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PARTIAL EIGENVALUES FOR BLOCK MATRICES

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Abstract. In this paper, we define extensions of the classical eigenvalues of the matrix $A \in \mathbb{M}_m(\mathbb{C})$. These extensions are eigenvalues matrices for the block matrix $A \in \mathbb{M}_m(\mathbb{M}_n)$, where $\mathbb{M}_m(\mathbb{M}_n)$ is the set of all $m \times m$ block complex matrices with each block in $\mathbb{M}_n(\mathbb{C})$, they are

$$\lambda_h^{(1)}(A) = [\lambda_h(G_{l,k})]_{l,k=1}^n, \quad \text{for } h = 1,2,\cdots,m$$

and

$$\lambda_h^{(2)}(A) = [\lambda_h(A_{i,j})]_{i,j=1}^m, \quad \text{for } h = 1,2,\cdots,n.$$

Among other equalities and inequalities, we prove equalities which relate our new definitions with $\mathrm{tr}_1(A)$ and $\mathrm{tr}_2(A)$ as follows,

$$\mathrm{tr}_1(A) = \sum_{h=1}^m \lambda_h^{(1)}(A)$$

and

$$\mathrm{tr}_2(A) = \sum_{h=1}^n \lambda_h^{(2)}(A).$$

These relations are extensions of the classical relation $\mathrm{tr}(A) = \sum_{h=1}^m \lambda_h(A)$, where $A \in \mathbb{M}_m(\mathbb{C})$. Several new relations and properties of our new definitions are also given.

1. Introduction

Let $\mathbb{M}_m(\mathbb{M}_n)$ be the set of all $m \times m$ block complex matrices with each block in $\mathbb{M}_n(\mathbb{C})$. In the special case, $\mathbb{M}_m(\mathbb{C})$ is the set of all $m \times m$ complex matrices. Recall that the matrix $A \in \mathbb{M}_m(\mathbb{C})$ is called positive semidefinite and it is denoted by $A \geqslant 0$ if $x^T Ax \geqslant 0$ for all $x \in \mathbb{C}^n$. Let $A \in \mathbb{M}_m(\mathbb{C})$, then A^T and A^* denote, respectively, the transpose and conjugate transpose of A. For $A \in \mathbb{M}_m(\mathbb{C})$, the absolute value of A is denoted by $|A| = (A^*A)^{\frac{1}{2}}$. For $A \in \mathbb{M}_m(\mathbb{C})$, let $s_1(A) \geqslant \cdots \geqslant s_m(A)$ be the singular values of A which are the eigenvalues of $|A|$. It should be mentioned here that $s_j(A) = s_j(A^*) = s_j(|A|)$ and $s_j(kA) = |k|s_j(A)$, $k \in \mathbb{C}$. To get acquainted with the characteristics of singular values and recent inequalities on this topic, we advise young authors to read $[1-11]$. We can measure the matrix by several measures, one of them is the unitarily invariant norm denoted by $|||.|||$, which satisfy

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$|||UAV||| = |||A|||$ for any $A \in \mathbb{M}_m(\mathbb{C})$ and any unitary matrices U and V. Typical example of unitarily invariant norms is the spectral (usual operator) norm denoted by $||.||$, where $||\, A\, || = \sup_{||x||=||y||=1} |\langle Ax, y\rangle|$. Given $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. Xu and Fu in [13] define the transpose and partial transpose of A, respectively, by $A^T = [A_{j,i}^T]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ and $A^\tau = [A_{j,i}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. Recently, [15], some attractive definitions are presented:

$$\mathrm{tr}_1(A) = \sum_{i=1}^m A_{i,i} \quad \text{and} \quad \mathrm{tr}_2(A) = [\mathrm{tr} A_{i,j}]_{i,j=1}^m$$

where $\mathrm{tr}_1(A) \in \mathbb{M}_n(\mathbb{C})$ and $\mathrm{tr}_2(A) \in \mathbb{M}_m(\mathbb{C})$.

The authors in [16] present some new definitions, where if $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$, $A_{i,j} = [a_{l,k}^{i,j}]_{l,k=1}^n$, then

1. $G_{l,k} = [a_{l,k}^{i,j}]_{i,j=1}^m$.
2. $\det_1(A) = [\det(G_{l,k})]_{l,k=1}^n$.
3. $\det_2(A) = [\det(A_{i,j})]_{i,j=1}^m$.
4. For $A = [[a_{l,k}^{i,j}]_{l,k=1}^n]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$, the matrix $\tilde{A} \in \mathbb{M}_n(\mathbb{M}_m)$ is defined as follows: $\tilde{A} = [G_{l,k}]_{l,k=1}^n = [[a_{l,k}^{i,j}]_{i,j=1}^m]_{l,k=1}^n$.

It is shown in [16] that A and $\tilde{A}$ are unitarily similar. Note that

1. $\tilde{\tilde{A}} = A$.
2. $\det_1(A) = \det_2(\tilde{A})$.
3. $\det_2(A) = \det_1(\tilde{A})$.

Tao has proved in[18] that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$, then

$$2s_j(B) \leqslant s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right), \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{1.1}$$

Audeh and Kittaneh in [11] have proved that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$, then

$$s_j(B) \leqslant s_j(A \oplus C), \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{1.2}$$

In this paper, we define an extensions of the classical eigenvalues of the matrix $A \in \mathbb{M}_m(\mathbb{C})$. These extensions are an eigenvalues matrices for the block matrix $A \in \mathbb{M}_m(\mathbb{M}_n)$. The classical relation that relate $\mathrm{tr}(A)$ with $\lambda_h(A)$ is

$$\mathrm{tr}(A) = \sum_{h=1}^m \lambda_h(A) \tag{1.3}$$

for $A \in \mathbb{M}_m(\mathbb{C})$. We prove generalizations of equality (1.3). Several applications and relations are also provided.

