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INEQUALITIES FOR POWER SERIES OF PRODUCT OF OPERATORS IN HILBERT SPACES WITH APPLICATIONS TO NUMERICAL RADIUS

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Abstract. Let H be a complex Hilbert space. We consider the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0,1,\ldots\}$. Suppose that this power series is convergent on the open disk $D(0,R) := \{z \in \mathbb{C} | \, z < R\}$. We define $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$, which has the same radius of convergence R. In this paper, we show among others that, if the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$ and A, B, $C, D \in B(H)$ with $\|AB\| < R$, then the following vector inequality holds

$$\begin{aligned} &|\langle D^* AB f(AB) Cx, y\rangle| \\ &\leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \end{aligned}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Application for norm and numerical radius inequalities for the composite operator $D^* AB f(AB) C$ are provided. Some examples for fundamental power series are also given.

1. Introduction

The main aim of the work is to develop vector and numerical radius inequalities for functions defined by power series of bounded linear operators on a complex Hilbert space H. The study of numerical range and the inequalities for numerical radius is useful in investigating many properties of linear operators and has various applications in numerous fields of sciences such as, see [2], application in quantum information theory, in particular, quantum error correction, additive uncertainty relations, multi-observable quantum uncertainty relations etc... By making use of the numerical radius inequalities one can also estimate the roots of polynomials using the notion of the Frobenius companion matrix, see [3] and [5].

In the following, we recall some fundaments notions and present some facts that will be used in this work.

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1}$$

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Obviously, by (1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leqslant w(T)\|x\|^2 . \tag{2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathscr{B}(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geqslant 0$ for any $T \in \mathscr{B}(H)$ and $\omega(T) = 0$ if and only if $T = 0$;

(ii) $\omega(\lambda T) = |\lambda|\,\omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in \mathscr{B}(H)$;

(iii) $\omega(T+V) \leqslant \omega(T) + \omega(V)$ for any $T, V \in \mathscr{B}(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leqslant \|T\| \leqslant 2\omega(T) \tag{3}$$

for any $T \in \mathscr{B}(H)$.

In the recent paper [2], P. Bhunia obtained the following interesting inequalities for the numerical radius of a product of two operators in Hilbert spaces.

Let $B,\ C \in \mathscr{B}(H)$, then

$$\omega^2(BC) \leqslant \frac{1}{2}\left\||B^*|^4 + |C|^4\right\|,$$

$$\omega^2(BC) \leqslant \frac{1}{2}\left(\|B\|^2\|C\|^2 + \omega\left(|B^*|^2|C|^2\right)\right),$$

$$\omega^2(BC) \leqslant \frac{1}{2}\left(\frac{1}{2}\left\||B^*|^4 + |C|^4\right\| + \omega\left(|B^*|^2|C|^2\right)\right)$$

and

$$\omega^2(BC) \leqslant \left\|\alpha|B^*|^2 + (1-\alpha)|C|^2\right\|\|B\|^{2(1-\alpha)}\|C\|^{2\alpha}$$

for all $\alpha \in [0,1]$.

In particular,

$$\omega^2(BC) \leqslant \frac{1}{2}\left\||B^*|^2 + |C|^2\right\|\|B\|\|C\|.$$

For more related results for the numerical radius of a product, see for instance the recent books [3], [9] and research papers [6] and [15]. However, Berezin transforms have been extensively applied in the field of reproducing kernel Hilbert spaces, addressing various problems. In their study of the boundedness of operators on reproducing kernel Hilbert spaces, Chalendar et al. [8] looked at more generic operators and investigated the Berezin symbols of their unitary orbits. Bhunia et al. [4] introduced a new norm, the α-Berezin norm, for the space of all bounded linear operators defined on a reproducing kernel Hilbert space, which extends the Berezin radius and Berezin norm. Additionally, the Banach algebra approaches are investigated by Karaev et al. [11]. For further information on Berezin transforms and Banach algebra structure, refer to [1, 10, 12, 17, 18, 19].

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [14]:

THEOREM 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. For any $T\in\mathscr{B}(H)$*

$$|\langle Tx,y\rangle|\leqslant\|f(|T|)x\|\,\|g(|T^*|)y\| \tag{4}$$

for all $x,\ y\in H$.

If we take $f(t)=t^{\lambda},\ g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$, then we obtain the famous *Kato's inequality* [13]

$$|\langle Tx,y\rangle|\leqslant\left\||T|^{\lambda}x\right\|\left\||T^*|^{1-\lambda}y\right\| \tag{5}$$

for all $x,\ y\in H$. The case $\lambda=1/2$ is also of interest,

$$|\langle Tx,y\rangle|\leqslant\left\||T|^{1/2}x\right\|\left\||T^*|^{1/2}y\right\|$$

for all $x,\ y\in H$.

We consider the power series with complex coefficients $f(z):=\sum_{k=0}^{\infty}a_kz^k$ with $a_k\in\mathbb{C}$ for $k\in\mathbb{N}:=\{0,1,\ldots\}$. Suppose that this power series is convergent on the open disk $D(0,R):=\{z\in\mathbb{C}|\ z<R\}$. If $R=\infty$ then $D(0,R)=\mathbb{C}$. We define $f_a(z):=\sum_{k=0}^{\infty}|a_k|z^k$ which has the same radius of convergence R. Motivated by Kittaneh's result (4), we show among others that, if the power series with complex coefficients $f(z):=\sum_{k=0}^{\infty}a_kz^k$ is convergent on $D(0,R)$ and $A,\ B,\ C,D\in\mathscr{B}(H)$ with $\|AB\|<R$, then the following vector inequality extending Kato's result (5) for power series holds

$$\begin{aligned}&|\langle D^*ABf(AB)Cx,y\rangle| \\ &\leqslant\|A\|^{\alpha}\|B\|^{1-\alpha}f_a(\|AB\|)\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2}\end{aligned} \tag{6}$$

for $\alpha\in[0,1]$ and $x,\ y\in H$.

Moreover, motivated by Bunia's inequalities for the numerical radius mentioned above, we provide norm and numerical radius inequalities for the composite operator $D^*ABf(AB)C$ under certain natural assumptions for the operators involved. Examples for polar decomposition of an operator T are considered. Some particular cases for fundamental power series $f(z)=(1\pm z)^{-1},\ |z|<1$ and $f(z)=\exp(z)$, with $z\in\mathbb{C}$, are also given.

