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LAI LAW FOR LINEAR PROCESSES WITH LONG MEMORY

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Abstract. The Lai law, characterizing the convergence rate of the single laws of logarithm for the sequence of independent and identically distributed random variables, has been extended to the linear processes with long memory.

1. Introduction and main result

The Lai law, associated with the single laws of logarithm, states that

THEOREM A. *Let $r>1$ and $\{X_n, n\geqslant 1\}$ be a sequence of independent and identically distributed random variables with partial sums $S_n=\sum_{k=1}^n X_k$, $n\geqslant 1$. Suppose that*

$$EX_1=0,\ \ EX_1^2=1 \text{ and } \ E(X_1^2/\log|X_1|)^r<\infty. \tag{1.1}$$

where, and in the following, $\log x=\log_e \max\{x,e\}, x>0$. Then for all $\varepsilon>\sqrt{r-1}$,

$$\sum_{n=1}^{\infty} n^{r-2}P\{|S_n|>\varepsilon\sqrt{2n\log n}\}<\infty. \tag{1.2}$$

Conversely, if (1.2) *holds for some $\varepsilon>0$, then $EX_1=0$ and $E(X_1^2/\log|X_1|)^r<\infty$.*

The result is first established by Lai (1974), Chen and Wang (2008) extended it to the linear processes with short memory partly, and furthermore showed that

$$\sum_{n=1}^{\infty} n^{r-2}P\{|S_n|>\varepsilon\sqrt{2n\log n}\}=\infty,\ \ \text{for all } \ \varepsilon<\sqrt{r-1}.$$

Combining the results of Lai (1974), Chen and Wang (2008),

$$\sum_{n=1}^{\infty} n^{r-2}P\{|S_n|>\varepsilon\sqrt{2n\log n}\}\begin{cases}<\infty, & \text{if } \varepsilon>\sqrt{r-1},\\ =\infty, & \text{if } \varepsilon<\sqrt{r-1}\end{cases} \tag{1.3}$$

if and only if (1.1) holds.

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Recently, Liu et al. (2022) extended (1.3) to the weighted sums under some conditions both on the weights and the moments. The first motivation of this note comes from the results of Lai (1974), Chen and Wang (2008), and Liu et al. (2022) to extent the Lai law to the linear processes with long memory.

Let $\{\zeta_i, i \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables and $\{a_i, i \in \mathbb{Z}\}$ a sequence of real numbers. Here and in the following, $\mathbb{Z}$ denotes the set of all integers. $\{X_n, n \geqslant 1\}$ is called a linear process or an infinite order moving average process, if X_n is defined by

$$X_n = \sum_{i=-\infty}^{\infty} a_{i+n}\zeta_i \tag{1.4}$$

for $n \geqslant 1$.

If $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, $\sum_{i=-\infty}^{\infty} a_i \neq 0$, $\{X_n, n \geqslant 1\}$ is a linear process with short memory. If $\sum_{i=-\infty}^{\infty} |a_i| = \infty$, $\{X_n, n \geqslant 1\}$ is a linear process with long memory (see Chapter 3 in Giraitis et al., 2012).

In the case of short memory, many authors have studied the limit properties. For example, Li et al. (1992) obtained the complete convergence for linear process, Zhang (1996) extended the result of Li et al. (1992) to φ-mixing random variables, and Chen et al. (2009) obtained the result of Zhang (1996) without any conditions on mixing rate, Chen and Wang (2008) obtained the convergence rates for probabilities of moderate deviations for linear processes including the Lai law, Liu et al. (2015) obtained the Davis-Gut law for linear processes, and so on.

For a convenience, if $\{a_i, i \in \mathbb{Z}\}$ a sequence of real numbers, set

$$W_n(t) = \left(\sum_{i=-\infty}^{\infty} |\omega_{ni}|^t \right)^{1/t}$$

for $n \geqslant 1$ and $t > 0$, where $\omega_{ni} = \sum_{j=1}^{n} a_{i+j}$.

In the case of long memory, few people studied the limit properties. Wang et al. (2003) obtained the strong approximation when $a_i, i \geqslant 0$, have some special expressions. Characiejus and Račkauskas (2016) first obtained the convergence rate in the Marcinkiewicz-Zygmund strong law of large numbers with the norming sequence similarly as $W_n(p)$, $1 < p < 2$, Zhang et al. (2017) extended the result of Characiejus and Račkauskas (2016) and obtained the Baum and Katz laws with the norming sequence $W_n(p)$, $1 < p < 2$. The second motivation of this note comes from the results of Wang et al. (2003), Characiejus and Račkauskas (2016), and Zhang et al. (2017) to obtain the Lai law for the linear processes with long memory.