2. Main result

DEFINITION 2.1. Let $A = [A_{i,j}]_{i,j=1}^{m} \in \mathbb{M}_m(\mathbb{M}_n)$, we define

$$\lambda_h^{(1)}(A) = [\lambda_h(G_{l,k})]_{l,k=1}^{n}, \quad \text{for} \quad h = 1,2,\cdots,m$$

and

$$\lambda_h^{(2)}(A) = [\lambda_h(A_{i,j})]_{i,j=1}^{m}, \quad \text{for} \quad h = 1,2,\cdots,n.$$

It is obvious that $\lambda_h^{(1)}(A) \in \mathbb{M}_n$ and $\lambda_h^{(2)}(A) \in \mathbb{M}_m$. Also, it is clear that

$$\lambda_h^{(1)}(A) = \lambda_h^{(2)}(\tilde{A}). \tag{2.1}$$

$$\lambda_h^{(2)}(A) = \lambda_h^{(1)}(\tilde{A}). \tag{2.2}$$

Now, we will generalize equality (1.3) to a one which relates $\mathrm{tr}_1(A)$ with $\lambda_h^{(1)}(A)$. This new equality is an amazing generalization of equality (1.3).

THEOREM 2.2. *Let $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. Then*

$$\mathrm{tr}_1(A) = \sum_{h=1}^{m} \lambda_h^{(1)}(A). \tag{2.3}$$

Proof.

$$\begin{aligned}
\mathrm{tr}_1(A) &= \sum_{i=1}^{m} A_{i,i} = [\mathrm{tr}(G_{l,k})]_{l,k=1}^{n} \\
&= \begin{bmatrix} \mathrm{tr}(G_{1,1}) & \mathrm{tr}(G_{1,2}) & \cdots & \mathrm{tr}(G_{1,n}) \\ \mathrm{tr}(G_{2,1}) & \mathrm{tr}(G_{2,2}) & \cdots & \mathrm{tr}(G_{2,n}) \\ \vdots & \vdots & \ldots & \vdots \\ \mathrm{tr}(G_{n,1}) & \mathrm{tr}(G_{n,2}) & \cdots & \mathrm{tr}(G_{n,n}) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{h=1}^{m} \lambda_h(G_{1,1}) & \sum_{h=1}^{m} \lambda_h(G_{1,2}) & \cdots & \sum_{h=1}^{m} \lambda_h(G_{1,n}) \\ \sum_{h=1}^{m} \lambda_h(G_{2,1}) & \sum_{h=1}^{m} \lambda_h(G_{2,2}) & \cdots & \sum_{h=1}^{m} \lambda_h(G_{2,n}) \\ \vdots & \vdots & \ldots & \vdots \\ \sum_{h=1}^{m} \lambda_h(G_{n,1}) & \sum_{h=1}^{m} \lambda_h(G_{n,2}) & \cdots & \sum_{h=1}^{m} \lambda_h(G_{n,n}) \end{bmatrix} \\
&= \sum_{h=1}^{m} \begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) & \cdots & \lambda_h(G_{1,n}) \\ \lambda_h(G_{2,1}) & \lambda_h(G_{2,2}) & \cdots & \lambda_h(G_{2,n}) \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_h(G_{n,1}) & \lambda_h(G_{n,2}) & \cdots & \lambda_h(G_{n,n}) \end{bmatrix} \\
&= \sum_{h=1}^{m} [\lambda_h(G_{l,k})]_{l,k=1}^{n} = \sum_{h=1}^{m} \lambda_h^{(1)}(A). \quad \square
\end{aligned}$$

Another attractive generalization of equality (1.3) is the following theorem which relates $\mathrm{tr}_2(A)$ with $\lambda_h^{(2)}(A)$.

THEOREM 2.3. *Let* $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. *Then*

$$\operatorname{tr}_2(A) = \sum_{h=1}^{n} \lambda_h^{(2)}(A). \tag{2.4}$$

Proof.

$$\begin{aligned}
\operatorname{tr}_2(A) &= [\operatorname{tr}(A_{i,j})]_{i,j=1}^m \\
&= \begin{bmatrix} \operatorname{tr}(A_{1,1}) & \operatorname{tr}(A_{1,2}) & \cdots & \operatorname{tr}(A_{1,m}) \\ \operatorname{tr}(A_{2,1}) & \operatorname{tr}(A_{2,2}) & \cdots & \operatorname{tr}(A_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \operatorname{tr}(A_{m,1}) & \operatorname{tr}(A_{m,2}) & \cdots & \operatorname{tr}(A_{m,m}) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{h=1}^n \lambda_h(A_{1,1}) & \sum_{h=1}^n \lambda_h(A_{1,2}) & \cdots & \sum_{h=1}^n \lambda_h(A_{1,m}) \\ \sum_{h=1}^n \lambda_h(A_{2,1}) & \sum_{h=1}^n \lambda_h(A_{2,2}) & \cdots & \sum_{h=1}^n \lambda_h(A_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=1}^n \lambda_h(A_{m,1}) & \sum_{h=1}^n \lambda_h(A_{m,2}) & \cdots & \sum_{h=1}^n \lambda_h(A_{m,m}) \end{bmatrix} \\
&= \sum_{h=1}^{n} \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_h(A_{m,1}) & \lambda_h(A_{m,2}) & \cdots & \lambda_h(A_{m,m}) \end{bmatrix} \\
&= \sum_{h=1}^{n} [\lambda_h(A_{i,j})]_{i,j=1}^m = \sum_{h=1}^{n} \lambda_h^{(2)}(A). \qquad \square
\end{aligned}$$

Interesting properties of $\lambda_h^{(1)}(A)$ and $\lambda_h^{(2)}(A)$ are listed below.

THEOREM 2.4. *If* $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$, *then*

$$\lambda_h^{(1)}(A^\tau) = \lambda_h^{(1)}(A) \quad \textit{and} \quad \lambda_h^{(2)}(A^T) = \lambda_h^{(2)}(A^\tau) = (\lambda_h^{(2)}(A))^T. \tag{2.5}$$

Proof. Since $A^\tau = [A_{j,i}]_{i,j=1}^m$ and $\tilde{A} = [G_{l,k}]_{l,k=1}^n$, we have

$$\begin{aligned}
\lambda_h^{(1)}(A^\tau) &= \lambda_h^{(2)}(\tilde{A^\tau}) \\
&= [\lambda_h(G_{l,k}^T)]_{l,k=1}^n \\
&= [\lambda_h[a_{l,k}^{j,i}]_{i,j=1}^m]_{l,k=1}^n \\
&= [\lambda_h[a_{l,k}^{i,j}]_{i,j=1}^m]_{l,k=1}^n \quad (\text{because } \lambda_h(A) = \lambda_h(A^T)) \\
&= [\lambda_h(G_{l,k})]_{l,k=1}^n \\
&= \lambda_h^{(1)}(A).
\end{aligned}$$