2. Some vector inequalities

In [14] F. Kittaneh obtained the following result of Kato's type for powers of an operator:

LEMMA 1. *Let T be an operator in $\mathscr{B}(H)$. Then for any integer $n\geqslant1$,*

$$|\langle T^nx,y\rangle|^2\leqslant\|T\|^{2n-2}\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2(1-\alpha)}y,y\right\rangle \tag{7}$$

for $\alpha\in[0,1]$ and $x,\ y\in H$.

In particular,

$$|\langle T^n x, y\rangle|^2 \leqslant \|T\|^{2n-2} \langle |T| x, x\rangle \langle |T^*| y, y\rangle \tag{8}$$

for $x,\ y \in H$.

We have the following result for product of two operators

COROLLARY 1. *Let* A, B *be two operators in* $\mathscr{B}(H)$. *Then for any integer* $n \geqslant 1$,

$$|\langle (AB)^n x, y\rangle| \leqslant \|AB\|^{n-1} \|A\|^{\alpha} \|B\|^{1-\alpha} \left\langle |B|^{2\alpha} x, x\right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{9}$$

for $\alpha \in [0,1]$ *and* $x,\ y \in H$.

In particular,

$$|\langle (AB)^n x, y\rangle| \leqslant \|AB\|^{n-1} \|A\|^{1/2} \|B\|^{1/2} \langle |B| x, x\rangle^{1/2} \langle |A^*| y, y\rangle^{1/2} \tag{10}$$

for $x,\ y \in H$.

Proof. If we take $T = AB$ in (7), then we get

$$|\langle (AB)^n x, y\rangle|^2 \leqslant \|AB\|^{2n-2} \left\langle |AB|^{2\alpha} x, x\right\rangle \left\langle \left|(AB)^*\right|^{2(1-\alpha)} y, y\right\rangle \tag{11}$$

for $\alpha \in [0,1]$ and $x,\ y \in H$.

Observe that

$$|AB|^2 = (AB)^* AB = B^* (A^* A) B \leqslant B^* \left(\|A\|^2 I\right) B = \|A\|^2 |B|^2$$

and by taking the power $\alpha \in [0,1]$, we get

$$|AB|^{2\alpha} \leqslant \|A\|^{2\alpha} |B|^{2\alpha}.$$

Also

$$\left|(AB)^*\right|^2 = (AB)(AB)^* = ABB^*A^* \leqslant \|B\|^2 |A^*|^2,$$

which implies that

$$\left|(AB)^*\right|^{2(1-\alpha)} \leqslant \|B\|^{2(1-\alpha)} |A^*|^{2(1-\alpha)}.$$

Therefore

$$\left\langle |AB|^{2\alpha} x, x\right\rangle \leqslant \|A\|^{2\alpha} \left\langle |B|^{2\alpha} x, x\right\rangle, x \in H$$

and

$$\left\langle \left|(AB)^*\right|^{2(1-\alpha)} y, y\right\rangle \leqslant \|B\|^{2(1-\alpha)} \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle,\ y \in H.$$

By utilizing (11) we deduce the desired result (9). □

LEMMA 2. *Let A, B, C, D be operators in $\mathscr{B}(H)$. Then for any integer $n \geqslant 1$,*

$$
\begin{aligned}
&\left|\left\langle D^*(AB)^n Cx, y\right\rangle\right| \\
&\leqslant \|AB\|^{n-1}\|A\|^{\alpha}\|B\|^{1-\alpha}\left\langle\left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2}
\end{aligned} \tag{12}
$$

for $\alpha \in [0,1]$ and x, $y \in H$.

In particular,

$$
\begin{aligned}
&\left|\left\langle D^*(AB)^n Cx, y\right\rangle\right| \\
&\leqslant \|AB\|^{n-1}\|A\|^{1/2}\|B\|^{1/2}\left\langle\left||B|^{1/2} C\right|^2 x, x\right\rangle^{1/2}\left\langle\left||A^*|^{1/2} D\right|^2 y, y\right\rangle^{1/2}.
\end{aligned} \tag{13}
$$

Proof. From (9) for Cx instead of x and Dy instead of y, we obtain

$$
\begin{aligned}
&\left|\left\langle D^*(AB)^n Cx, y\right\rangle\right| \\
&\leqslant \|AB\|^{n-1}\|A\|^{\alpha}\|B\|^{1-\alpha}\left\langle C^*|B|^{2\alpha} Cx, x\right\rangle^{1/2}\left\langle D^*|A^*|^{2(1-\alpha)} Dy, y\right\rangle^{1/2}
\end{aligned} \tag{14}
$$

for all x, $y \in H$.

Since

$$C^*|B|^{2\alpha} C = \left||B|^{\alpha} C\right|^2$$

and

$$D^*|A^*|^{2(1-\alpha)} D = \left||A^*|^{1-\alpha} D\right|^2,$$

hence by (14) we get (12). □

As some natural examples that are useful for applications, we can point out that, if

$$
\begin{aligned}
f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda},\ \lambda \in D(0,1); \qquad (15)\\
g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda,\ \lambda \in \mathbb{C};\\
h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda,\ \lambda \in \mathbb{C};\\
l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda},\ \lambda \in D(0,1);
\end{aligned}
$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$
\begin{aligned}
f_a(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda},\ \lambda \in D(0,1); \qquad (16)\\
g_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda,\ \lambda \in \mathbb{C};
\end{aligned}
$$

$$h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \; \lambda \in \mathbb{C};$$

$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \; \lambda \in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{17}$$

$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \;\; \alpha,\beta,\gamma > 0, \;\; \lambda \in D(0,1);$$

where Γ is *Gamma function*.

The following result provides an extension of Kato's result (5) for functions defined by power series as above:

THEOREM 2. *Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$ and $A,\ B,\ C, D \in \mathscr{B}(H)$ with $\|AB\| < R$, then*

$$|\langle D^* ABf(AB)Cx, y\rangle| \tag{18}$$
$$\leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and $x,\ y \in H$.

In particular,

$$|\langle D^* ABf(AB)Cx, y\rangle| \tag{19}$$
$$\leqslant \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \left\langle \left||B|^{1/2} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1/2} D\right|^2 y, y\right\rangle^{1/2}$$

for $x,\ y \in H$.

Proof. If we take $n = k+1,\ k \in \mathbb{N}$ in (12), then we get

$$\left|\left\langle D^* AB(AB)^k Cx, y\right\rangle\right| \tag{20}$$
$$\leqslant \|AB\|^k \|A\|^{\alpha} \|B\|^{1-\alpha} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2}$$

for all $x, \; y \in H$.