We now state our main result. Some auxiliary lemmas and the proof of the main result will be detailed in the next section.

THEOREM 1.1. *Let $r > 1$. Let $\{\zeta_i, i \in \mathbb{Z}\}$ be i.i.d. random variables with*

$$E\zeta_0 = 0, \ E\zeta_0^2 = 1, \ E(\zeta_0^2/\log|\zeta_0|)^r < \infty. \tag{1.5}$$

Assume that $\{a_i, i \in \mathbb{Z}\}$ is sequence of real numbers with

$$\sum_{i=-\infty}^{\infty} a_i^2 < \infty, \tag{1.6}$$

and

$$W_n(q)/W_n(2) = O(n^{1/q-1/2}) \text{ for some } q > 2r. \tag{1.7}$$

Then

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2W_n^2(2)\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > \sqrt{r-1}, \\ = \infty, & \text{if } \varepsilon < \sqrt{r-1}, \end{cases} \tag{1.8}$$

where $S_n = \sum_{i=k}^{n} X_k$, *and* X_n *is defined as* (1.4), $n \geqslant 1$.

REMARK 1.1. Since $\{\zeta_i, i \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables, we can easily prove, under conditions (1.5) and (1.6), that the series $\sum_{i=-\infty}^{\infty} a_{i+n}\zeta_i$ converges almost surely (see, for example, Lemma 3.1 in Sung, 2009). Hence the linear process, $\{X_n, n \geqslant 1\}$, are well defined under conditions of Theorem 1.1.

REMARK 1.2. Theorem 1.1 includes the corresponding result of Chen and Wang (2008). In fact, suppose that $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, and $a := \sum_{i=-\infty}^{\infty} a_i \neq 0$, then (1.6) holds, and by Lemma 2.1 in Burton and Dehling (1990),

$$\frac{1}{n} W_n^t(t) \to |a|^t$$

holds for any $t \geqslant 1$, which follows (1.7), and hence (1.8) holds by Theorem 1.1. Note that $W_n^2(2) \sim na^2$ by the above formula, then (1.8) is equivalent to the corresponding result of Chen and Wang (2008) as follow.

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2a^2 n\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > \sqrt{r-1}, \\ = \infty, & \text{if } \varepsilon < \sqrt{r-1}. \end{cases}$$

REMARK 1.3. Let $\{\zeta_i, i \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables, and $\{a_i, i \in \mathbb{Z}\}$ a sequence of real numbers defined by $a_i = (i+1)^{-d}$ if $i \geqslant 0$ and $a_i = 0$ if $i < 0$, where $1/2 < d < 1$. Then it is obvious that (1.6) holds, $\sum_{i=-\infty}^{\infty} |a_i| = \infty$, and hence the linear processes $\{X_n, n \geqslant 1\}$ is long memory. When $t > 1$ and $1/t < d < 1$, Characiejus and Račkauskas (2016) proved that $W_n(t) \sim Cn^{1/t+1-d}$ as $n \to \infty$, where C is a positive constant. From this, it is easily checked that condition (1.7) holds. Suppose that the moment condition (1.5) holds for some $r > 1$, then (1.8) follows from Theorem 1.1.

Throughout this paper, C denotes a positive constant which very often may vary at each occurrence. For events A, $I(A)$ denotes the indicator function of the event A.

2. Lemma and proof

The main idea in the proof of the main result is from the invariance principle's way to estimate the rate of convergence (see Sakhanenko, 1980, 1984, 1985), which is a powerful tool in the field of limit theory (for example, see Chen and Wang (2008), Liu et al. (2015), etc.) and is listed as the following lemma.

