Journal of
Mathematical
Inequalities
Volume 18, Number 3 (2024), 971–982 doi:10.7153/jmi-2024-18-53

RADII OF LEMNISCATE STARLIKENESS AND CONVEXITY OF THE FUNCTIONS INCLUDING DERIVATIVES OF BESSEL FUNCTIONS

ERHAN DENIZ, ADEM KIZILTEPE AND LUMINITA-IOANA COTIRLA

(*Communicated by T. Burić*)

Abstract. In this paper, our aim is to determine the radii of starlikeness and convexity associated with lemniscate of Bernoulli for three different kinds of normalizations of the function $N_\nu(z) = az^2J_\nu''(z)+bzJ_\nu'(z)+cJ_\nu(z)$, where J_ν is the Bessel function of the first kind of order ν. The key tools in the proof of our main results are the Mittag-Leffler expansion for the function $N_\nu(z)$ and properties of real zeros of it. Also, we give tables related with special cases of parameters.

1. Introduction

Denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\} \quad (r > 0)$ the disk of radius r and let $\mathbb{D} = \mathbb{D}_1$. Let $\mathscr{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ which satisfy the usual normalization conditions $f(0) = f'(0)-1 = 0$. Traditionally, the subclass of $\mathscr{A}$ consisting of univalent functions is denoted by $\mathscr{S}$. We say that the function $f \in \mathscr{A}$ is starlike in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}_r$. For $\beta \in [0,1)$ we say that the function f is starlike of order β in $\mathbb{D}_r$ if and only if $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{D}_r$. We define by the real number

$$r_\beta^*(f) = \sup\left\{ r > 0 : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r \right\}$$

the radius of starlikeness of order β of the function f. Note that $r^*(f) = r_0^*(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r_\beta^*(f)})$ is a starlike domain with respect to the origin.

The function $f \in \mathscr{A}$ is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}_r$. For $\beta \in [0,1)$ we say that the function f is convex of

Mathematics subject classification (2020): Primary 33C10; Secondary 30C45.
Keywords and phrases: Normalized Bessel functions of the fist kind, lemniscate convex functions, lemniscate starlike functions, zeros of Bessel function, radius.

order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \mathbb{D}_r$. The radius of convexity of order β of the function f is defined by the real number

$$r^c_\beta(f) = \sup\left\{r > 0 : \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r\right\}.$$

Note that $r^c(f) = r^c_0(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c_\beta(f)})$ is a convex domain.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. In terms of subordination, starlikeness and convexity conditions are, respectively, equivalent to $zf'(z)/f(z) \prec (1+z)/(1-z)$ and $1 + (zf''(z)/f'(z)) \prec (1+z)/(1-z)$. Ma and Minda [15] gave a unified presentation of various subclasses of starlike and convex functions by replacing the subordinate function $(1+z)/(1-z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0) = 1,\ \varphi'(0) > 0$ and φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. They introduced the following general classes that envelopes several well-known classes as special cases:

$$\mathscr{S}^*[\varphi] = \left\{f \in \mathscr{A} :\ \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\} \text{ and } \mathscr{C}[\varphi] = \left\{f \in \mathscr{A} :\ 1+\frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\}.$$

In literature, the functions belonging to these classes are called Ma-Minda starlike and Ma-Minda convex, respectively.

We note that:

(1) $\mathscr{S}^*[(1+(1-2\beta)z)/(1-z)] = \mathscr{S}^*(\beta)$ $(0 \leqslant \beta < 1)$ is class of starlike functions of order β.

(2) $\mathscr{C}[(1+(1-2\beta)z)/(1-z)]) = \mathscr{C}(\beta)$ $(0 \leqslant \beta < 1)$ is class of convex functions of order β.

(3) $\mathscr{S}^*\left[1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right] = \mathscr{S}^*_P = \left\{f \in \mathscr{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|\right\}$ is class of parabolik starlike functions.

(4) $\mathscr{C}\left[1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right] = \mathscr{UCV} = \left\{f \in \mathscr{A} : \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|\right\}$ is class of uniformly convex functions.