Further, if we multiply by $|a_k| \geqslant 0, \; k \in \{0,1,\ldots\}$ and sum over k from 0 to m, then we get

$$\left|\left\langle D^* AB\left(\sum_{k=0}^{m} a_k (AB)^k\right) Cx, y\right\rangle\right| \tag{21}$$
$$= \left|\sum_{k=0}^{m} a_k \left\langle D^* AB (AB)^k Cx, y\right\rangle\right|$$
$$\leqslant \sum_{k=0}^{m} |a_k| \left|\left\langle D^* AB (AB)^k Cx, y\right\rangle\right|$$
$$\leqslant \left(\sum_{k=0}^{m} |a_k| \|AB\|^k\right) \|A\|^{\alpha} \|B\|^{1-\alpha} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2}$$

for all $x, \; y \in H$.

Since $\|AB\| < R$, then series $\sum_{k=0}^{\infty} a_k (AB)^k$ and $\sum_{k=0}^{\infty} |a_k| \|AB\|^k$ are convergent and

$$\sum_{k=0}^{\infty} a_k (AB)^k = f(AB) \text{ and } \sum_{k=0}^{\infty} |a_k| \|AB\|^k = f_a(\|AB\|).$$

By taking now the limit over $m \to \infty$ in (21) we deduce the desired result (18). □

COROLLARY 2. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$ *and* $A, \; B \in \mathscr{B}(H)$ *with* $\|AB\| < R$, *then*

$$|\langle ABf(AB)x, y\rangle| \tag{22}$$
$$\leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\langle |B|^{2\alpha} x, x\right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle^{1/2}$$

for $\alpha \in [0,1]$ *and* $x, \; y \in H$. *In particular,*

$$|\langle ABf(AB)x, y\rangle| \leqslant \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \langle |B| x, x\rangle^{1/2} \langle |A^*| y, y\rangle^{1/2} \tag{23}$$

for $x, \; y \in H$.

REMARK 1. If we take $B = A$ in (22), then we get

$$\left|\left\langle A^2 f\left(A^2\right) x, y\right\rangle\right| \leqslant \|A\| f_a\left(\left\|A^2\right\|\right) \left\langle |A|^{2\alpha} x, x\right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{24}$$

for $\alpha \in [0,1]$ and $x, \; y \in H$. In particular,

$$\left|\left\langle A^2 f\left(A^2\right) x, y\right\rangle\right| \leqslant \|A\| f_a\left(\left\|A^2\right\|\right) \langle |A| x, x\rangle^{1/2} \langle |A^*| y, y\rangle^{1/2} \tag{25}$$

for $x, \; y \in H$.

Also, if we take in (22) $A = B^*$, then we get

$$\left|\left\langle |B|^2 f\left(|B|^2\right) x, y\right\rangle\right| \leqslant \|B\| f_a\left(\|B\|^2\right) \left\langle |B|^{2\alpha} x, x\right\rangle^{1/2} \left\langle |B|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{26}$$

for $\alpha \in [0,1]$ and $x,\ y \in H$. In particular,

$$\left|\left\langle |B|^2 f\left(|B|^2\right) x, y\right\rangle\right| \leqslant \|B\| f_a\left(\|B\|^2\right) \langle |B| x, x\rangle^{1/2} \langle |B| y, y\rangle^{1/2} \tag{27}$$

for $x,\ y \in H$.

EXAMPLE 1. Let $A,\ B,\ C,\ D \in \mathscr{B}(H)$ with $\|AB\| < 1$, then by taking $f(z) = (1 \pm z)^{-1},\ |z| < 1$ in Theorem 2 we get

$$\left|\left\langle D^* AB (1 \pm AB)^{-1} Cx, y\right\rangle\right| \leqslant \frac{\|A\|^{\alpha} \|B\|^{1-\alpha}}{1 - \|AB\|} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \tag{28}$$

for $\alpha \in [0,1]$ and $x,\ y \in H$. In particular,

$$\left|\left\langle D^* AB (1 \pm AB)^{-1} Cx, y\right\rangle\right| \leqslant \frac{\sqrt{\|A\| \|B\|}}{1 - \|AB\|} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \tag{29}$$

for $x,\ y \in H$.

Also, if $A,\ B,\ C \in \mathscr{B}(H)$ and $\alpha \in [0,1]$, then by Theorem 2 for $f(z) = \exp(wz)$, with $w,\ z \in \mathbb{C},\ w \neq 0$

$$|\langle D^* AB \exp(wAB) Cx, y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} \exp(|w| \|AB\|) \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \tag{30}$$

for $\alpha \in [0,1]$ and $x,\ y \in H$. In particular,

$$\begin{aligned} &|\langle D^* AB \exp(wAB) Cx, y\rangle| \\ &\leqslant \|A\|^{1/2} \|B\|^{1/2} \exp(|w| \|AB\|) \\ &\quad \times \left\langle \left||B|^{1/2} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1/2} D\right|^2 y, y\right\rangle^{1/2} \end{aligned} \tag{31}$$

for $x,\ y \in H$.

3. Norm and numerical radius inequalities

The following vector inequality for positive operators $A \geqslant 0$, obtained by C. A. McCarthy in [16] is well known,

$$\langle Ax,x\rangle^{p} \leqslant \langle A^{p}x,x\rangle , \quad p \geqslant 1$$

for $x \in H$, $\|x\| = 1$.

Buzano's inequality [7],

$$|\langle x,e\rangle \langle e,y\rangle| \leqslant \frac{1}{2}\left[\|x\| \|y\| + |\langle x,y\rangle|\right] \tag{32}$$

that holds for any x, y, $e \in H$ with $\|e\| = 1$ will also be used in the sequel.

