}\right|^{\frac{1}{3}} \\ &= \Big(\frac{2}{1-t}\Big)^{\frac{1}{3}} - 1 - \left(\frac{1-u}{\sqrt{1+u^2-2ut}}\right)^{\frac{1}{3}} \\ &= \phi(t), \ \ t \in \Big[\frac{1}{4},1\Big). \end{aligned}$$

A simple computation gives that $\phi'(t) > 0$, $t \in [\frac{1}{4},1)$, $u \in [0,1]$ is equivalent to

$$2(1+u^2-2ut)^{\frac{7}{2}} > u^3(1-u)(1-t)^4, \ t \in \Big[\frac{1}{4},1\Big), \ u \in [0,1]. \tag{33}$$

On the other hand the inequalities hold

$$2(1+u^2-2ut)^{\frac{7}{2}} \geqslant 2(2u-2ut)^{\frac{7}{2}} = 2^{\frac{9}{2}} u^{\frac{7}{2}} (1-t)^{\frac{7}{2}} \geqslant u^3(1-u)(1-t)^4,$$

provided that $t \in [\frac{1}{4},1)$, $u \in [0,1]$ and $u \geqslant \frac{1}{2^9}$. If $\frac{1}{2^9} \geqslant u$, then

$$2(1+u^2-2ut)^{\frac{7}{2}} \geqslant 1 \geqslant u^3(1-u)(1-t)^4, \ \ t \in \Big[\frac{1}{4},1\Big), \ u \in [0,1].$$

Thus the inequality holds $\phi'(t) > 0$, $t \in [\frac{1}{4},1)$ and consequently $\phi(t) \geqslant \phi(\frac{1}{4}) = \left(\frac{8}{3}\right)^{\frac{1}{3}} - 1 - \left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}$, $t \in [\frac{1}{4},1)$.

Finally we infer the inequalities

$$\Big(\frac{2}{|1+x|}\Big)^{\alpha} - 1 - \left|\frac{1-u}{x+u}\right|^{\alpha+\beta-1} \geqslant \Big(\frac{8}{3}\Big)^{\frac{1}{3}} - 1 - \left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}$$

and

$$M \geqslant |1+x|^{\alpha+1-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left|\frac{1}{x+u}\right|^{1-\beta}\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}\right]$$
$$\times(1+u+u^2+\ldots+u^{2n})du \text{ provided that } \arg(x)\in\left[\arccos\left(-\frac{1}{4}\right),\pi\right]. \quad (34)$$

We have

$$\left(\frac{1}{1-u}\right)^{\alpha}\left|\frac{1}{x+u}\right|^{1-\beta}\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}\right](1+u+u^2+\ldots+u^{2n})$$
$$= f(u)g(u), \quad (35)$$

where

$$f(u)=\frac{1+u+\ldots+u^{2n}}{1+u+\ldots+u^6}\left(\frac{1+u}{|x+u|}\right)^{1-\beta}(1+u)^{\beta-\frac{2}{3}}\left(\frac{1}{1-u}\right)^{\alpha-\frac{2}{3}},$$

$$g(u)=\frac{1+u+u^2+\ldots+u^6}{(1+u)^{\frac{1}{3}}(1-u)^{\frac{2}{3}}}\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}\right].$$

Since the function $\tilde{f}:[0,1]\to\mathbb{R}$ defined by $\tilde{f}(t)=\frac{1+u+\ldots+u^{2n}}{1+u+\ldots+u^6}$ is increasing provided that $n\geqslant 3$, it follows that the mapping $f:[0,1]\to(0,+\infty)$ is increasing because, it is a product of four positive, increasing functions, and the equation $g(u)=0$ has an unique root $x_0\in(0,1)$, $g(u)\geqslant 0$, $u\in(x_0,1)$ and $g(u)\leqslant 0$, $u\in(0,x_0)$ because the mapping $g_1:[0,1]\to\mathbb{R}$, $g_1(u)=\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}$ is strictly increasing. Consequently the conditions of the Lemma 6 are fulfilled and we get

$$M\geqslant|1+x|^{1+\alpha-\beta}\int_0^1 f(u)g(u)du\geqslant|1+x|^{1+\alpha-\beta}f(x_0)\int_0^1 g(u)du.$$

Since $H_1=\int_0^1 g(u)du=0,294\ldots$, it follows that $M\geqslant 0$ and the proof is done for $n\geqslant 3$.