(5) $\mathscr{S}^*\left[\left(\frac{1+z}{1-z}\right)^\beta\right] = \mathscr{S}^*_\beta = \left\{f \in \mathscr{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}\right\}$ is class of strongly starlike functions of order β.

(6) $\mathscr{C}\left[\left(\frac{1+z}{1-z}\right)^\beta\right] = \mathscr{C}_\beta = \left\{f \in \mathscr{A} : \left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\beta\pi}{2}\right\}$ is class of strongly convex functions of order β.

(7) $\mathscr{S}^*\left[\sqrt{1+z}\right] = \mathscr{S}^*_L = \left\{f \in \mathscr{A} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1\right\}$ is class of lemniscate starlike functions.

(8) $\mathscr{C}\left[\sqrt{1+z}\right] = \mathscr{C}_L = \left\{f \in \mathscr{A} : \left|\left(1+\frac{zf''(z)}{f'(z)}\right)^2 - 1\right| < 1\right\}$ is class of lemniscate convex functions.

The classes $\mathscr{S}_L^*$ and $\mathscr{C}_L$ studied by Sokol and Stankiewicz [20].We define the radius of lemniscate starlikeness and lemniscate convexity by the real numbers

$$r_L^*(f) = \sup\left\{ r > 0 : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \text{ for all } z \in \mathbb{D}_r \right\}$$

and

$$r_L^c(f) = \sup\left\{ r > 0 : \left| \left(1 + \frac{zf''(z)}{f'(z)} \right)^2 - 1 \right| < 1 \text{ for all } z \in \mathbb{D}_r \right\}.$$

The Bessel function of the first kind of order ν is defined by [18, p. 217]

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2} \right)^{2n+\nu} \quad (z \in \mathbb{C}). \tag{1.1}$$

We know that it has all its zeros real for $\nu > -1$. Here now we consider mainly the general function

$$N_\nu(z) = az^2 J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z)$$

studied by Mercer [17]. Here, as in [17], $q = b - a$ and

$$(c = 0 \text{ and } q \neq 0) \text{ or } (c > 0 \text{ and } q > 0).$$

From (1.1), we have the power series representation

$$N_\nu(z) = \sum_{n=0}^{\infty} \frac{Q(2n+\nu)(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2} \right)^{2n+\nu} \quad (z \in \mathbb{C}) \tag{1.2}$$

where $Q(\nu) = a\nu(\nu-1) + b\nu + c$ $(a,b,c \in \mathbb{R})$. There are a few important works on the function N_ν. First, Mercer's paper [17] which it has been proved that the kth positive zero of N_ν increases with ν in $\nu > 0$. Second, Ismail and Muldoon [10] showed that under the conditions $a,b,c \in \mathbb{R}$ such that $c = 0$ and $b \neq a$ or $c > 0$ and $b > a$;

(i) For $\nu > 0$, the zeros of $N_\nu(z)$ are either real or purely imaginary.

(ii) For $\nu \geqslant \max\{0, \nu_0\}$, where ν_0 is the largest real root of the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$, the the zeros of $N_\nu(z)$ are real.

(iii) If $\nu > 0$, $\left(a\nu^2 + (b-a)\nu + c\right) / (b-a) > 0$ and $a/(b-a) < 0$, the zeros of $N_\nu(z)$ are all real except for a single pair which are conjugate purely imaginary.

Lastly, Baricz, Çağlar and Deniz [4] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ν by using results of Mercer [17], Ismail and Muldoon [10] and Shah and Trimble [19]. Also they proved Mittag-Leffler expansion of $N_\nu(z)$ as follows

$$N_\nu(z) = \alpha z^2 J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z) = \frac{Q(\nu)z^\nu}{2^\nu\Gamma(\nu+1)} \prod_{n \geqslant 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2} \right) \tag{1.3}$$

where $Q(\nu) = a\nu(\nu-1) + b\nu + c, \quad (a,b,c \in \mathbb{R})$ and $\lambda_{\nu,n}$ is the nth positive zero of $N_\nu(z) \ (n \in \mathbb{N})$. Very recently, Deniz and his co-authors [11–13] studied the radii of starlikeness and convexity of order β for the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ defined by (1.4) using some new Mittag-Leffler expansions for quotients of the function N_ν, special properties of the zeros of the function N_ν and its derivatives. Moreover, radii problem for some subclasses of analytic functions for Bessel functions studied by [1–3, 5–9, 14, 16, 21].