THEOREM 3. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$, $\alpha \in [0,1]$ *and* A, B, $C, D \in \mathscr{B}(H)$ *with* $\|AB\| < R$, *then we have the norm inequality*

$$\|D^{*}ABf(AB)C\| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\||B|^{\alpha} C\right\| \left\||A^{*}|^{1-\alpha} D\right\|. \tag{33}$$

We also have the numerical radius inequalities

$$\omega(D^{*}ABf(AB)C) \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\| \frac{\left||B|^{\alpha} C\right|^{2} + \left||A^{*}|^{1-\alpha} D\right|^{2}}{2} \right\| \tag{34}$$

and

$$\omega(D^{*}ABf(AB)C) \leqslant \frac{\sqrt{2}}{2} \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \times \left[\left\||B|^{\alpha} C\right\|^{2} \left\||A^{*}|^{1-\alpha} D\right\|^{2} + \omega\left(\left||A^{*}|^{1-\alpha} D\right|^{2} \left||B|^{\alpha} C\right|^{2} \right) \right]^{1/2}. \tag{35}$$

Proof. From (18) we get

$$\|D^{*}ABf(AB)C\| = \sup_{\|x\|=\|y\|=1} |\langle D^{*}ABf(AB)Cx,y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \times \sup_{\|x\|=\|y\|=1} \left[\left\langle \left||B|^{\alpha} C\right|^{2} x,x \right\rangle^{1/2} \left\langle \left||A^{*}|^{1-\alpha} D\right|^{2} y,y \right\rangle^{1/2} \right]$$

$$\begin{aligned}&\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha} f_a(\|AB\|)\\&\quad\times \sup_{\|x\|=1}\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^{1/2}\sup_{\|y\|=1}\left\langle \left||A^*|^{1-\alpha}D\right|^2 y,y\right\rangle^{1/2}\\&= \|A\|^{\alpha}\|B\|^{1-\alpha} f_a(\|AB\|)\left\|\left||B|^{\alpha}C\right|^2\right\|^{1/2}\left\|\left||A^*|^{1-\alpha}D\right|^2\right\|^{1/2}\\&= \|A\|^{\alpha}\|B\|^{1-\alpha} f_a(\|AB\|)\left\||B|^{\alpha}C\right\|\left\||A^*|^{1-\alpha}D\right\|,\end{aligned}$$

which gives (33).

From (18) we get for $y=x$ that

$$\begin{aligned}&|\langle D^*ABf(AB)Cx,x\rangle|\\&\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha} f_a(\|AB\|)\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle^{1/2}\end{aligned} \tag{36}$$

for all $x\in H$.

By using the *A-G-mean inequality*, we obtain that

$$\begin{aligned}&\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle^{1/2}\\&\leqslant \frac{1}{2}\left[\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle+\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle\right]=\left\langle \frac{\left||B|^{\alpha}C\right|^2+\left||A^*|^{1-\alpha}D\right|^2}{2}x,x\right\rangle\end{aligned}$$

and then by (36) we derive

$$\begin{aligned}&\omega(D^*ABf(AB)C)\\&=\sup_{\|x\|=1}|\langle D^*ABf(AB)Cx,x\rangle|\\&\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha} f_a(\|AB\|)\sup_{\|x\|=1}\left\langle \frac{\left||B|^{\alpha}C\right|^2+\left||A^*|^{1-\alpha}D\right|^2}{2}x,x\right\rangle\\&= \|A\|^{\alpha}\|B\|^{1-\alpha} f_a(\|AB\|)\left\|\frac{\left||B|^{\alpha}C\right|^2+\left||A^*|^{1-\alpha}D\right|^2}{2}\right\|\end{aligned}$$

and the inequality (34) is proved.

From (18) for $y=x$ and Buzano's inequality we derive that

$$\begin{aligned}&|\langle D^*ABf(AB)Cx,x\rangle|^2\\&\leqslant \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2(\|AB\|)\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle\left\langle x,\left||A^*|^{1-\alpha}D\right|^2 x\right\rangle\end{aligned} \tag{37}$$

$$
\begin{aligned}
&\leqslant \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2(\|AB\|) \\
&\quad \times \frac{1}{2}\left[\left\|\left||B|^{\alpha}C\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}D\right|^2 x\right\| + \left|\left\langle \left||B|^{\alpha}C\right|^2 x, \left||A^*|^{1-\alpha}D\right|^2 x\right\rangle\right|\right] \\
&= \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2(\|AB\|) \\
&\quad \times \frac{1}{2}\left[\left\|\left||B|^{1/2}C\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}D\right|^2 x\right\| + \left|\left\langle \left||A^*|^{1-\alpha}D\right|^2\left||B|^{1/2}C\right|^2 x, x\right\rangle\right|\right]
\end{aligned}
$$

for all $x \in H,\ \|x\| = 1.$

By taking the supremum over $\|x\| = 1$ in (37) we get that

$$
\begin{aligned}
&\omega^2(D^*ABf(AB)C) \\
&= \sup_{\|x\|=1} |\langle D^*ABf(AB)Cx, x\rangle|^2 \\
&\leqslant \frac{1}{2}\|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2(\|AB\|) \\
&\quad \times \sup_{\|x\|=1}\left[\left\|\left||B|^{\alpha}C\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}D\right|^2 x\right\| + \left|\left\langle \left||A^*|^{1-\alpha}D\right|^2\left||B|^{\alpha}C\right|^2 x, x\right\rangle\right|\right] \\
&\leqslant \frac{1}{2}\|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2(\|AB\|)\left[\sup_{\|x\|=1}\left\|\left||B|^{\alpha}C\right|^2 x\right\| \sup_{\|x\|=1}\left\|\left||A^*|^{1-\alpha}D\right|^2 x\right\|\right. \\
&\quad \left. + \sup_{\|x\|=1}\left|\left\langle \left||A^*|^{1-\alpha}D\right|^2\left||B|^{\alpha}C\right|^2 x, x\right\rangle\right|\right] \\
&= \frac{1}{2}\|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2(\|AB\|) \\
&\quad \times \left[\left\||B|^{\alpha}C\right\|^2\left\||A^*|^{1-\alpha}D\right\|^2 + \omega\left(\left||A^*|^{1-\alpha}D\right|^2\left||B|^{\alpha}C\right|^2\right)\right],
\end{aligned}
$$

which proves (35). □

We notice that, in the simplest case when $f \equiv 1,$ we get the following numerical radius inequalities for four operators

$$
\omega(D^*ABC) \leqslant \|A\|^{\alpha}\|B\|^{1-\alpha}\left\|\frac{\left||B|^{\alpha}C\right|^2 + \left||A^*|^{1-\alpha}D\right|^2}{2}\right\|
$$

and

$$
\begin{aligned}
\omega(D^*ABC) &\leqslant \frac{\sqrt{2}}{2}\|A\|^{\alpha}\|B\|^{1-\alpha} \\
&\quad \times \left[\left\||B|^{\alpha}C\right\|^2\left\||A^*|^{1-\alpha}D\right\|^2 + \omega\left(\left||A^*|^{1-\alpha}D\right|^2\left||B|^{\alpha}C\right|^2\right)\right]^{1/2},
\end{aligned}
$$

where $A,\ B,\ C, D \in \mathscr{B}(H).$

COROLLARY 3. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$ *and* $A,\ B \in \mathscr{B}(H)$ *with* $\|AB\| < R$, *then we have the numerical radius inequalities for* $\alpha \in [0,1]$

$$\omega(ABf(AB)) \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\| \frac{|B|^{2\alpha} + |A^*|^{2(1-\alpha)}}{2} \right\| \tag{38}$$

and

$$\omega(ABf(AB)) \leqslant \frac{\sqrt{2}}{2} \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \times \left[\|B\|^{2\alpha} \|A^*\|^{2(1-\alpha)} + \omega\left(|A^*|^{2(1-\alpha)} |B|^{2\alpha} \right) \right]^{1/2}. \tag{39}$$