Also,

$$\begin{aligned}\lambda_h^{(2)}(A^T) &= [\lambda_h(A^T_{j,i})]_{i,j=1}^m \\ &= [\lambda_h[a^{j,i}_{k,l}]_{l,k=1}^n]_{i,j=1}^m \\ &= [\lambda_h[a^{j,i}_{l,k}]_{l,k=1}^n]_{i,j=1}^m \quad \text{(because } \lambda_h(A)=\lambda_h(A^T)\text{)} \\ &= \lambda_h^{(2)}(A^\tau) \\ &= [\lambda_h(A_{j,i})]_{i,j=1}^m \\ &= ([\lambda_h(A_{i,j})]_{i,j=1}^m)^T \\ &= (\lambda_h^{(2)}(A))^T. \quad \square\end{aligned}$$

THEOREM 2.5. *If* $A=[A_{i,j}]\in\mathbb{M}_m(\mathbb{M}_n)$ *is Hermitian matrix, then* $\lambda_h^{(1)}(A)$ *and* $\lambda_h^{(2)}(A)$ *are Hermitian matrices.*

Proof. If $A=[A_{i,j}]\in\mathbb{M}_m(\mathbb{M}_n)$ is Hermitian matrix, then $\tilde{A}$ is Hermitian (since A and $\tilde{A}$ are unitarily similar). This implies that

$$A=\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A^*_{1,2} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \cdots & \vdots \\ A^*_{1,m} & A^*_{2,m} & \cdots & A_{m,m}\end{bmatrix} \quad \text{and} \quad \tilde{A}=\begin{bmatrix} G_{1,1} & G_{1,2} & \cdots & G_{1,n} \\ G^*_{1,2} & G_{2,2} & \cdots & G_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ G^*_{1,n} & G^*_{2,n} & \cdots & G_{n,n}\end{bmatrix}.$$

Then

$$\lambda_h^{(1)}(A)=\begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) & \cdots & \lambda_h(G_{1,n}) \\ \overline{\lambda_h(G_{1,2})} & \lambda_h(G_{2,2}) & \cdots & \lambda_h(G_{2,n}) \\ \vdots & \vdots & \cdots & \vdots \\ \overline{\lambda_h(G_{1,n})} & \overline{\lambda_h(G_{2,n})} & \cdots & \lambda_h(G_{n,n})\end{bmatrix}$$

and

$$\lambda_h^{(2)}(A)=\begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \overline{\lambda_h(A_{1,2})} & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \overline{\lambda_h(A_{1,m})} & \overline{\lambda_h(A_{2,m})} & \cdots & \lambda_h(A_{m,m})\end{bmatrix}.$$

This shows that $\lambda_h^{(1)}(A)$ and $\lambda_h^{(2)}(A)$ are Hermitian matrices. $\square$

This following lemma is essential for our next conclusions. This lemma is given in [18].

LEMMA 2.6. *Let* $A,B,C,D\in\mathbb{M}_n(\mathbb{C})$. *Then*

$$\left|\left|\left|\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right|\right|\right| \leqslant \left|\left|\left|\begin{bmatrix} ||A|| & ||B|| \\ ||C|| & ||D||\end{bmatrix}\right|\right|\right|. \tag{2.6}$$

THEOREM 2.7. *Let* $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. *Then*

$$||\lambda_h^{(1)}(A)|| \leqslant \left|\left| \begin{bmatrix} |\lambda_h(G_{1,1})| & |\lambda_h(G_{1,2})| & \cdots & |\lambda_h(G_{1,n})| \\ |\lambda_h(G_{2,1})| & |\lambda_h(G_{2,2})| & \cdots & |\lambda_h(G_{2,n})| \\ \vdots & \vdots & \cdots & \vdots \\ |\lambda_h(G_{n,1})| & |\lambda_h(G_{n,2})| & \cdots & |\lambda_h(G_{n,n})| \end{bmatrix} \right|\right|.$$

In the special case where $n = 2$, *we give*

$$||\lambda_h^{(1)}(A)|| \leqslant \frac{1}{2}\left(|\lambda_h(G_{1,1})| + |\lambda_h(G_{2,2})|\right) + \frac{1}{2}\sqrt{(|\lambda_h(G_{1,1})| - |\lambda_h(G_{2,2})|)^2 + 4|\lambda_h(G_{1,2})||\lambda_h(G_{2,1})|}.$$

Proof. If $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$, then

$$||\lambda_h^{(1)}(A)|| = ||[\lambda_h(G_{l,k})]_{l,k=1}^n||$$

$$= \left|\left| \begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) & \cdots & \lambda_h(G_{1,n}) \\ \lambda_h(G_{2,1}) & \lambda_h(G_{2,2}) & \cdots & \lambda_h(G_{2,n}) \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_h(G_{n,1}) & \lambda_h(G_{n,2}) & \cdots & \lambda_h(G_{n,n}) \end{bmatrix} \right|\right|$$

$$\leqslant \left|\left| \begin{bmatrix} |\lambda_h(G_{1,1})| & |\lambda_h(G_{1,2})| & \cdots & |\lambda_h(G_{1,n})| \\ |\lambda_h(G_{2,1})| & |\lambda_h(G_{2,2})| & \cdots & |\lambda_h(G_{2,n})| \\ \vdots & \vdots & \cdots & \vdots \\ |\lambda_h(G_{n,1})| & |\lambda_h(G_{n,2})| & \cdots & |\lambda_h(G_{n,n})| \end{bmatrix} \right|\right| \quad \text{(by inequality (2.6))}.$$

In the special case where $n = 2$, we give

$$||\lambda_h^{(1)}(A)|| = \left|\left| \begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) \\ \lambda_h(G_{2,1}) & \lambda_h(G_{2,2}) \end{bmatrix} \right|\right|$$

$$\leqslant \left|\left| \begin{bmatrix} |\lambda_h(G_{1,1})| & |\lambda_h(G_{1,2})| \\ |\lambda_h(G_{2,1})| & |\lambda_h(G_{2,2})| \end{bmatrix} \right|\right| \quad \text{(by inequality (2.6))}$$

$$= \frac{1}{2}\left(|\lambda_h(G_{1,1})| + |\lambda_h(G_{2,2})|\right) + \frac{1}{2}\sqrt{(|\lambda_h(G_{1,1})| - |\lambda_h(G_{2,2})|)^2 + 4|\lambda_h(G_{1,2})||\lambda_h(G_{2,1})|}. \quad \square$$