Note that N_ν is not belongs to $\mathscr{A}$. To prove the main results we need normalizations of the function N_ν. In this paper we focus on the following normalized forms

$$f_\nu(z) = \left[\frac{2^\nu \Gamma(\nu+1)}{Q(\nu)} N_\nu(z)\right]^{\frac{1}{\nu}}, \tag{1.4}$$

$$g_\nu(z) = \frac{2^\nu \Gamma(\nu+1) z^{1-\nu}}{Q(\nu)} N_\nu(z),$$

$$h_\nu(z) = \frac{2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}}}{Q(\nu)} N_\nu(\sqrt{z}).$$

In the rest of this paper, the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$ will always provide on $a,b,c \in \mathbb{R}$ $(c = 0$ and $a \neq b)$ or $(c > 0$ and $a < b)$. Moreover, ν_0 is the largest real root of the quadratic $Q(\nu)$ defined according to the above conditions.

In this paper, we intend to look the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ for which our aim is to find the radii of lemniscate starlikeness and lemniscate convexity.

The following result is a key tool in the proof of main results.

LEMMA 1.1. [12] *If* $\nu \geqslant \max\{0, \nu_0\}$ *then the functions* $z \longmapsto \Psi_\nu(z) = \frac{2^\nu \Gamma(\nu+1)}{Q(\nu) z^\nu} N_\nu(z)$ *has infinitely many zeros and all of them are positive. Denoting by* $\lambda_{\nu,n}$ *the* n*th positive zero of* $\Psi_\nu(z)$, *under the same conditions the Weierstrassian decomposition*

$$\Psi_\nu(z) = \prod_{n \geqslant 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right)$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\lambda'_{\nu,n}$ *the* n*th positive zero of* $\Phi'_\nu(z)$, *where* $\Phi_\nu(z) = z^\nu \Psi_\nu(z)$, *then the positive zeros of* $\Psi_\nu(z)$ *are interlaced with those of* $\Phi'_\nu(z)$. *In the other words, the zeros satisfy the chain of inequalities*

$$\lambda'_{\nu,1} < \lambda_{\nu,1} < \lambda'_{\nu,2} < \lambda_{\nu,2} < \lambda'_{\nu,3} < \lambda_{\nu,3} < \cdots.$$

2. Main results

2.1. Radii of lemniscate starlikeness and lemniscate convexity of the functions $f_\nu,\ g_\nu$ and h_ν

This section deals with the problem to find the radii of lemniscate starlikeness and lemniscate convexity of the functions $f_\nu(z),\ g_\nu(z)$ and $h_\nu(z)$ defined by (1.4).

THEOREM 2.1. *Let $\nu \geqslant \max\{0,\nu_0\}$. The following statements hold:*

a) *If $\nu \neq 0$, then the radius of lemniscate starlikeness of the function f_ν is the smallest positive root of the equation*

$$\left(\frac{rN_\nu'(r)}{N_\nu(r)}\right)^2 - 4\nu\frac{rN_\nu'(r)}{N_\nu(r)} + 2\nu^2 = 0.$$

b) *The radius of lemniscate starlikeness of the function g_ν is the smallest positive root of the equation*

$$\left(\frac{rN_\nu'(r)}{N_\nu(r)}\right)^2 - 2(1+\nu)\frac{rN_\nu'(r)}{N_\nu(r)} + \nu^2 + 2\nu - 1 = 0.$$

c) *The radius of lemniscate starlikeness of the function h_ν is the smallest positive root of the equation*

$$\left(\frac{rN_\nu'(r)}{N_\nu(r)}\right)^2 - 2(2+\nu)\frac{rN_\nu'(r)}{N_\nu(r)} + \nu^2 + 4\nu - 4 = 0.$$

Proof. We need to show that the following inequalities

$$\left|\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)^2 - 1\right| < 1, \quad \left|\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)^2 - 1\right| < 1 \quad \text{and} \quad \left|\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)^2 - 1\right| < 1 \qquad (2.1)$$

are valid for $z \in \mathbb{D}_{r_L^*(f_\nu)}$, $z \in \mathbb{D}_{r_L^*(g_\nu)}$ and $z \in \mathbb{D}_{r_L^*(h_\nu)}$ respectively, and each of the above inequalities does not hold in larger disks.