REMARK 2. If we take $\alpha = 1/2$ in Theorem 3, then we get

$$\|D^* ABf(AB)C\| \leqslant \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \left\| |B|^{1/2} C \right\| \left\| |A^*|^{1/2} D \right\|. \tag{40}$$

We also have the numerical radius inequalities

$$\omega(D^* ABf(AB)C) \leqslant \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \left\| \frac{\left| |B|^{1/2} C \right|^2 + \left| |A^*|^{1/2} D \right|^2}{2} \right\| \tag{41}$$

and

$$\omega(D^* ABf(AB)C) \leqslant \frac{\sqrt{2}}{2} \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \times \left[\left\| |B|^{1/2} C \right\|^2 \left\| |A^*|^{1/2} D \right\|^2 + \omega\left(\left| |A^*|^{1/2} D \right|^2 \left| |B|^{1/2} C \right|^2 \right) \right]^{1/2}. \tag{42}$$

Moreover, if we take $D = C = I$ in (41) and (42), then we get

$$\omega(ABf(AB)) \leqslant \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \left\| \frac{|B| + |A^*|}{2} \right\| \tag{43}$$

and

$$\omega(ABf(AB)) \leqslant \frac{\sqrt{2}}{2} \|A\|^{1/2} \|B\|^{1/2} f_a(\|AB\|) \left[\|B\| \|A\| + \omega(|A^*| |B|) \right]^{1/2}. \tag{44}$$

The second main result is as follows:

THEOREM 4. *Assume that the conditions of Theorem 3 are satisfied. If* $\alpha \in [0,1]$, $r > 0$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geqslant 1$, *then*

$$\begin{aligned} &\omega(D^* ABf(AB)C) \\ &\leqslant \|A\|^\alpha \|B\|^{1-\alpha} f_a(\|AB\|) \left\| \frac{1}{p} \left| |B|^\alpha C \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} D \right|^{2qr} \right\|^{\frac{1}{2r}}. \end{aligned} \tag{45}$$

If $r \geqslant 1$, *then*

$$\begin{aligned} &\omega(D^* ABf(AB)C) \\ &\leqslant \|A\|^\alpha \|B\|^{1-\alpha} f_a(\|AB\|) \\ &\quad \times \left(\frac{\left\| |B|^\alpha C \right\|^{2r} \left\| |A^*|^{1-\alpha} D \right\|^{2r} + \omega^r \left(\left| |A^*|^{1-\alpha} D \right|^2 \left| |B|^\alpha C \right|^2 \right)}{2} \right)^{\frac{1}{2r}}. \end{aligned} \tag{46}$$

If $r \geqslant 1$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geqslant 2$, *then also*

$$\begin{aligned} \omega^{2r}(D^* ABf(AB)C) \leqslant \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\ \times \left\{ \frac{1}{2} \left[\left\| \frac{1}{p} \left| |B|^\alpha C \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} D \right|^{2qr} \right\| \right. \right. \\ \left. \left. + \omega^r \left(\left| |A^*|^{1-\alpha} D \right|^2 \left| |B|^\alpha C \right|^2 \right) \right] \right\}. \end{aligned} \tag{47}$$

Proof. If we take the power $2r > 0$ in (18) written for $y = x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned} &|\langle D^* ABf(AB)Cx, x\rangle|^{2r} \\ &\leqslant \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \left\langle \left| |B|^\alpha C \right|^2 x, x \right\rangle^r \left\langle \left| |A^*|^{1-\alpha} D \right|^2 x, x \right\rangle^r \\ &\leqslant \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\ &\quad \times \left[\frac{1}{p} \left\langle \left| |B|^\alpha C \right|^2 x, x \right\rangle^{pr} + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} D \right|^2 x, x \right\rangle^{qr} \right] \\ &\leqslant \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\ &\quad \times \left[\frac{1}{p} \left\langle \left| |B|^\alpha C \right|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} D \right|^{2qr} x, x \right\rangle \right] \\ &= \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\ &\quad \times \left[\left\langle \left(\frac{1}{p} \left| |B|^\alpha C \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} D \right|^{2qr} \right) x, x \right\rangle \right] \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
&\omega^{2r}\left(D^{*}ABf\left(AB\right)C\right)\\
&=\sup_{\|x\|=1}\left|\left\langle D^{*}ABf\left(AB\right)Cx,x\right\rangle\right|^{2r}\\
&\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}f_{a}^{2r}\left(\|AB\|\right)\\
&\quad\times\sup_{\|x\|=1}\left[\left\langle\left(\frac{1}{p}\left||B|^{\alpha}C\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}D\right|^{2qr}\right)x,x\right\rangle\right]\\
&=\|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}f_{a}^{2r}\left(\|AB\|\right)\left\|\frac{1}{p}\left||B|^{\alpha}C\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}D\right|^{2qr}\right\|,
\end{aligned}$$

which proves (45).

By taking the power $r\geqslant 1$ in (37) and by using the convexity of the power function, we get

$$\begin{aligned}
&\left|\left\langle D^{*}ABf\left(AB\right)Cx,x\right\rangle\right|^{2r}\\
&\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}f_{a}^{2r}\left(\|AB\|\right)\\
&\quad\times\left[\frac{\left\|\left||B|^{\alpha}C\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\right|}{2}\right]^{r}\\
&\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}f_{a}^{2r}\left(\|AB\|\right)\\
&\quad\times\left(\frac{\left\|\left||B|^{\alpha}C\right|^{2}x\right\|^{r}\left\|\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\|^{r}+\left|\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\right|^{r}}{2}\right)
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, we get that

$$\begin{aligned}
&\omega^{2r}\left(D^{*}ABf\left(AB\right)C\right)\\
&\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}f_{a}^{2r}\left(\|AB\|\right)\\
&\quad\times\frac{\left\||B|^{\alpha}C\right\|^{2r}\left\||A^{*}|^{1-\alpha}D\right\|^{2r}+\omega^{r}\left(\left||A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}\right)}{2},
\end{aligned}$$

which proves (46).