THEOREM 2.8. *Let* $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. *Then*

$$||\lambda_h^{(2)}(A)|| \leqslant \left|\left| \begin{bmatrix} |\lambda_h(A_{1,1})| & |\lambda_h(A_{1,2})| & \cdots & |\lambda_h(A_{1,m})| \\ |\lambda_h(A_{2,1})| & |\lambda_h(A_{2,2})| & \cdots & |\lambda_h(A_{2,m})| \\ \vdots & \vdots & \cdots & \vdots \\ |\lambda_h(A_{m,1})| & |\lambda_h(A_{m,2})| & \cdots & |\lambda_h(A_{m,m})| \end{bmatrix} \right|\right|.$$

In the special case where $m=2$, we give

$$||\lambda_h^{(2)}(A)|| \leqslant \frac{1}{2}\left(|\lambda_h(A_{1,1})|+|\lambda_h(A_{2,2})|\right) \\ +\frac{1}{2}\sqrt{(|\lambda_h(A_{1,1})|-|\lambda_h(A_{2,2})|)^2+4\,|\lambda_h(A_{1,2})|\,|\lambda_h(A_{2,1})|}.$$

Proof. If $A=[A_{i,j}]\in\mathbb{M}_m(\mathbb{M}_n)$, then

$$||\lambda_h^{(2)}(A)|| = ||[\lambda_h(A_{i,j})]_{i,j=1}^m||$$

$$= \left|\left|\begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_h(A_{m,1}) & \lambda_h(A_{m,2}) & \cdots & \lambda_h(A_{m,m}) \end{bmatrix}\right|\right|$$

$$\leqslant \left|\left|\begin{bmatrix} |\lambda_h(A_{1,1})| & |\lambda_h(A_{1,2})| & \cdots & |\lambda_h(A_{1,m})| \\ |\lambda_h(A_{2,1})| & |\lambda_h(A_{2,2})| & \cdots & |\lambda_h(A_{2,m})| \\ \vdots & \vdots & \ldots & \vdots \\ |\lambda_h(A_{m,1})| & |\lambda_h(A_{m,2})| & \cdots & |\lambda_h(A_{m,m})| \end{bmatrix}\right|\right| \quad \text{(by inequality (2.6))}.$$

In the special case where $m=2$, we give

$$||\lambda_h^{(2)}(A)|| = \left|\left|\begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) \end{bmatrix}\right|\right|$$

$$\leqslant \left|\left|\begin{bmatrix} |\lambda_h(A_{1,1})| & |\lambda_h(A_{1,2})| \\ |\lambda_h(A_{2,1})| & |\lambda_h(A_{2,2})| \end{bmatrix}\right|\right| \quad \text{(by inequality (2.6))}$$

$$= \frac{1}{2}\left(|\lambda_h(A_{1,1})|+|\lambda_h(A_{2,2})|\right)$$

$$+\frac{1}{2}\sqrt{(|\lambda_h(A_{1,1})|-|\lambda_h(A_{2,2})|)^2+4\,|\lambda_h(A_{1,2})|\,|\lambda_h(A_{2,1})|}. \qquad \square$$

THEOREM 2.9. *Let $A=[A_{i,j}]\in\mathbb{M}_n(\mathbb{M}_2)$. Then the eigenvalues of $\lambda_h^{(1)}(A)$ are denoted by $\lambda_j(\lambda_h^{(1)}(A))$ where*

$$\lambda_j(\lambda_h^{(1)}(A)) = \frac{\mathrm{tr}(\lambda_h^{(1)}(A)) \pm \sqrt{(\mathrm{tr}\lambda_h^{(1)}(A))^2-4\det(\lambda_h^{(1)}(A))}}{2}, \quad \textit{for} \quad j=1,2. \tag{2.7}$$

Proof. If $A = [A_{i,j}] \in \mathbb{M}_n(\mathbb{M}_2)$, then $\lambda_h^{(1)}(A) = \begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) \\ \lambda_h(G_{2,1}) & \lambda_h(G_{2,2}) \end{bmatrix}$. The eigenvalues of this matrix are:

$$\lambda_j(\lambda_h^{(1)}(A)) = \det(\lambda I - \lambda_h^{(1)}(A))$$

$$= \det \begin{bmatrix} \lambda - \lambda_h(G_{1,1}) & -\lambda_h(G_{1,2}) \\ -\lambda_h(G_{2,1}) & \lambda - \lambda_h(G_{2,2}) \end{bmatrix}$$

$$= (\lambda - \lambda_h(G_{1,1}))(\lambda - \lambda_h(G_{2,2})) - (\lambda_h(G_{2,1})\lambda_h(G_{1,2}))$$

$$= \lambda^2 - \lambda(\mathrm{tr}(\lambda_h^{(1)}(A))) + \det(\lambda_h^{(1)}(A)) = 0.$$

Thus

$$\lambda_j(\lambda_h^{(1)}(A)) = \frac{\mathrm{tr}(\lambda_h^{(1)}(A)) \pm \sqrt{(\mathrm{tr}\lambda_h^{(1)}(A))^2 - 4\det(\lambda_h^{(1)}(A))}}{2}. \quad \square$$

THEOREM 2.10. *Let $A = [A_{i,j}] \in \mathbb{M}_2(\mathbb{M}_n)$. Then the eigenvalues of $\lambda_h^{(2)}(A)$ are denoted by $\lambda_j(\lambda_h^{(2)}(A))$, where*

$$\lambda_j(\lambda_h^{(2)}(A)) = \frac{\mathrm{tr}(\lambda_h^{(2)}(A)) \pm \sqrt{(\mathrm{tr}(\lambda_h^{(2)}(A)))^2 - 4\det(\lambda_h^{(2)}(A))}}{2}, \quad \textit{for} \quad j = 1,2. \tag{2.8}$$