LEMMA 2.1. *For any $q > 2$, there exists $B = B(q) > 0$ satisfying that for any sequence of independent random variables $\{\xi_i, 1 \leqslant i \leqslant n\}$ with mean zero and $E|\xi_i|^q < \infty, 1 \leqslant i \leqslant n$, there is a sequence of independent normal random variables $\{\eta_i, 1 \leqslant i \leqslant n\}$ with $E\eta_i = 0$, $E\eta_i^2 = E\xi_i^2$ and for all $y > 0$,*

$$P\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}\xi_i - \sum_{i=1}^{k}\eta_i\right| > y\right\} \leqslant By^{-q}\sum_{i=1}^{n}E|\xi_i|^q.$$

Proof of Theorem 1.1. Set

$$\zeta'_{ni} = \zeta_i I(|\zeta_i| > \sqrt{n\log n}) - E\zeta_i I(|\zeta_i| > \sqrt{n\log n}), \quad \zeta''_{ni} = \zeta_i - \zeta'_{ni}.$$

We first prove that for all $\varepsilon > \sqrt{r-1}$,

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2W_n^2(2)\log n}\} < \infty. \tag{2.1}$$

Note that

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_{i+k}\zeta_i = \sum_{i=-\infty}^{\infty}\left(\sum_{k=i+1}^{i+n} a_k\right)\zeta_i = \sum_{i=-\infty}^{\infty}\omega_{ni}\zeta_i,$$

and for every $\varepsilon > \sqrt{r-1}$

$$\left\{|S_n| > \varepsilon\sqrt{2W_n^2\log n}\right\} \subset \left\{\left|\sum_{i=-\infty}^{\infty}\omega_{ni}\zeta'_{ni}\right| > \varepsilon_1\sqrt{2W_n^2\log n}\right\} \cup \left\{|\sum_{i=-\infty}^{\infty}\omega_{ni}\zeta''_{ni}| > \varepsilon_2\sqrt{2W_n^2\log n}\right\},$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > \sqrt{r-1}$ with $\varepsilon = \varepsilon_1 + \varepsilon_2$. Hence for all $\varepsilon > \sqrt{r-1}$ and some $\varepsilon_1 > 0$, $\varepsilon_2 > \sqrt{r-1}$ with $\varepsilon = \varepsilon_1 + \varepsilon_2$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-2}P\left\{|S_n| > \varepsilon\sqrt{2W_n^2\log n}\right\} &\leqslant \sum_{n=1}^{\infty} n^{r-2}P\left\{\left|\sum_{i=-\infty}^{\infty}\omega_{ni}\zeta'_{ni}\right| > \varepsilon_1\sqrt{2W_n^2\log n}\right\}\\ &\quad + \sum_{n=1}^{\infty} n^{r-2}P\left\{\left|\sum_{i=-\infty}^{\infty}\omega_{ni}\zeta''_{ni}\right| > \varepsilon_2\sqrt{2W_n^2\log n}\right\}\\ &= I_1 + I_2.\end{aligned}$$

By the Markov inequality, and a standard computation, we have

$$\begin{aligned} I_1 &\leqslant C\sum_{n=1}^{\infty} n^{r-2}W_n^{-2}(2)(\log n)^{-1}E\left|\sum_{i=-\infty}^{\infty}\omega_{ni}\zeta'_{ni}\right|^2\\ &= C\sum_{n=1}^{\infty} n^{r-2}(\log n)^{-1}E\zeta_0^2 I(|\zeta_0| > \sqrt{n\log n})\\ &\leqslant CE(\zeta_0^2/\log|\zeta_0|)^r < \infty.\end{aligned}$$

From Lemma 2.1 that for any $n \geqslant 1$, and $i \in \mathbb{Z}$, there exists normal random variables Z_{ni} with $EZ_{ni} = 0$ and $EZ_{ni}^2 = E|a_{ni}\zeta''_{ni}|^2$, such that for all $y > 0$

$$P\left\{\left|\sum_{i=-\infty}^{\infty} \omega_{ni}\zeta''_{ni} - \sum_{i=-\infty}^{\infty} Z_{ni}\right| > y\right\} \leqslant Ay^{-q} \sum_{i=-\infty}^{\infty} E|\omega_{ni}\zeta''_{ni}|^q. \tag{2.2}$$