Also, observe that

$$\begin{aligned}\left\| \left| |B|^{\alpha} C \right|^{2} x \right\|^{r} \left\| \left| |A^{*}|^{1-\alpha} D \right|^{2} x \right\|^{r} &\leqslant \frac{1}{p} \left\| \left| |B|^{\alpha} C \right|^{2} x \right\|^{pr} + \frac{1}{q} \left\| \left| |A^{*}|^{1-\alpha} D \right|^{2} x \right\|^{qr} \\ &= \frac{1}{p} \left\| \left| |B|^{\alpha} C \right|^{2} x \right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| \left| |A^{*}|^{1-\alpha} D \right|^{2} x \right\|^{2\frac{qr}{2}} \\ &= \frac{1}{p} \left\langle \left| |B|^{\alpha} C \right|^{4} x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle \left| |A^{*}|^{1-\alpha} D \right|^{4} x \right\rangle^{\frac{qr}{2}} \\ &\leqslant \frac{1}{p} \left\langle \left| |B|^{\alpha} C \right|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle \left| |A^{*}|^{1-\alpha} D \right|^{2qr} x \right\rangle \\ &= \left\langle \left(\frac{1}{p} \left| |B|^{\alpha} C \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} D \right|^{2qr} \right) x, x \right\rangle\end{aligned}$$

for $x \in H$ with $\|x\| = 1$. Then

$$\begin{aligned}&\frac{\left\| \left| |B|^{\alpha} C \right|^{2} x \right\|^{r} \left\| \left| |A^{*}|^{1-\alpha} D \right|^{2} x \right\|^{r} + \left| \left\langle \left| |A^{*}|^{1-\alpha} D \right|^{2} \left| |B|^{\alpha} C \right|^{2} x, x \right\rangle \right|^{r}}{2} \\ &\leqslant \frac{1}{2} \left[\left\langle \left(\frac{1}{p} \left| |B|^{\alpha} C \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} D \right|^{2qr} \right) x, x \right\rangle \right. \\ &\quad \left. + \left| \left\langle \left| |A^{*}|^{1-\alpha} D \right|^{2} \left| |B|^{\alpha} C \right|^{2} x, x \right\rangle \right|^{r} \right],\end{aligned}$$

which gives that

$$\begin{aligned}|\langle D^{*}ABf(AB)Cx, x\rangle|^{2r} &\leqslant \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_{a}^{2r}(\|AB\|) \\ &\quad \times \left\{ \frac{1}{2} \left[\left\langle \left(\frac{1}{p} \left| |B|^{\alpha} C \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} D \right|^{2qr} \right) x, x \right\rangle \right. \right. \\ &\quad \left. \left. + \left| \left\langle \left| |A^{*}|^{1-\alpha} D \right|^{2} \left| |B|^{\alpha} C \right|^{2} x, x \right\rangle \right|^{r} \right] \right\}\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $x \in H$ with $\|x\| = 1$, which gives (47). □

COROLLARY 4. *Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$, A, $B \in \mathscr{B}(H)$ with $\|AB\| < R$ and $\alpha \in [0,1]$. If $\alpha \in [0,1]$, $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geqslant 1$, then*

$$\omega(ABf(AB)) \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_{a}(\|AB\|) \left\| \frac{1}{p} |B|^{2\alpha pr} + \frac{1}{q} |A^{*}|^{2(1-\alpha)qr} \right\|^{\frac{1}{2r}}. \tag{48}$$

If $r \geqslant 1$, *then*

$$\omega(ABf(AB)) \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \times \left(\frac{\|B\|^{2\alpha r} \|A\|^{2(1-\alpha)r} + \omega^r \left(|A^*|^{2(1-\alpha)} |B|^{2\alpha} \right)}{2} \right)^{\frac{1}{2r}}. \tag{49}$$

If $r \geqslant 1,\ p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr,\ qr \geqslant 2$, *then also*

$$\omega^{2r}(ABf(AB)) \leqslant \|A\|^{2\alpha r} \|B\|^{2(1-\alpha)r} f_a^{2r}(\|AB\|) \times \left\{ \frac{1}{2} \left[\left\| \frac{1}{p} |B|^{2\alpha pr} + \frac{1}{q} |A^*|^{2(1-\alpha)qr} \right\| + \omega^r \left(|A^*|^{2(1-\alpha)} |B|^{2\alpha} \right) \right] \right\}. \tag{50}$$

As above, if we take $f \equiv 1$ then we get several numerical radius inequalities for the product of four and two operators, respectively.

REMARK 3. If we take $\alpha = 1/2$ in Corollary 4, then we get

$$\omega(ABf(AB)) \leqslant \sqrt{\|A\| \|B\|} f_a(\|AB\|) \left\| \frac{1}{p} |B|^{pr} + \frac{1}{q} |A^*|^{qr} \right\|^{\frac{1}{2r}} \tag{51}$$

for $r > 0,\ p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr,\ qr \geqslant 1$.

If $r \geqslant 1$, then

$$\omega(ABf(AB)) \leqslant \sqrt{\|A\| \|B\|} f_a(\|AB\|) \left(\frac{\|B\|^r \|A\|^r + \omega^r(|A^*| |B|)}{2} \right)^{\frac{1}{2r}}. \tag{52}$$

If $r \geqslant 1,\ p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr,\ qr \geqslant 2$, then also

$$\omega^{2r}(ABf(AB)) \leqslant \|A\|^r \|B\|^r f_a^{2r}(\|AB\|) \times \left\{ \frac{1}{2} \left[\left\| \frac{1}{p} |B|^{pr} + \frac{1}{q} |A^*|^{qr} \right\| + \omega^r(|A^*| |B|) \right] \right\}. \tag{53}$$

We also have:

THEOREM 5. *With the assumptions of Theorem 3, we have for* $r \geqslant 1,\ \lambda \in [0,1]$ *that*

$$\begin{aligned} &\omega(D^*ABf(AB)C) \\ &\leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\| |B|^{\alpha} C \right\|^{\lambda} \left\| |A^*|^{1-\alpha} D \right\|^{1-\lambda} \\ &\quad \times \left\| (1-\lambda) \left| |B|^{\alpha} C \right|^{2r} + \lambda \left| |A^*|^{1-\alpha} D \right|^{2r} \right\|^{\frac{1}{2r}} \end{aligned} \tag{54}$$

for all $\alpha \in [0,1]$.