From the equation (1.3) we get

$$\frac{zf_\nu'(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad (\nu \geqslant \max\{0,\nu_0\},\ \nu \neq 0),$$

$$\frac{zg_\nu'(z)}{g_\nu(z)} = (1-\nu) + \frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad (\nu \geqslant \max\{0,\nu_0\}),$$

$$\frac{zh_\nu'(z)}{h_\nu(z)} = 1 - \frac{\nu}{2} + \frac{1}{2}\frac{\sqrt{z}N_\nu'(\sqrt{z})}{N_\nu(\sqrt{z})} = 1 - \sum_{n\geqslant 1}\frac{z}{\lambda_{\nu,n}^2 - z}, \quad (\nu \geqslant \max\{0,\nu_0\}).$$

For $z \in \mathbb{D}_{\lambda_{\nu,1}}$ by simple computations, we have

$$\begin{aligned}
\left|\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)^2 - 1\right| &\leqslant \frac{1}{\nu^2}\left(\sum_{n\geqslant 1}\frac{2|z|^2}{\lambda_{\nu,n}^2-|z|^2}\right)\left(\sum_{n\geqslant 1}\frac{2|z|^2}{\lambda_{\nu,n}^2-|z|^2}+2\nu\right)\\
&= \left(\frac{|z|f_\nu'(|z|)}{f_\nu(|z|)}\right)^2 - 4\frac{|z|f_\nu'(|z|)}{f_\nu(|z|)}+3,\\
\left|\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)^2 - 1\right| &\leqslant \left(\sum_{n\geqslant 1}\frac{2|z|^2}{\lambda_{\nu,n}^2-|z|^2}\right)\left(\sum_{n\geqslant 1}\frac{2|z|^2}{\lambda_{\nu,n}^2-|z|^2}+2\right)\\
&= \left(\frac{|z|g_\nu'(|z|)}{g_\nu(|z|)}\right)^2 - 4\frac{|z|g_\nu'(|z|)}{g_\nu(|z|)}+3,\\
\left|\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)^2 - 1\right| &\leqslant \left(\sum_{n\geqslant 1}\frac{2|z|}{\lambda_{\nu,n}^2-|z|}\right)\left(\sum_{n\geqslant 1}\frac{2|z|}{\lambda_{\nu,n}^2-|z|}+2\right)\\
&= \left(\frac{|z|h_\nu'(|z|)}{h_\nu(|z|)}\right)^2 - 4\frac{|z|h_\nu'(|z|)}{h_\nu(|z|)}+3,
\end{aligned}$$

where equalities are attained only when $z=|z|=r$. Thus, for $r\in(0,\lambda_{\nu,1})$ it follows that

$$\begin{aligned}
\inf_{z\in\mathbb{D}_r}\left\{\left|\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)^2-1\right|-1\right\} &= \left(\frac{rf_\nu'(r)}{f_\nu(r)}\right)^2-4\frac{rf_\nu'(r)}{f_\nu(r)}+2,\\
\inf_{z\in\mathbb{D}_r}\left\{\left|\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)^2-1\right|-1\right\} &= \left(\frac{rg_\nu'(r)}{g_\nu(r)}\right)^2-4\frac{rg_\nu'(r)}{g_\nu(r)}+2
\end{aligned}$$

and

$$\inf_{z\in\mathbb{D}_r}\left\{\left|\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)^2-1\right|-1\right\} = \left(\frac{rh_\nu'(r)}{h_\nu(r)}\right)^2-4\frac{rh_\nu'(r)}{h_\nu(r)}+2.$$