Note that

$$\left\{\left|\sum_{i=-\infty}^{\infty} \omega_{ni}\zeta''_{ni}\right| > \varepsilon_2\sqrt{2W_n^2(2)\log n}\right\} \subset \left\{\left|\sum_{i=-\infty}^{\infty} \omega_{ni}\zeta''_{ni} - \sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_3\sqrt{2W_n^2(2)\log n}\right\}$$
$$\cup \left\{\left|\sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_4\sqrt{2W_n^2(2)\log n}\right\}$$

where $\varepsilon_3 > 0$, $\varepsilon_4 > \sqrt{r-1}$ with $\varepsilon_2 = \varepsilon_3 + \varepsilon_4$. Hence

$$\begin{aligned} I_2 &\leqslant \sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{i=-\infty}^{\infty} \omega_{ni}\zeta''_{ni} - \sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_3\sqrt{2W_n^2(2)\log n}\right\} \\ &\quad + \sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_4\sqrt{2W_n^2(2)\log n}\right\} \\ &= I_3 + I_4. \end{aligned}$$

By (2.2), (1.7) and a standard computation we can derive that for $q > 2r$

$$\begin{aligned} I_3 &\leqslant C\sum_{n=1}^{\infty} n^{r-2}(W_n^2(2)\log n)^{-q/2} \sum_{i=-\infty}^{\infty} E|\omega_{ni}\zeta''_{ni}|^q \\ &\leqslant C\sum_{n=1}^{\infty} n^{r-2}(W_n(q)/W_n(2)^q(\log n)^{-q/2} E|\zeta_0|^q I(|\zeta_0| \leqslant \sqrt{n\log n}) \\ &\leqslant C\sum_{n=1}^{\infty} n^{r-1-q/2}(\log n)^{-q/2} E|\zeta_0|^q I(|\zeta_0| \leqslant \sqrt{n\log n}) \\ &\leqslant CE(\zeta_0^2/\log|\zeta_0|)^r < \infty. \end{aligned}$$

Let N be a standard normal random variable. Note that $E\{\zeta_0 I(|\zeta_0| \leqslant \sqrt{n\log n}) - E\zeta I(|\zeta| \leqslant \sqrt{n\log n})\}^2 \leqslant 1$ and $P\{|N| > x\} \sim \pi^{-1}x^{-1}e^{-x^2/2}$. Hence for large enough n,

$$\begin{aligned} P\left\{\left|\sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_4\sqrt{2W_n^2(2)\log n}\right\} &\leqslant P\left\{|N| > \frac{\varepsilon_4\sqrt{2W_n^2(2)\log n}}{\sqrt{\sum_{i=-\infty}^{\infty}\omega_{ni}^2}}\right\} \\ &= P\left\{|N| > \varepsilon_4\sqrt{2\log n}\right\} \\ &\leqslant C\exp\{-\varepsilon_4^2\log n\} \\ &= Cn^{-\varepsilon_4^2}, \end{aligned}$$

which follows that $I_4 \leqslant C\sum_{-\infty}^{\infty} n^{r-2-\varepsilon_4^2} < \infty$ by the fact $\varepsilon_4 > \sqrt{r-1}$. Hence, (2.1) holds.

Now we prove that for all $\varepsilon \in (0, \sqrt{r-1})$,

$$\sum_{n=1}^{\infty} n^{r-1} P\left\{|S_n| > \varepsilon\sqrt{2W_n^2(2)\log n}\right\} = \infty. \tag{2.3}$$

Note that $\varepsilon \in (0, \sqrt{r-1})$, there exist $\varepsilon_5 > 0$, $\varepsilon_6 > 0$, and $\varepsilon_7 \in (0, \sqrt{r-1})$ with $\varepsilon_7 = \varepsilon_5 + \varepsilon_6 + \varepsilon$,

$$\begin{aligned}\left\{\left|\sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_7\sqrt{2W_n^2(2)\log n}\right\} \subset & \left\{\left|\sum_{i=-\infty}^{\infty} \omega_{ni}\zeta_{ni}'' - \sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_5\sqrt{2W_n^2(2)\log n}\right\} \\ & \cup \left\{\left|\sum_{i=-\infty}^{\infty} \omega_{ni}\zeta_{ni}'\right| > \varepsilon_6\sqrt{2W_n^2(2)\log n}\right\} \\ & \cup \left\{|S_n| > \varepsilon\sqrt{2W_n^2(2)\log n}\right\}.\end{aligned}$$

Using $P\{|N| > x\} \sim \pi^{-1}x^{-1}e^{-x^2/2}$, it is easy to show that

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{i=-\infty}^{\infty} Z_{ni}\right| > \varepsilon_7\sqrt{2W_n^2(2)\log n}\right\} = \infty.$$

Recall $I_1 < \infty$, $I_3 < \infty$, (2.3) holds. The proof is completed. □

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