In order to prove the inequality (27) in case $n\in\{1,2\}$, we will prove $M+m\geqslant 0$. Since $3(u^2-u+1)\geqslant u^2+u+1$ in case $n=1$ we have

$$m\geqslant\frac{2^{1+\alpha-\beta}}{3}\int_0^1\left(\frac{1}{1+u}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1+u+u^2)du$$

and it follows that

$$M+m\geqslant|1+x|^{\alpha}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha}\left|\frac{1+x}{x+u}\right|^{1-\beta}\left[\left(\frac{2}{|1+x|}\right)^{\alpha}-1-\left|\frac{1-u}{x+u}\right|^{\alpha+\beta-1}\right.$$
$$\left.+\frac{2^{1+\alpha-\beta}}{3}\left|\frac{x+u}{1+x}\right|^{1-\beta}\left(\frac{1}{1+u}\right)^{\alpha}(1-u)^{\alpha+\beta-1}\right](1+u+u^2)du \quad (36)$$

The conditions $|x|=1$, $\mathrm{Re}(x)\leqslant -\frac{1}{4}$, $\frac{2}{3}\leqslant \alpha\leqslant\beta\leqslant 1$ imply $\left|\frac{x+u}{1+x}\right|\geqslant\sqrt{\frac{1+u^2-\frac{1}{2}u}{\frac{3}{2}}}$

$$\begin{aligned}&\frac{2^{1+\alpha-\beta}}{3}\left|\frac{x+u}{1+x}\right|^{1-\beta}\left(\frac{1}{1+u}\right)^{\alpha}(1-u)^{\alpha+\beta-1}\\ &\geqslant\frac{2^{\alpha}}{3}\left|\frac{8+8u^2-4u}{3(1-u)^2}\right|^{\frac{1-\beta}{2}}\left(\frac{1-u}{1+u}\right)^{\alpha}\geqslant\frac{2^{\frac{2}{3}}}{3}\left(\frac{1-u}{1+u}\right)^{\frac{2}{3}}\geqslant\frac{2^{\frac{2}{3}}}{3}\frac{1-u}{1+u}.\end{aligned}\tag{37}$$

Finally from (36) and (37) we get

$$\begin{aligned}M+m\geqslant|1+x|^{\alpha}\int_0^1&\left(\frac{1}{1-u}\right)^{\alpha}\left|\frac{1+x}{x+u}\right|^{1-\beta}\\ &\times\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}+\frac{2^{\frac{2}{3}}}{3}\frac{1-u}{1+u}\right](1+u+u^2)du.\end{aligned}$$

This inequality can be rewritten in the following form

$$\begin{aligned}M+m\geqslant|1+x|^{1+\alpha-\beta}\int_0^1&\left(\frac{1}{1-u}\right)^{\alpha-\frac{2}{3}}(1+u)^{\beta-\frac{2}{3}}\left|\frac{1+u}{x+u}\right|^{1-\beta}\frac{1+u+u^2}{(1+u)^{\frac{1}{3}}(1-u)^{\frac{2}{3}}}\\ &\times\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}+\frac{2^{\frac{2}{3}}}{3}\frac{1-u}{1+u}\right]du.\end{aligned}\tag{38}$$

We use again Lemma 6 choosing $f(u)=\left(\frac{1}{1-u}\right)^{\alpha-\frac{2}{3}}(1+u)^{\beta-\frac{2}{3}}\left|\frac{1+u}{x+u}\right|^{1-\beta}$ and $g(u)=\frac{1+u+u^2}{(1+u)^{\frac{1}{3}}(1-u)^{\frac{2}{3}}}\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}+\frac{2^{\frac{2}{3}}}{3}\frac{1-u}{1+u}\right]$. The function $f:[0,1]\to(0,\infty)$ is increasing and the mapping $h:[0,1]\to\mathbb{R}$, $h(u)=\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}+\frac{2^{\frac{2}{3}}}{3}\frac{1-u}{1+u}$ has a unique root $u_0\in(0,1)$ with $g(u)\leqslant 0$, $u\in[0,u_0]$ and $g(u)\geqslant 0$, $u\in[u_0,1]$. Inequality (38) can be rewritten as follows

$$M+m\geqslant|1+x|^{1+\alpha-\beta}\int_0^1 f(u)g(u)du>0,$$

and Lemma 6 implies

$$M+m\geqslant|1+x|^{1+\alpha-\beta}f(u_0)\int_0^1 g(u)du>0,$$

because $H_2=\int_0^1 g(u)du=0,161\ldots$, and the proof is done in case $n=1$ too.