On the other hand, we consider the mappings $\psi_\nu,\ \varphi_\nu,\ \phi_\nu : (0,\lambda_{\nu,1}) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
\psi_\nu(r) &= \left(\frac{rf_\nu'(r)}{f_\nu(r)}\right)^2-4\frac{rf_\nu'(r)}{f_\nu(r)}+2,\\
\varphi_\nu(r) &= \left(\frac{rg_\nu'(r)}{g_\nu(r)}\right)^2-4\frac{rg_\nu'(r)}{g_\nu(r)}+2
\end{aligned}$$

and

$$\phi_\nu(r) = \left(\frac{rh_\nu'(r)}{h_\nu(r)}\right)^2-4\frac{rh_\nu'(r)}{h_\nu(r)}+2.$$

Since,

$$\psi_\nu'(r) = \frac{2}{\nu}\sum_{n\geqslant 1}\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2 - r^2\right)^2}\left(1+\frac{1}{\nu}\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2}\right) > 0,$$

$$\varphi_\nu'(r) = 2\sum_{n\geqslant 1}\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2 - r^2\right)^2}\left(1+\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2}\right) > 0$$

and

$$\phi_\nu'(r) = 4\sum_{n\geqslant 1}\frac{\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2 - r\right)^2}\left(1+\sum_{n\geqslant 1}\frac{2r}{\lambda_{\nu,n}^2 - r}\right) > 0$$

the functions ψ_ν, φ_ν and ϕ_ν are strictly increasing for all $\nu \geqslant \max\{0,\nu_0\}, \nu \neq 0$. Also from $\lim_{r\searrow 0}\psi_\nu(r) = \lim_{r\searrow 0}\varphi_\nu(r) = \lim_{r\searrow 0}\phi_\nu(r) = -1 < 0$ and $\lim_{r\nearrow\lambda_{\nu,1}}\psi_\nu(r) = \lim_{r\nearrow\lambda_{\nu,1}}\varphi_\nu(r) = \lim_{r\nearrow\lambda_{\nu,1}}\phi_\nu(r) = \infty$, in view of the minimum principle for harmonic functions imply that the corresponding inequalities in (2.1) for $\nu \geqslant \max\{0,\nu_0\}, \nu \neq 0$ hold if only if $z \in \mathbb{D}_{r_1}$, $z \in \mathbb{D}_{r_2}$ and $z \in \mathbb{D}_{r_3}$, respectively, where r_1, r_2 and r_3 is the smallest positive roots of euations

$$\psi_\nu(r) = 0, \quad \varphi_\nu(r) = 0 \text{ and } \phi_\nu(r) = 0,$$

which are equivalent to equalities in the parts of **a**, **b**, and **c** of Theorem. This completes the proof of Thereom. □

	$r_L^*(f_{3/2})$				
	$b=1$ and $c=0$		$a=1$ and $c=0$		$a=1$ and $b=2$
$a=2$	0.599617	$b=2$	0.785637	$c=2$	0.929432
$a=3$	0.561499	$b=3$	0.844262	$c=3$	0.985163
$a=4$	0.539752	$b=4$	0.882675	$c=4$	1.03333

Table 1. *Radii of lemniscate starlikeness for f_ν when $\nu = 1.5$*

	$r_L^*(g_{3/2})$				
	$b=1$ and $c=0$		$a=1$ and $c=0$		$a=1$ and $b=2$
$a=2$	0.506144	$b=2$	0.660389	$c=2$	0.777979
$a=3$	0.474373	$b=3$	0.708826	$c=3$	0.823245
$a=4$	0.456218	$b=4$	0.740562	$c=4$	0.862248

Table 2. *Radii of lemniscate starlikeness for g_ν when $\nu = 1.5$*

	$r_L^*(h_{3/2})$				
	$b=1$ and $c=0$		$a=1$ and $c=0$		$a=1$ and $b=2$
$a=2$	0.67085	$b=2$	0.882142	$c=2$	1.04753
$a=3$	0.627753	$b=3$	0.948955	$c=3$	1.11205
$a=4$	0.6032	$b=4$	0.992741	$c=4$	1.16799

Table 3. *Radii of lemniscate starlikeness for h_ν when $\nu = 1.5$*

For $\nu = 1.5$, considering the special values of $a,b,c \in \mathbb{R}$, radii of lemniscate starlikeness of the functions f_ν, g_ν and h_ν are seen from the tables above. If the values of b and c are fixed and the values of a is increased, radii of lemniscate starlikeness of the functions f_ν, g_ν and h_ν are monotone decreasing. If the values of a and c are fixed and the values of b is increased or the values of a and b are fixed and the values of c is increased radii of lemniscate starlikeness of the functions f_ν, g_ν and h_ν are monotone increasing.