Proof. Let $A = [A_{i,j}] \in \mathbb{M}_2(\mathbb{M}_n)$. Then $\lambda_h^{(2)}(A) = \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) \end{bmatrix}$. The eigenvalues of this matrix are:

$$\lambda_j(\lambda_h^{(2)}(A)) = \det(\lambda I - \lambda_h^{(2)}(A))$$

$$= \det \begin{bmatrix} \lambda - \lambda_h(A_{1,1}) & -\lambda_h(A_{1,2}) \\ -\lambda_h(A_{2,1}) & \lambda - \lambda_h(A_{2,2}) \end{bmatrix}$$

$$= (\lambda - \lambda_h(A_{1,1}))(\lambda - \lambda_h(A_{2,2})) - (\lambda_h(A_{2,1})\lambda_h(A_{1,2}))$$

$$= \lambda^2 - \lambda(\mathrm{tr}(\lambda_h^{(2)}(A))) + \det(\lambda_h^{(2)}(A)) = 0.$$

Thus

$$\lambda_j(\lambda_h^{(2)}(A)) = \frac{\mathrm{tr}(\lambda_h^{(2)}(A)) \pm \sqrt{(\mathrm{tr}(\lambda_h^{(2)}(A)))^2 - 4\det(\lambda_h^{(2)}(A))}}{2}. \quad \square$$

THEOREM 2.11. *Let $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$ and $B = [B_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. Then*

$$\lambda_h^{(2)}(AB^T) = (\lambda_h^{(2)}(BA^T))^T. \tag{2.9}$$

Proof. The equality (2.9) can be reached directly by applying equality (2.5). □

Bhatia in [12] present the definition of Hadamard product of matrices, where if $A = [A_{i,j}]$ and $B = [B_{i,j}]$ are matrices of the same size, the Hadamard product of A and B, denoted by $A \, o \, B$, is the matrix $[A_{i,j}B_{i,j}]$, $A^{o^n} = [A^n_{i,j}]$ and $A^{o^{-1}} = [A^{-1}_{i,j}]$.

The author in [17], shows that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\det(A \, o \, B) \geqslant \det(A)\det(B). \tag{2.10}$$

DEFINITION 2.12. If $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$, then $\hat{A} = [A^*_{i,j}]^m_{i,j=1}$.

THEOREM 2.13. *Let* $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$ *and* $\hat{A} = [A^*_{i,j}]^m_{i,j=1}$. *Then*

$$\lambda^{(2)}_h(A \, o \, \hat{A}) = \lambda^{(2)}_h(\hat{A} \, o \, A).$$

Proof. Let $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$ and $\hat{A} = [A^*_{i,j}]^m_{i,j=1}$. Then

$$\lambda^{(2)}_h(A \, o \, \hat{A}) = \lambda^{(2)}_h\left(\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{bmatrix} o \begin{bmatrix} A^*_{1,1} & A^*_{1,2} & \cdots & A^*_{1,m} \\ A^*_{2,1} & A^*_{2,2} & \cdots & A^*_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A^*_{m,1} & A^*_{m,2} & \cdots & A^*_{m,m} \end{bmatrix}\right)$$

$$= \lambda^{(2)}_h\left(\begin{bmatrix} A_{1,1}A^*_{1,1} & A_{1,2}A^*_{1,2} & \cdots & A_{1,m}A^*_{1,m} \\ A_{2,1}A^*_{2,1} & A_{2,2}A^*_{2,2} & \cdots & A_{2,m}A^*_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1}A^*_{m,1} & A_{m,2}A^*_{m,2} & \cdots & A_{m,m}A^*_{m,m} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \lambda_h(A_{1,1}A^*_{1,1}) & \lambda_h(A_{1,2}A^*_{1,2}) & \cdots & \lambda_h(A_{1,m}A^*_{1,m}) \\ \lambda_h(A_{2,1}A^*_{2,1}) & \lambda_h(A_{2,2}A^*_{2,2}) & \cdots & \lambda_h(A_{2,m}A^*_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_h(A_{m,1}A^*_{m,1}) & \lambda_h(A_{m,2}A^*_{m,2}) & \cdots & \lambda_h(A_{m,m}A^*_{m,m}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_h(A^*_{1,1}A_{1,1}) & \lambda_h(A^*_{1,2}A_{1,2}) & \cdots & \lambda_h(A^*_{1,m}A_{1,m}) \\ \lambda_h(A^*_{2,1}A_{2,1}) & \lambda_h(A^*_{2,2}A_{2,2}) & \cdots & \lambda_h(A^*_{2,m}A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_h(A^*_{m,1}A_{m,1}) & \lambda_h(A^*_{m,2}A_{m,2}) & \cdots & \lambda_h(A^*_{m,m}A_{m,m}) \end{bmatrix}$$

$$= \lambda^{(2)}_h\left(\begin{bmatrix} A^*_{1,1} & A^*_{1,2} & \cdots & A^*_{1,m} \\ A^*_{2,1} & A^*_{2,2} & \cdots & A^*_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A^*_{m,1} & A^*_{m,2} & \cdots & A^*_{m,m} \end{bmatrix} o \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{bmatrix}\right) = \lambda^{(2)}_h(\hat{A} \, o \, A).$$ □

THEOREM 2.14. *Let $A=[A_{i,j}]\in\mathbb{M}_m(\mathbb{M}_n)$. Then*

$$\det_1(A)=o_{h=1}^{m}\lambda_h^{(1)}(A). \tag{2.11}$$

If $\lambda_h^{(1)}(A)$ is a positive semidefinite for all $h=1,2,\cdots,m$, then

$$\det(\det_1 A)\geqslant\prod_{h=1}^{m}\det(\lambda_h^{(1)}(A)).$$

Proof.