In order to finish the proof, we have to prove the inequality in case $n=2$. A simple calculation shows that

$$5(u^4-u^3+u^2+u+1)\geqslant u^4+u^3+u^2+u+1,\ \ u\in[0,1],$$

thus we get $m \geqslant \frac{2^{1+\alpha-\beta}}{5}\int_0^1\left(\frac{1}{1+u}\right)^{\alpha}\left(\frac{1}{1-u}\right)^{1-\beta}(1+u+u^2+u^3+u^4)du.$

A completely analogous calculations to the case $n=1,$ lead to

$$M+m \geqslant |1+x|^{1+\alpha-\beta}\int_0^1\left(\frac{1}{1-u}\right)^{\alpha-\frac{2}{3}}(1+u)^{\beta-\frac{2}{3}}\left|\frac{1+u}{x+u}\right|^{1-\beta} \times\frac{1+u+u^2+u^3+u^4}{(1+u)^{\frac{1}{3}}(1-u)^{\frac{2}{3}}}\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}+\frac{2^{\frac{2}{3}}}{5}\frac{1-u}{1+u}\right]du.$$

Now, we use again Lemma 6, putting $f(u)=\left(\frac{1}{1-u}\right)^{\alpha-\frac{2}{3}}(1+u)^{\beta-\frac{2}{3}}\left|\frac{1+u}{x+u}\right|^{1-\beta}$ and $g(u)=\frac{1+u+u^2+u^3+u^4}{(1+u)^{\frac{1}{3}}(1-u)^{\frac{2}{3}}}\left[\left(\frac{8}{3}\right)^{\frac{1}{3}}-1-\left(\frac{1-u}{\sqrt{1+u^2-\frac{1}{2}u}}\right)^{\frac{1}{3}}+\frac{2^{\frac{2}{3}}}{5}\frac{1-u}{1+u}\right]$. We get

$$M+m \geqslant |1+x|^{1+\alpha-\beta}f(u_0)\int_0^1 g(u)du>0,$$

because $H_3=\int_0^1 g(u)du=0,276\ldots$ and the proof is done. □

LEMMA 9. *The equation*

$$\frac{3}{4}-(1+u)\frac{(1-u)^{\frac{1}{2}}}{(1+u^2-\frac{1}{2}u)^{\frac{3}{4}}}=0, \quad u\in[0,1] \tag{39}$$

has a unique root $u_0\in(0,1)$, $\frac{3}{4}-(1+u)\frac{(1-u)^{\frac{1}{2}}}{(1+u^2-\frac{1}{2}u)^{\frac{3}{4}}}<0,\ u\in(0,u_0)$ *and* $\frac{4}{5}-(1+u)\frac{(1-u)^{\frac{1}{2}}}{(1+u^2-\frac{1}{2}u)^{\frac{3}{4}}}>0,\ u\in(u_0,1).$

Proof. The equation (39) is equivalent to $v(u)=w(u),\ u\in[0,1],$ where

$$v(u)=\frac{3}{4}(1+u^2-\frac{1}{2}u)^{\frac{3}{4}}, \quad w(u)=(1+u)(1-u)^{\frac{1}{2}}.$$

We have $v(0)=\frac{3}{4},\ w(0)=1,\ v(1)=\frac{3}{4}(\frac{3}{2})^{\frac{3}{4}},\ w(1)=0$. Since v is strictly decreasing on $(0,\frac{1}{4})$ and strictly increasing on $(\frac{1}{4},1)$, w is strictly increasing on $(0,\frac{1}{3})$ and strictly decreasing on $(\frac{1}{3},1)$, and $v(\frac{1}{3})=\frac{3}{4}\left(\frac{17}{18}\right)^{\frac{3}{4}}$, $v(\frac{1}{4})=\frac{3}{4}\left(\frac{15}{16}\right)^{\frac{3}{4}}$, $w(\frac{1}{3})=\frac{4}{3}\left(\frac{2}{3}\right)^{\frac{1}{2}}$, $w(\frac{1}{4})=\frac{5}{4}\left(\frac{3}{4}\right)^{\frac{1}{2}}$, it follows that the equation $v(u)=w(u)$ has a unique root $u_0\in(\frac{1}{3},1)$ and $v(u)<w(u),\ u\in(0,u_0)$ and $v(u)>w(u),\ u\in(u_0,1).$ □

4. The proof of the main result

Proof. According to Lemma 5, it is enough to prove the desired inequality in the case $n=0$. We will prove the inequality (15) provided that $n=0$, $\frac{3}{4}\leqslant\alpha\leqslant\beta\leqslant 1$, $x\in\mathbb{C}$, $x=e^{i\theta}$, $\theta\in(-\pi,\pi]$.