The second principal result related with radii of the lemniscate convexity.

THEOREM 2.2. *Let $\nu \geqslant \max\{0, \nu_0\}$. The following statements hold:*

a) *If $\nu \neq 0$ then, the radius $r_L^c(f_\nu)$ is the smallest positive root of the equation*

$$\left(\frac{rN_\nu''(r)}{N_\nu'(r)} + \left(\frac{1}{\nu} - 1\right)\frac{rN_\nu'(r)}{N_\nu(r)}\right)^2 - 2\left(\frac{rN_\nu''(r)}{N_\nu'(r)} + \left(\frac{1}{\nu} - 1\right)\frac{rN_\nu'(r)}{N_\nu(r)}\right) - 1 = 0.$$

Moreover, $r_L^c(f_\nu) < \lambda'_{\nu,1} < \lambda_{\nu,1}$.

b) *The radius $r_L^c(g_\nu)$ is the smallest positive root of the equation*

$$\left(\frac{z^2N_\nu''(z) + 2(1-\nu)zN_\nu'(z) + (\nu^2 - \nu)N_\nu(z)}{zN_\nu'(z) + (1-\nu)N_\nu(z)}\right)^2 - 2\left(\frac{z^2N_\nu''(z) + 2(1-\nu)zN_\nu'(z) + (\nu^2 - \nu)N_\nu(z)}{zN_\nu'(z) + (1-\nu)N_\nu(z)}\right) - 1 = 0.$$

c) *The radius $r_L^c(h_\nu)$ is the smallest positive root of the equation*

$$\left(\frac{rN_\nu''(\sqrt{r}) + (3-2\nu)\sqrt{r}N_\nu'(\sqrt{r}) + \left(\nu^2 - 2\nu\right)N_\nu(\sqrt{r})}{2\sqrt{r}N_\nu'(\sqrt{r}) + 2(2-\nu)N_\nu(\sqrt{r})}\right)^2 - 2\left(\frac{rN_\nu''(\sqrt{r}) + (3-2\nu)\sqrt{r}N_\nu'(\sqrt{r}) + \left(\nu^2 - 2\nu\right)N_\nu(\sqrt{r})}{2\sqrt{r}N_\nu'(\sqrt{r}) + 2(2-\nu)N_\nu(\sqrt{r})}\right) - 1 = 0.$$

Proof. **a)** In [12], authors obtained

$$1 + \frac{zf_\nu''(z)}{f_\nu'(z)} = 1 + \frac{zN_\nu''(z)}{N_\nu'(z)} + \left(\frac{1}{\nu} - 1\right)\frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \left(\frac{1}{\nu} - 1\right)\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2} - \sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}'^2 - z^2}, \tag{2.2}$$

where $\lambda'_{\nu,n}$ denotes the nth positive zero of the function N'_ν. Now, by using the triangle inequality, for all $z \in \mathbb{D}_{\lambda'_{\nu,1}}$ and $1 \geqslant \nu > \max\{0, \nu_0\}$ we obtain the inequality

$$\left|\left(1 + \frac{zf_\nu''(z)}{f_\nu'(z)}\right)^2 - 1\right| \leqslant \left(\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}'^2 - r^2} + \left(\frac{1}{\nu} - 1\right)\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2}\right)^2 + 2\left(\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}'^2 - r^2} + \left(\frac{1}{\nu} - 1\right)\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2 - r^2}\right), \tag{2.3}$$

where $|z| = r$. Thus, from (2.2) and (2.3) we have

$$\left| \left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)^2 - 1 \right| \leqslant \left(\frac{rf_\nu''(r)}{f_\nu'(r)}\right)^2 - 2\left(\frac{rf_\nu''(r)}{f_\nu'(r)}\right). \tag{2.4}$$

Moreover, observe that if we use the inequality [2, Lemma 2.1]

$$\left|\frac{z}{a-z} - \mu\left(\frac{z}{b-z}\right)\right| \leqslant \frac{|z|}{a-|z|} - \mu\frac{|z|}{b-|z|},$$

where $b > a > 0$, $\mu \in [0,1]$ and $z \in \mathbb{C}$ such that $|z| < a$, then we get that the inequality (2.4) is also valid when $\nu \geqslant 1$. Thus, for $\nu > \max\{0, \nu_0\}$ and $z \in \mathbb{D}_{\lambda'_{\nu,1}}$, the relation (2.4) holds.