$$\det_1(A)=[\det(G_{l,k})]_{l,k=1}^{n}$$

$$=\begin{bmatrix}\det(G_{1,1}) & \det(G_{1,2}) & \cdots & \det(G_{1,n})\\ \det(G_{2,1}) & \det(G_{2,2}) & \cdots & \det(G_{2,m})\\ \vdots & \vdots & \cdots & \vdots\\ \det(G_{n,1}) & \det(G_{n,2}) & \cdots & \det(G_{n,n})\end{bmatrix}$$

$$=\begin{bmatrix}\prod_{h=1}^{m}\lambda_h(G_{1,1}) & \prod_{h=1}^{m}\lambda_h(G_{1,2}) & \cdots & \prod_{h=1}^{m}\lambda_h(G_{1,n})\\ \prod_{h=1}^{m}\lambda_h(G_{2,1}) & \prod_{h=1}^{m}\lambda_h(G_{2,2}) & \cdots & \prod_{h=1}^{m}\lambda_h(G_{2,n})\\ \vdots & \vdots & \cdots & \vdots\\ \prod_{h=1}^{m}\lambda_h(G_{n,1}) & \prod_{h=1}^{m}\lambda_h(G_{n,2}) & \cdots & \prod_{h=1}^{m}\lambda_h(G_{n,n})\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_1(G_{1,1}) & \lambda_1(G_{1,2}) & \cdots & \lambda_1(G_{1,n})\\ \lambda_1(G_{2,1}) & \lambda_1(G_{2,2}) & \cdots & \lambda_1 2,n)\\ \vdots & \vdots & \cdots & \vdots\\ \lambda_1(G_{n,1}) & \lambda_1(G_{n,2}) & \cdots & \lambda_1(G_{n,n})\end{bmatrix} o \begin{bmatrix}\lambda_2(G_{1,1}) & \lambda_2(G_{1,2}) & \cdots & \lambda_2(G_{1,n})\\ \lambda_2(G_{2,1}) & \lambda_2(G_{2,2}) & \cdots & \lambda_2(G_{2,n})\\ \vdots & \vdots & \cdots & \vdots\\ \lambda_2(G_{n,1}) & \lambda_2(G_{n,2}) & \cdots & \lambda_2(G_{n,n})\end{bmatrix} o\cdots o \begin{bmatrix}\lambda_m(G_{1,1}) & \lambda_m(G_{1,2}) & \cdots & \lambda_m(G_{1,n})\\ \lambda_m(G_{2,1}) & \lambda_m(G_{2,2}) & \cdots & \lambda_m(G_{2,n})\\ \vdots & \vdots & \cdots & \vdots\\ \lambda_m(G_{n,1}) & \lambda_m(G_{n,2}) & \cdots & \lambda_m(G_{n,n})\end{bmatrix}$$

$$=\lambda_1^{(1)}(A)\ o\ \lambda_2^{(1)}(A)\ o\ \lambda_3^{(1)}(A)\ o\ \cdots\ o\ \lambda_m^{(1)}(A)$$

$$=o_{h=1}^{m}\lambda_h^{(1)}(A).$$

Also, if $\lambda_h^{(1)}(A)$ is a positive semidefinite for all $h=1,2,\cdots,m$, then

$$\det(\det_1 A)=\det(o_{h=1}^{m}\lambda_h^{(1)}(A))\geqslant\prod_{h=1}^{m}\det(\lambda_h^{(1)}(A)),\quad\text{(by inequality (2.10))}.\qquad\square$$

THEOREM 2.15. *Let $A=[A_{i,j}]\in\mathbb{M}_m(\mathbb{M}_n)$. Then*

$$\det_2(A)=o_{h=1}^{n}\lambda_h^{(2)}(A). \tag{2.12}$$

If $\lambda_h^{(2)}(A)$ is a positive semidefinite for all $h=1,2,\cdots,n$, then

$$\det(\det_2 A)\geqslant\prod_{h=1}^{n}\det(\lambda_h^{(2)}(A)).$$

Proof.

$$\det_2(A) = [\det(A_{i,j})]_{i,j=1}^{m}$$

$$= \begin{bmatrix} \det(A_{1,1}) & \det(A_{1,2}) & \cdots & \det(A_{1,m}) \\ \det(A_{2,1}) & \det(A_{2,2}) & \cdots & \det(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \det(A_{m,1}) & \det(A_{m,2}) & \cdots & \det(A_{m,m}) \end{bmatrix}$$

$$= \begin{bmatrix} \prod_{h=1}^{n}\lambda_h(A_{1,1}) & \prod_{h=1}^{n}\lambda_h(A_{1,2}) & \cdots & \prod_{h=1}^{n}\lambda_h(A_{1,m}) \\ \prod_{h=1}^{n}\lambda_h(A_{2,1}) & \prod_{h=1}^{n}\lambda_h(A_{2,2}) & \cdots & \prod_{h=1}^{n}\lambda_h(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \prod_{h=1}^{n}\lambda_h(A_{m,1}) & \prod_{h=1}^{n}\lambda_h(A_{m,2}) & \cdots & \prod_{h=1}^{n}\lambda_h(A_{m,m}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1(A_{1,1}) & \lambda_1(A_{1,2}) & \cdots & \lambda_1(A_{1,m}) \\ \lambda_1(A_{2,1}) & \lambda_1(A_{2,2}) & \cdots & \lambda_1(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1(A_{m,1}) & \lambda_1(A_{m,2}) & \cdots & \lambda_1(A_{m,m}) \end{bmatrix} o \begin{bmatrix} \lambda_2(A_{1,1}) & \lambda_2(A_{1,2}) & \cdots & \lambda_2(A_{1,m}) \\ \lambda_2(A_{2,1}) & \lambda_2(A_{2,2}) & \cdots & \lambda_2(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_2(A_{m,1}) & \lambda_2(A_{m,2}) & \cdots & \lambda_2(A_{m,m}) \end{bmatrix} o \cdots o \begin{bmatrix} \lambda_n(A_{1,1}) & \lambda_n(A_{1,2}) & \cdots & \lambda_n(A_{1,m}) \\ \lambda_n(A_{2,1}) & \lambda_n(A_{2,2}) & \cdots & \lambda_n(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_n(A_{m,1}) & \lambda_n(A_{m,2}) & \cdots & \lambda_n(A_{m,m}) \end{bmatrix}$$

$$= \lambda_1^{(2)}(A)\; o\; \lambda_2^{(2)}(A)\; o\; \lambda_3^{(2)}(A)\; o\; \cdots\; o\; \lambda_n^{(2)}(A)$$

$$= o_{h=1}^{n}\lambda_h^{(2)}(A).$$

Also, if $\lambda_h^{(2)}(A)$ is a positive semidefinite for all $h = 1,2,\cdots,n$, then

$$\det(\det_2 A) = \det(o_{h=1}^{n}\lambda_h^{(2)}(A)) \geqslant \prod_{h=1}^{n}\det(\lambda_h^{(2)}(A)) \quad \text{(by inequality (2.10)).} \quad \square$$