Moreover, we have

$$\omega\left(D^{*}ABf\left(AB\right)C\right) \leqslant \|A\|^{\alpha}\|B\|^{1-\alpha} f_{a}\left(\|AB\|\right) \times \left\|(1-\lambda)\left||B|^{\alpha}C\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}D\right|^{2r}\right\|^{\frac{1}{2r}} \times \left\|\lambda\left||B|^{\alpha}C\right|^{2r}+(1-\lambda)\left||A^{*}|^{1-\alpha}D\right|^{2r}\right\|^{\frac{1}{2r}} \tag{55}$$

for all $\alpha \in [0,1]$ *and* $r \geqslant 1$.

Proof. From (36) we get

$$\begin{aligned}
&\left|\langle D^{*}ABf\left(AB\right)Cx,x\rangle\right|^{2} \\
&\leqslant \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_{a}^{2}\left(\|AB\|\right)\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle \\
&= \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_{a}^{2}\left(\|AB\|\right)\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{\lambda}\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{1-\lambda} \\
&\quad \times \left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{\lambda}\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{1-\lambda} \\
&= \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_{a}^{2}\left(\|AB\|\right)\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{1-\lambda}\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{\lambda} \\
&\quad \times \left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{\lambda}\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{1-\lambda} \\
&\leqslant \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_{a}^{2}\left(\|AB\|\right) \\
&\quad \times \left[(1-\lambda)\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle+\lambda\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle\right] \\
&\quad \times \left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{\lambda}\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{1-\lambda}
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the power $r \geqslant 1$, then we get by the convexity of power r that

$$\begin{aligned}
&\left|\langle D^{*}ABf\left(AB\right)Cx,x\rangle\right|^{2r} \\
&\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} f_{a}^{2r}\left(\|AB\|\right) \\
&\quad \times \left[(1-\lambda)\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle+\lambda\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle\right]^{r} \\
&\quad \times \left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{r\lambda}\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{r(1-\lambda)}
\end{aligned} \tag{56}$$

$$\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \times \left[(1-\lambda)\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^r + \lambda\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle^r\right] \times \left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^{r\lambda}\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle^{r(1-\lambda)}$$

for all $x \in H$, $\|x\| = 1$.

If we use McCarthy inequality for power $r \geqslant 1$, then we get

$$\begin{aligned}
&(1-\lambda)\left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^r + \lambda\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle^r \\
&\leqslant (1-\lambda)\left\langle \left||B|^{\alpha}C\right|^{2r} x,x\right\rangle + \lambda\left\langle \left||A^*|^{1-\alpha}D\right|^{2r} x,x\right\rangle \\
&= \left\langle \left[(1-\lambda)\left||B|^{\alpha}C\right|^{2r} + \lambda\left||A^*|^{1-\alpha}D\right|^{2r}\right] x,x\right\rangle
\end{aligned}$$

and by (56) we get

$$\begin{aligned}
&|\langle D^*ABf(AB)Cx,x\rangle|^{2r} \\
&\leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\
&\quad \times \left\langle \left[(1-\lambda)\left||B|^{\alpha}C\right|^{2r} + \lambda\left||A^*|^{1-\alpha}D\right|^{2r}\right] x,x\right\rangle \\
&\quad \times \left\langle \left||B|^{\alpha}C\right|^2 x,x\right\rangle^{r\lambda}\left\langle \left||A^*|^{1-\alpha}D\right|^2 x,x\right\rangle^{r(1-\lambda)}
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$\begin{aligned}
\omega^{2r}(D^*ABf(AB)C) \leqslant{}& \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\
&\times \left\|(1-\lambda)\left||B|^{\alpha}C\right|^{2r} + \lambda\left||A^*|^{1-\alpha}D\right|^{2r}\right\| \\
&\times \left\||B|^{\alpha}C\right\|^{2r\lambda}\left\||A^*|^{1-\alpha}D\right\|^{2r(1-\lambda)},
\end{aligned}$$

which gives (54).

We also have

$$\begin{aligned}
|\langle D^*ABf(AB)Cx,x\rangle|^{2r} \leqslant{}& \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} f_a^{2r}(\|AB\|) \\
&\times \left\langle \left[(1-\lambda)\left||B|^{\alpha}C\right|^{2r} + \lambda\left||A^*|^{1-\alpha}D\right|^{2r}\right] x,x\right\rangle \\
&\times \left\langle \left[\lambda\left||B|^{\alpha}C\right|^{2r} + (1-\lambda)\left||A^*|^{1-\alpha}D\right|^{2r}\right] x,x\right\rangle
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$, which gives (55). □

COROLLARY 5. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$, $A,\ B \in \mathscr{B}(H)$ *with* $\|AB\| < R$ *and* $\alpha \in [0,1]$. *Then we have for* $r \geqslant 1,\ \lambda \in [0,1]$ *that*

$$\omega(ABf(AB)) \leqslant \|A\|^{\alpha+(1-\alpha)(1-\lambda)} \|B\|^{1-\alpha+\alpha\lambda} f_a(\|AB\|) \times \left\|(1-\lambda)|B|^{2\alpha r} + \lambda |A^*|^{2(1-\alpha)r}\right\|^{\frac{1}{2r}} \tag{57}$$

for all $\alpha \in [0,1]$.

Also, we have

$$\omega(ABf(AB)) \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \times \left\|(1-\lambda)|B|^{2\alpha r} + \lambda |A^*|^{2(1-\alpha)r}\right\|^{\frac{1}{2r}} \times \left\|\lambda |B|^{2\alpha r} + (1-\lambda)|A^*|^{2(1-\alpha)r}\right\|^{\frac{1}{2r}}. \tag{58}$$

REMARK 4. If we take $\alpha = 1/2$ in Corollary 5, then we get for $r \geqslant 1$ that

$$\omega(ABf(AB)) \leqslant \|A\|^{\frac{2-\lambda}{2}} \|B\|^{\frac{1+\lambda}{2}} f_a(\|AB\|) \|(1-\lambda)|B|^r + \lambda |A^*|^r\|^{\frac{1}{2r}} \tag{59}$$

for all $\lambda \in [0,1]$.