On the other hand, we define the function $\Lambda_\nu : (0, \lambda'_{\nu,1}) \to \mathbb{R}$,

$$\Lambda_\nu(r) = \left(\frac{rf_\nu''(r)}{f_\nu'(r)}\right)^2 - 2\left(\frac{rf_\nu''(r)}{f_\nu'(r)}\right) - 1.$$

Since the zeros of N_ν and N'_ν are interlacing according to Lemma 1.1 and $r < \lambda'_{\nu,1} < \lambda_{\nu,1}$ $\left(\text{or } r < \sqrt{\lambda_{\nu,1}\lambda'_{\nu,1}}\right)$ for all $\nu > \max\{0, \nu_0\}$ we have

$$(\lambda_{\nu,n})\left(\lambda'^2_{\nu,n} - r^2\right) - (\lambda'_{\nu,n})\left(\lambda^2_{\nu,n} - r^2\right) < 0.$$

Thus following inequality

$$\Lambda'_\nu(r) > 8r\left(\sum_{n\geqslant 1}\left(\frac{\lambda'^2_{\nu,n}}{\left(\lambda'^2_{\nu,n} - r^2\right)^2} - \frac{\lambda^2_{\nu,n}}{\left(\lambda^2_{\nu,n} - r^2\right)^2}\right)\right)$$
$$\times 2r^2\left(\sum_{n\geqslant 1}\left(\frac{1}{\lambda'^2_{\nu,n} - r^2} - \frac{1}{\lambda^2_{\nu,n} - r^2}\right)\right) > 0$$

is satisfied. Consequently, the function Λ_ν is strictly increasing. Observe also that $\lim_{r\searrow 0}\Lambda_\nu(r) = -1$ and $\lim_{r\nearrow\lambda'_{\nu,1}}\Lambda_\nu(r) = \infty$, which means that for $z \in \mathbb{D}_{r_4}$ we have

$$\left|\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)^2 - 1\right| < 1$$

if and ony if r_4 is the unique root of

$$\left(\frac{rf_\nu''(r)}{f_\nu'(r)}\right)^2 - 2\left(\frac{rf_\nu''(r)}{f_\nu'(r)}\right) - 1 = 0$$

situated in $(0, \lambda'_{\nu,1})$. The above equation is equivalent to equation in **a)**.

b) Observe that

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)} = 1+\frac{z^2N_\nu''(z) + 2(1-\nu)zN_\nu'(z) + (\nu^2-\nu)N_\nu(z)}{zN_\nu'(z) + (1-\nu)N_\nu(z)}.$$

In [12], authors obtained

$$g_\nu'(z)=\prod_{n\geqslant 1}\left(1-\frac{z^2}{\delta_{\nu,n}^2}\right), \tag{2.5}$$

where $\delta_{\nu,n}$ denotes the nth positive zero of the function g_ν'. By means of (2.5) we have

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)}=1-\sum_{n\geqslant 1}\left(\frac{2z^2}{\delta_{\nu,n}^2-z^2}\right).$$

By using the triangle inequaliy, for all $z\in\mathbb{D}_{\delta_{\nu,n}}$ we obtain that

$$\left|\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)^2-1\right|\leqslant\left(\frac{rg_\nu''(r)}{g_\nu'(r)}\right)^2-2\left(\frac{rg_\nu''(r)}{g_\nu'(r)}\right),$$

where $|z|=r.$ Similarly to the proof in part a), the lemniscate convex radius $r_L^c(g_\nu)$ is the unique positive root of the equation

$$\left(\frac{rg_\nu''(r)}{g_\nu'(r)}\right)^2-2\left(\frac{rg_\nu''(r)}{g_\nu'(r)}\right)-1=0$$

in $(0,\delta_{\nu,1})$. The above equation is equivalent to equation in **b)**.