THEOREM 2.16. *Let $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$, $S = [S_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$ where $s_{i,j}$ are invertible matrices for $i,j = 1,2,\cdots,m$. Then*

$$\lambda_h^{(2)}(S\; o\; A\; o\; S^{o^{-1}}) = \lambda_h^{(2)}(A) = \lambda_h^{(2)}(S^{o^{-1}}\; o\; A\; o\; S). \tag{2.13}$$

Proof. Let $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$, $S = [S_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$ where $S_{i,j}$ are invertible matrices for $i,j = 1,2,\cdots,m$. Then

$$\lambda_h^{(2)}(S\; o\; A\; o\; S^{o^{-1}}) = \lambda_h^{(2)}\left(\begin{bmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,m} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ S_{m,1} & S_{m,2} & \cdots & S_{m,m} \end{bmatrix} o \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{bmatrix} o \begin{bmatrix} S_{1,1}^{-1} & S_{1,2}^{-1} & \cdots & S_{1,m}^{-1} \\ S_{2,1}^{-1} & S_{2,2}^{-1} & \cdots & S_{2,m}^{-1} \\ \vdots & \vdots & \dots & \vdots \\ S_{m,1}^{-1} & S_{m,2}^{-1} & \cdots & S_{m,m}^{-1} \end{bmatrix} \right)$$

$$= \lambda_h^{(2)}\left(\begin{bmatrix} S_{1,1}A_{1,1}S_{1,1}^{-1} & S_{1,2}A_{1,2}S_{1,2}^{-1} & \cdots & S_{1,m}A_{1,m}S_{1,m}^{-1} \\ S_{2,1}A_{2,1}S_{2,1}^{-1} & S_{2,2}A_{2,2}S_{2,2}^{-1} & \cdots & S_{2,m}A_{2,m}S_{2,m}^{-1} \\ \vdots & \vdots & \dots & \vdots \\ S_{m,1}A_{m,1}S_{m,1}^{-1} & S_{m,2}A_{m,2}S_{m,2}^{-1} & \cdots & S_{m,m}A_{m,m}S_{m,m}^{-1} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \lambda_h(S_{1,1}A_{1,1}S_{1,1}^{-1}) & \lambda_h(S_{1,2}A_{1,2}S_{1,2}^{-1}) & \cdots & \lambda_h(S_{1,m}A_{1,m}S_{1,m}^{-1}) \\ \lambda_h(S_{2,1}A_{2,1}S_{2,1}^{-1}) & \lambda_h(S_{2,2}A_{2,2}S_{2,2)}^{-1} & \cdots & \lambda_h(S_{2,m}A_{2,m}S_{2,m}^{-1}) \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_h(S_{m,1}A_{m,1}S_{m,1}^{-1}) & \lambda_h(S_{m,2}A_{m,2}S_{m,2}^{-1}) & \cdots & \lambda_h(S_{m,m}A_{m,m}S_{m,m}^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_h(A_{m,1}) & \lambda_h(A_{m,2}) & \cdots & \lambda_h(A_{m,m}) \end{bmatrix}$$

(since the eigenvalues of the matrices AB and BA are the same)

$$= \lambda_h^{(2)}(A).$$

Moreover,

$$\lambda_h^{(2)}(S^{o^{-1}}\ o\ A\ o\ S) = \lambda_h^{(2)}\left(\begin{bmatrix} S_{1,1}^{-1} & S_{1,2}^{-1} & \cdots & S_{1,m}^{-1} \\ S_{2,1}^{-1} & S_{2,2}^{-1} & \cdots & S_{2,m}^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m,1}^{-1} & S_{m,2}^{-1} & \cdots & S_{m,m}^{-1} \end{bmatrix} o \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{bmatrix} o \begin{bmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,m} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,m} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m,1} & S_{m,2} & \cdots & S_{m,m} \end{bmatrix}\right)$$

$$= \lambda_h^{(2)}\left(\begin{bmatrix} S_{1,1}^{-1}A_{1,1}S_{1,1} & S_{1,2}^{-1}A_{1,2}S_{1,2} & \cdots & S_{1,m}^{-1}A_{1,m}S_{1,m} \\ S_{2,1}^{-1}A_{2,1}S_{2,1} & S_{2,2}^{-1}A_{2,2}S_{2,2} & \cdots & S_{2,m}^{-1}A_{2,m}S_{2,m} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m,1}^{-1}A_{m,1}S_{m,1} & S_{m,2}^{-1}A_{m,2}S_{m,2} & \cdots & S_{m,m}^{-1}A_{m,m}S_{m,m} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \lambda_h(S_{1,1}^{-1}A_{1,1}S_{1,1}) & \lambda_h(S_{1,2}^{-1}A_{1,2}S_{1,2}) & \cdots & \lambda_h(S_{1,m}^{-1}A_{1,m}S_{1,m}) \\ \lambda_h(S_{2,1}^{-1}A_{2,1}S_{2,1}) & \lambda_h(S_{2,2}^{-1}A_{2,2}S_{2,2)} & \cdots & \lambda_h(S_{2,m}^{-1}A_{2,m}S_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_h(S_{m,1}^{-1}A_{m,1}S_{m,1}) & \lambda_h(S_{m,2}^{-1}A_{m,2}S_{m,2}) & \cdots & \lambda_h(S_{m,m}^{-1}A_{m,m}S_{m,m}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_h(A_{m,1}) & \lambda_h(A_{m,2}) & \cdots & \lambda_h(A_{m,m}) \end{bmatrix} = \lambda_h^{(2)}(A). \quad \square$$