Also, we have

$$\omega(ABf(AB)) \leqslant \sqrt{\|A\|\|B\|} f_a(\|AB\|) \times \|(1-\lambda)|B|^r + \lambda |A^*|^r\|^{\frac{1}{2r}} \|\lambda |B|^r + (1-\lambda)|A^*|^r\|^{\frac{1}{2r}} \tag{60}$$

for all $\lambda \in [0,1]$.

From (59) we get for $\lambda = 1/2$ that

$$\omega(ABf(AB)) \leqslant \|A\|^{\frac{3}{4}} \|B\|^{\frac{3}{4}} f_a(\|AB\|) \left\|\frac{|B|^r + |A^*|^r}{2}\right\|^{\frac{1}{2r}}. \tag{61}$$

By taking $f \equiv 1,\ f(z) = (1 \pm z)^{-1},\ |z| < 1$ or $f(z) = \exp(wz)$, with $w,\ z \in \mathbb{C}$, $w \neq 0$ in the numerical radius inequalities obtained above, one can obtain several results concerning these fundamental functions.

4. Some examples for polar decomposition

Let $T \in \mathscr{B}(H)$ where $T = U|T|$ is the polar decomposition of T and U is the partial isometry. By taking $A = U|T|^{1-t}$ and $B = |T|^t$, see also [2], we have that

$$AB = T,\ \|A\| = \|T\|^{1-t},\ \|B\| = \|T\|^t,\ |B|^{\alpha} = |T|^{\alpha t}$$

and

$$|A^*|^{1-\alpha} = |T^*|^{(1-\alpha)(1-t)}$$

for $t \in [0,1]$.

Now, if we use the inequality (18) for the above choice of A and B, then we get

$$|\langle D^* T f(T) Cx, y\rangle| \leqslant \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \times \left\langle \left| |T|^{\alpha t} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left| |T^*|^{(1-\alpha)(1-t)} D\right|^2 y, y\right\rangle^{1/2} \tag{62}$$

for $\alpha,\ t \in [0,1]$ and $x,\ y \in H$.

The case $t = 1/2$ gives

$$|\langle D^* T f(T) Cx, y\rangle| \leqslant \|T\|^{1/2} f_a(\|T\|) \left\langle \left| |T|^{\alpha/2} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left| |T^*|^{(1-\alpha)/2} D\right|^2 y, y\right\rangle^{1/2},$$

which for $\alpha = 1/2$ provides

$$|\langle D^* T f(T) Cx, y\rangle| \leqslant \|T\|^{1/2} f_a(\|T\|) \left\langle \left| |T|^{1/4} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left| |T^*|^{1/4} D\right|^2 y, y\right\rangle^{1/2}$$

where $x,\ y \in H$..

From the inequalities (34) and (35) we derive

$$\omega(D^* T f(T) C) \leqslant \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \left\| \frac{\left| |T|^{\alpha t} C\right|^2 + \left| |T^*|^{(1-\alpha)(1-t)} D\right|^2}{2} \right\| \tag{63}$$

and

$$\omega(D^* T f(T) C) \leqslant \frac{\sqrt{2}}{2} \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \times \left[\left\| |T|^{\alpha t} C\right\|^4 \left\| |T^*|^{(1-\alpha)(1-t)} D\right\|^4 + \omega\left(\left| |T^*|^{(1-\alpha)(1-t)} D\right|^2 \left| |T|^{\alpha t} C\right|^2 \right) \right]^{1/2}. \tag{64}$$

for $\alpha,\ t \in [0,1]$

From Theorem 4, if $\alpha, t \in [0,1]$, $r > 0$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr,\ qr \geqslant 1$, then

$$\omega(D^* T f(T) C) \leqslant \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \left\| \frac{1}{p} \left| |T|^{\alpha t} C\right|^{2pr} + \frac{1}{q} \left| |T^*|^{(1-\alpha)(1-t)} D\right|^{2qr} \right\|^{\frac{1}{2r}}. \tag{65}$$

If $r \geqslant 1$, then

$$\omega(D^*Tf(T)C) \leqslant \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \times \left(\frac{\left\| |T|^{\alpha t} C \right\|^{2r} \left\| |T^*|^{(1-\alpha)(1-t)} D \right\|^{2r} + \omega^r \left(\left| |T^*|^{(1-\alpha)(1-t)} D \right|^2 \left| |T|^{\alpha t} C \right|^2 \right)}{2} \right)^{\frac{1}{2r}}. \tag{66}$$

If $r \geqslant 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geqslant 2$, then also

$$\omega^{2r}(D^*Tf(T)C) \leqslant \|T\|^{2r[\alpha(1-t)+t(1-\alpha)]} f_a^{2r}(\|T\|) \times \left\{ \frac{1}{2} \left[\left\| \frac{1}{p} \left| |T|^{\alpha t} C \right|^{2pr} + \frac{1}{q} \left| |T^*|^{(1-\alpha)(1-t)} D \right|^{2qr} \right\| + \omega^r \left(\left| |T^*|^{(1-\alpha)(1-t)} D \right|^2 \left| |T|^{\alpha t} C \right|^2 \right) \right] \right\}. \tag{67}$$

From Theorem 5 we obtain

$$\omega(D^*Tf(T)C) \leqslant \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \left\| |T|^{\alpha t} C \right\|^{\lambda} \left\| |T^*|^{(1-\alpha)(1-t)} D \right\|^{1-\lambda} \times \left\| (1-\lambda) \left| |B|^{\alpha} C \right|^{2r} + \lambda \left| |T^*|^{(1-\alpha)(1-t)} D \right|^{2r} \right\|^{\frac{1}{2r}} \tag{68}$$

for all $\alpha, \lambda, t \in [0,1]$. Moreover, we have

$$\omega(D^*Tf(T)C) \leqslant \|T\|^{\alpha(1-t)+t(1-\alpha)} f_a(\|T\|) \times \left\| (1-\lambda) \left| |T|^{\alpha t} C \right|^{2r} + \lambda \left| |T^*|^{(1-\alpha)(1-t)} D \right|^{2r} \right\|^{\frac{1}{2r}} \times \left\| \lambda \left| |T|^{\alpha t} C \right|^{2r} + (1-\lambda) \left| |T^*|^{(1-\alpha)(1-t)} D \right|^{2r} \right\|^{\frac{1}{2r}} \tag{69}$$

for all $\alpha, t \in [0,1]$ and $r \geqslant 1$.

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