c) Observe that

$$1+\frac{zh_\nu''(z)}{h_\nu'(z)}=1+\frac{zN_\nu''(\sqrt{z})+(3-2\nu)\sqrt{z}N_\nu'(\sqrt{z})+\left(\nu^2-2\nu\right)N_\nu(\sqrt{z})}{2\sqrt{z}N_\nu'(\sqrt{z})+2\left(2-\nu\right)N_\nu(\sqrt{z})}.$$

In [12], authors obtained the infinite product representation of $h_\nu'(z)$ as follows:

$$h_\nu'(z)=\prod_{n\geqslant 1}\left(1-\frac{z}{\gamma_{\nu,n}^2}\right), \tag{2.6}$$

where γ_ν denotes the nth positive zero of the function $h_{\nu,n}'$. By using the triangle inequaliy, for all $z\in\mathbb{D}_{\delta_{\nu,n}}$ we have

$$\left|\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)^2-1\right|\leqslant\left(\frac{rh_\nu''(r)}{h_\nu'(r)}\right)^2-2\left(\frac{rh_\nu''(r)}{h_\nu'(r)}\right),$$

where $|z|=r.$ Similarly to the proof in part **a)**, the lemniscate convexity radius $r_L^c(h_\nu)$ is the unique positive root of the equation

$$\left(\frac{rh_\nu''(r)}{h_\nu'(r)}\right)^2-2\left(\frac{rh_\nu''(r)}{h_\nu'(r)}\right)-1=0$$

in $(0,\gamma_{\nu,1})$. The above equation is equivalent to equation in **c)**. $\square$

	$r_L^c(f_{5/2})$					
	$b=1$ and $c=0$		$a=1$ and $c=0$		$a=1$ and $b=2$
$a=2$	0.733439	$b=2$	0.83426	$c=2$	0.894764
$a=3$	0.716228	$b=3$	0.872489	$c=3$	0.920737
$a=4$	0.706905	$b=4$	0.899487	$c=4$	0.944387

Table 4. *Radii of lemniscate convexity for f_ν when $\nu = 2.5$*

	$r_L^c(g_{5/2})$				
	$b=1$ and $c=0$		$a=1$ and $c=0$		$a=1$ and $b=2$
$a=2$	0.481843	$b=2$	0.545538	$c=2$	0.583111
$a=3$	0.470937	$b=3$	0.569667	$c=3$	0.599173
$a=4$	0.465023	$b=4$	0.586727	$c=4$	0.613771

Table 5. *Radii of lemniscate convexity for g_ν when $\nu = 2.5$*

	$r_L^c(h_{5/2})$				
	$b=1$ and $c=0$		$a=1$ and $c=0$		$a=1$ and $b=2$
$a=2$	0.628517	$b=2$	0.814203	$c=2$	0.93746
$a=3$	0.59923	$b=3$	0.890901	$c=3$	0.993068
$a=4$	0.583658	$b=4$	0.947138	$c=4$	1.04511

Table 6. *Radii of lemniscate convexity for h_ν when $\nu = 2.5$*

For $\nu = 2.5$, considering the special values of $a,b,c \in \mathbb{R}$, radii of lemniscate convexity of the functions f_ν, g_ν and h_ν are seen from the tables above. If the values of b and c are fixed and the values of a is increased, radii of lemniscate convexity of the functions f_ν, g_ν and h_ν are monotone decreasing. If the values of a and c are fixed and the values of b is increased or the values of a and b are fixed and the values of c is increased radii of lemniscate convexity of the functions f_ν, g_ν and h_ν are monotone increasing.

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(Received November 7, 2023)

Erhan Deniz
Department of Mathematics
Faculty of Science and Letters
Kafkas University
Kars, Turkey
e-mail: edeniz36@gmail.com

Adem Kızıltepe
Department of Mathematics
Faculty of Science and Letters
Kafkas University
Kars, Turkey
e-mail: admkzltp@gmail.com

Luminita-Ioana Cotirla
Department of Mathematics
Technical University of Cluj-Napoca
400114 Cluj-Napoca, Romania
e-mail: luminita.cotirla@math.utcluj.ro