THEOREM 2.17. *Let $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. Then*

$$\lambda_h^{(2)}(A^{o^k}) = (\lambda_h^{(2)}(A))^{o^k}.$$

Proof. If $A = [A_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. Then

$$\lambda_h^{(2)}(A^{o^k}) = \lambda_h^{(2)}(\underbrace{A\ o\ A\ o\ \cdots\ o\ A}_{k\text{ -times}})$$

$$= \lambda_h^{(2)}\left(\underbrace{\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{bmatrix} o \cdots o \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{bmatrix}}_{k\text{ -times}}\right)$$

$$= \lambda_h^{(2)} \begin{bmatrix} A_{1,1}^k & A_{1,2}^k & \cdots & A_{1,m}^k \\ A_{2,1}^k & A_{2,2}^k & \cdots & A_{2,m}^k \\ \vdots & \vdots & \dots & \vdots \\ A_{m,1}^k & A_{m,2}^k & \cdots & A_{m,m}^k \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_h(A_{1,1}^k) & \lambda_h(A_{1,2}^k) & \cdots & \lambda_h(A_{1,m}^k) \\ \lambda_h(A_{2,1}^k) & \lambda_h(A_{2,2}^k) & \cdots & \lambda_h(A_{2,m}^k) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_h(A_{m,1}^k) & \lambda_h(A_{m,2}^k) & \cdots & \lambda_h(A_{m,m}^k) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_h^k(A_{1,1}) & \lambda_h^k(A_{1,2}) & \cdots & \lambda_h^k(A_{1,m}) \\ \lambda_h^k(A_{2,1}) & \lambda_h^k(A_{2,2}) & \cdots & \lambda_h^k(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_h^k(A_{m,1}) & \lambda_h^k(A_{m,2}) & \cdots & \lambda_h^k(A_{m,m}) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_h(A_{m,1}) & \lambda_h(A_{m,2}) & \cdots & \lambda_h(A_{m,m}) \end{bmatrix} o \cdots o \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) & \cdots & \lambda_h(A_{1,m}) \\ \lambda_h(A_{2,1}) & \lambda_h(A_{2,2}) & \cdots & \lambda_h(A_{2,m}) \\ \vdots & \vdots & \dots & \vdots \\ \lambda_h(A_{m,1}) & \lambda_h(A_{m,2}) & \cdots & \lambda_h(A_{m,m}) \end{bmatrix}}_{k\text{ -times}}$$

$$= \underbrace{\lambda_h^{(2)}(A)\ o\ \lambda_h^{(2)}(A)\ o\ \cdots\ o\ \lambda_h^{(2)}(A)}_{k\text{ -times}} = (\lambda_h^{(2)}(A))^{o^k}. \quad \square$$

THEOREM 2.18. *Let* $A = [A_{i,j}] \in \mathbb{M}_2(\mathbb{M}_n)$ *and* $\lambda_h^{(2)}(A)$ *are positive semidefinite. Then*

1. $2|\lambda_h(A_{1,2})| \leqslant s_j\left(\begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) \\ \overline{\lambda_h(A_{1,2})} & \lambda_h(A_{2,2}) \end{bmatrix}\right)$, *for* $j = 1,2$ *and* $h = 1,2,\cdots,n$.
2. $|\lambda_h(A_{1,2})| \leqslant max\{\lambda_h(A_{1,1}), \lambda_h(A_{2,2})\}$.

Proof. If $A = [A_{i,j}] \in \mathbb{M}_2(\mathbb{M}_n)$ and $\lambda_h^{(2)}(A)$ are positive semidefinite, then A is Hermitian. This implies that $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^* & A_{2,2} \end{bmatrix}$. Now, $\lambda_h^{(2)}(A) = \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) \\ \lambda_h(A_{1,2}^*) & \lambda_h(A_{2,2}) \end{bmatrix} = \begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) \\ \overline{\lambda_h(A_{1,2})} & \lambda_h(A_{2,2}) \end{bmatrix} \geqslant 0$. Hence, by inequality (1.1), we get

$$2|\lambda_h(A_{1,2})| \leqslant s_j\left(\begin{bmatrix} \lambda_h(A_{1,1}) & \lambda_h(A_{1,2}) \\ \overline{\lambda_h(A_{1,2})} & \lambda_h(A_{2,2}) \end{bmatrix}\right), \quad \text{for } j = 1,2 \text{ and } h = 1,2,\cdots,n. \tag{2.14}$$

Also, by inequality (1.2), we give

$$|\lambda_h(A_{1,2})| \leqslant max\{\lambda_h(A_{1,1}), \lambda_h(A_{2,2})\}. \quad \square \tag{2.15}$$

COROLLARY 2.19. *Let $\tilde{A} = [G_{l,k}] \in \mathbb{M}_2(\mathbb{M}_m)$ and $\lambda_h^{(1)}(A)$ are positive semidefinite. Then*

1. $2|\lambda_h(G_{1,2})| \leqslant s_j\left(\begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) \\ \overline{\lambda_h(G_{1,2})} & \lambda_h(G_{2,2}) \end{bmatrix}\right)$, *for* $j = 1,2$ *and* $h = 1,2,\cdots,m$.
2. $|\lambda_h(G_{1,2})| \leqslant max\{\lambda_h(G_{1,1}), \lambda_h(G_{2,2})\}$.

Proof. If $\tilde{A} = [G_{l,k}] \in \mathbb{M}_2(\mathbb{M}_m)$ and $\lambda_h^{(1)}(A)$ are positive semidefinite, then $\tilde{A}$ is Hermitian. This implies that $\tilde{A} = \begin{bmatrix} G_{1,1} & G_{1,2} \\ G_{1,2}^* & G_{2,2} \end{bmatrix}$. Now, $\lambda_h^{(1)}(A) = \begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) \\ \lambda_h(G_{1,2}^*) & \lambda_h(G_{2,2}) \end{bmatrix} = \begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) \\ \overline{\lambda_h(G_{1,2})} & \lambda_h(G_{2,2}) \end{bmatrix} \geqslant 0$. Hence, by inequality (1.1), we get

$$2|\lambda_h(G_{1,2})| \leqslant s_j\left(\begin{bmatrix} \lambda_h(G_{1,1}) & \lambda_h(G_{1,2}) \\ \overline{\lambda_h(G_{1,2})} & \lambda_h(G_{2,2}) \end{bmatrix}\right), \quad \text{for } j = 1,2 \text{ and } h = 1,2,\cdots,m. \tag{2.16}$$

Also, by inequality (1.2), we get

$$|\lambda_h(G_{1,2})| \leqslant max\{\lambda_h(G_{1,1}), \lambda_h(G_{2,2})\}. \quad \square \tag{2.17}$$

Data availability statement

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