We have to determine the maximum value of the integral

$$\begin{aligned} I(x) &= \int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\Upsilon_0\left(\left|\frac{x+u}{1-u}\right|\right)du \\ &= \int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\left[\Big(\frac{\sqrt{1+u^2+2u\cos\theta}}{1-u}\Big)^{\alpha}+\Big(\frac{1-u}{\sqrt{1+u^2+2u\cos\theta}}\Big)^{\beta}\frac{1+u}{1-u}\right]du. \end{aligned} \tag{40}$$

We will prove the inequality

$$D_0(\alpha,\beta,x)=I(1)-I(x)\geqslant 0,\ |x|=1 \text{ and } \arg(x)\in\left[0,\arccos\left(-\frac{1}{4}\right)\right].$$

The integral $\int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\left[\Big(\frac{\sqrt{1+u^2+2u\cos\theta}}{1-u}\Big)^{\alpha}+\Big(\frac{1-u}{\sqrt{1+u^2+2u\cos\theta}}\Big)^{\beta}\frac{1+u}{1-u}\right]du$ is an even function with respect to θ, consequently it is enough to determine the maximum value provided that $\theta\in[0,\pi]$.

The notation $t=\cos\theta$ suggest to define the mapping $g:[-1,1]\to\mathbb{R}$ by the equality

$$g(t)=\int_0^1\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\left[\Big(\frac{\sqrt{1+u^2+2ut}}{1-u}\Big)^{\alpha}+\Big(\frac{1-u}{\sqrt{1+u^2+2ut}}\Big)^{\beta}\frac{1+u}{1-u}\right]du.$$

We have

$$\max_{|x|=1} I(x)=\max_{t\in[-1,1]} g(t)$$

and

$$\begin{aligned} g'(t) = \beta\int_0^1 &\left[\Big(\frac{\alpha}{\beta}\Big)\Big(\frac{2}{1+u}\Big)^{1+\alpha-\beta}\frac{1}{(1-u)^{\alpha}}(1+u^2+2ut)^{\frac{\alpha}{2}-1}u\right. \\ &\left.-\left(\frac{2}{1+u}\right)^{1+\alpha-\beta}\frac{1}{(1-u)^{1-\beta}}(1+u)(1+u^2+2ut)^{-\frac{\beta}{2}-1}u\right]du. \end{aligned} \tag{41}$$

We will prove $g'(t)>0$, provided that $t\in[-\frac{1}{4},1]$ and $\frac{3}{4}\leqslant\alpha\leqslant\beta\leqslant 1$.

The equality (41) can be rewritten as follows:

$$\begin{aligned} g'(t) = \beta 2^{1+\alpha-\beta}\int_0^1 &(1+u)^{\beta-\alpha}\Big(\frac{1}{1-u}\Big)^{\alpha-\frac{2}{3}}\Big(\frac{u}{1+u^2+2ut}\Big)^{1-\frac{\alpha}{2}}u^{\frac{\alpha}{2}-\frac{3}{8}} \\ &\times\frac{u^{\frac{1}{3}}}{(1+u)(1-u)^{\frac{2}{3}}}\left[\frac{\alpha}{\beta}-\frac{1+u}{1-u}\left(\frac{1-u}{\sqrt{1+u^2+2ut}}\right)^{\alpha+\beta}\right]dt. \end{aligned}$$

Since for $t\in[-\frac{1}{4},1]$ and $u\in[0,1]$ we have $\frac{1-u}{\sqrt{1+u^2+2tu}}\in(0,1)$, thus the conditions $\frac{3}{4}\leqslant\alpha\leqslant\beta\leqslant 1$, imply:

$$g'(t)\geqslant\beta 2^{1+\alpha-\beta}\int_0^1(1+u)^{\beta-\alpha}\Big(\frac{1}{1-u}\Big)^{\alpha-\frac{3}{4}}\Big(\frac{u}{1+u^2+2ut}\Big)^{1-\frac{\alpha}{2}}u^{\frac{\alpha}{2}-\frac{3}{8}}$$
$$\times\frac{u^{\frac{3}{8}}}{(1+u)(1-u)^{\frac{3}{4}}}\left[\frac{3}{4}-(1+u)\frac{(1-u)^{\frac{1}{2}}}{(1+u^2-\frac{1}{2}u)^{\frac{3}{4}}}\right]dt. \tag{42}$$

We get

$$g'(t)\geqslant\beta 2^{1+\alpha-\beta}\int_0^1\kappa(u)\lambda(u)du,$$

where

$$\kappa(u)=(1+u)^{\beta-\alpha}\Big(\frac{1}{1-u}\Big)^{\alpha-\frac{3}{4}}\Big(\frac{u}{1+u^2+2ut}\Big)^{1-\frac{\alpha}{2}}u^{\frac{\alpha}{2}-\frac{3}{8}}$$

and

$$\lambda(u)=\frac{u^{\frac{3}{8}}}{(1+u)(1-u)^{\frac{3}{4}}}\left[\frac{3}{4}-(1+u)\frac{(1-u)^{\frac{1}{2}}}{(1+u^2-\frac{1}{2}u)^{\frac{3}{4}}}\right],\ u\in[0,1].$$

According to Lemma 9, there is a real number $u_0\in(0,1)$ such that $\lambda(u)\leqslant 0$, $u\in[0,u_0]$ and $\lambda(u)\geqslant 0$, $u\in[u_0,1)$. The mapping κ is increasing, since it is a product of positive increasing functions on the interval $[0,1]$. It follows that we may apply Lemma 6 and we get:

$$g'(t)\geqslant\beta 2^{1+\alpha-\beta}\kappa(u_0)\int_0^1\lambda(u)du. \tag{43}$$

Taking into the account that

$$I_1=\int_0^1\lambda(u)du=0.5577\ldots$$

according to (43) we obtain

$$g'(t)\geqslant 0,\ t\in\left[-\frac{1}{4},1\right]. \tag{44}$$

Summarizing, we have proved that the mapping g is strictly increasing on the interval $[-\frac{1}{4},1]$ and consequently $g(1)\geqslant g(t)$, $(\forall)\,t\in[-\frac{1}{4},1]$. This means that $D_0(\alpha,\beta,x)\geqslant 0$ holds provided that $\frac{3}{4}\leqslant\alpha\leqslant\beta\leqslant 1$ and $x\in\mathbb{C}$, $|x|=1$, $0\leqslant\arg(x)\leqslant\arccos(-\frac{1}{4})$. According to Lemma 5 the inequality $D_n(\alpha,\beta,x)\geqslant 0$ holds provided that $\frac{3}{4}\leqslant\alpha\leqslant\beta\leqslant 1$ and $x\in\mathbb{C}$, $|x|=1$, $0\leqslant\arg(x)\leqslant\arccos(-\frac{1}{4})$. Finally, Lemma 8 shows that the inequality $D_n(\alpha,\beta,x)\geqslant 0$ holds for $\arg(x)\in[\arccos\big(-\frac{1}{4}\big),\pi]$ too and we are done.

Thus we have proved that $D_n(\alpha,\beta,x)\geqslant 0$ provided that $\frac{3}{4}\leqslant\alpha\leqslant\beta\leqslant 1$ and $x\in\mathbb{C}$, $|x|=1$. This result is equivalent to Theorem 1. □

REMARK 1. Numerical approach suggest that inequality $g'(t) > 0$, $t \in \left[-\frac{1}{4}, 1\right]$ holds for $\frac{2}{3} \leqslant \alpha \leqslant \beta \leqslant 1$. Taking into account that we proved Lemma 8 in case $\frac{2}{3} \leqslant \alpha \leqslant \beta \leqslant 1$, we think that the presented approach could lead to the following better result:

If $\frac{2}{3} \leqslant \alpha \leqslant \beta \leqslant 1$ and $x \in \mathbb{C}$ with $|x| = 1$, then the inequality holds

$$|A_{2n+1}(\alpha, \beta, x)| \leqslant A_{2n+1}(\alpha, \beta, 1), \quad (\forall)\, n \in \mathbb{N}. \qquad (*)$$

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