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REVERSED HERMITE–HADAMARD INEQUALITY WITH APPLICATIONS

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Abstract. The Hermite-Hadamard inequality is one of the most interesting inequalities that give lower and upper bounds of the mean value of a convex function in a way that refines the convex characteristic of the function.

This paper presents a new reversed version of this outstanding result, with applications toward means of positive numbers, operator inequalities, and the Riemann-Liouville fractional integrals.

1. Introduction

Let $x,y \in \mathbb{R}$ and let $f:[x,y] \to \mathbb{R}$ be a given function. We say that f is convex, on the interval $[x,y]$, if for $0 \leqslant t \leqslant 1$ and $a,b \in [x,y]$, the following inequality holds

$$f((1-t)a+tb) \leqslant (1-t)f(a)+tf(b). \tag{1.1}$$

In particular, when $t=\frac{1}{2}$, the above inequality reads

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}. \tag{1.2}$$

When f is continuous, the convexity of f is implied by the mid-convexity, represented in (1.2).

Convex functions play an important role in exposing the theory of real functions and are behind numerous mathematical phenomena. This is a sufficient reason for Mathematicians to investigate these functions further.

In [20], it was mentioned that Hermite proved the following refinement of (1.2) in 1881:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2}, \tag{1.3}$$

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where $f:[a,b]\to\mathbb{R}$ is convex. Since this was not known to most Mathematicians at the time, Hadamard proved the same result in 1893, which is why we call it the Hermite-Hadamard inequality these days. We refer the reader to [1, 5, 9, 11, 12, 14, 16, 17, 20, 23, 24, 27, 28, 30, 32, 33, 34] as a list of recent references that treat (1.3) in terms of applications, refinements, and reverses.

Another inequality that governs convex functions is the so called "Jensen-Mercer's inequality", which asserts that if $f:[x,y]\to\mathbb{R}$ is a convex function and $x\leqslant a_1,a_2,\dots,a_n \leqslant y$, then [19]

$$f\left(x+y-\sum_{i=1}^{n}w_ia_i\right)\leqslant f(x)+f(y)-\sum_{i=1}^{n}w_if(a_i), \tag{1.4}$$

where $w_1,w_2,\dots,w_n$ are any positive scalars such that $\sum_{i=1}^{n}w_i=1$.

To establish (1.4), Mercer [19] proved that

$$f(a+b-x)\leqslant f(a)+f(b)-f(x). \tag{1.5}$$

Ten years later, under the same assumptions, in [13], the following two inequalities were proved:

$$f\left(a+b-\frac{x+y}{2}\right)\leqslant f(a)+f(b)-\int_0^1 f(tx+(1-t)y)\,dt, \tag{1.6}$$

$$f\left(a+b-\frac{x+y}{2}\right)\leqslant \frac{1}{y-x}\int_x^y f(a+b-t)\,dt. \tag{1.7}$$

In [26, Theorem 2.1], the authors proved the following refinement of (1.1), for $t\in[0,1]$,

$$\begin{aligned} f((1-t)a+tb) &\leqslant (1-t)\int_0^1 f(t\alpha(b-a)+a)d\alpha \\ &\quad +t\int_0^1 f((1-t)\alpha(b-a)+tb+(1-t)a)d\alpha \\ &\leqslant (1-t)f(a)+tf(b). \end{aligned} \tag{1.8}$$

Then they used this inequality to show some results concerning the weighted logarithmic and identric means.

In this paper, we present a simple reverse of (1.3), then a reverse of (1.8). These obtained results will then give new reversed versions of various mean inequalities. Further, some of these results will be used to show reversed operator Hermite-Hadamard inequality and reversed inequalities for the Riemann-Liouville fractional integrals.

In our analysis, the difference between the two sides in (1.2) will play an important role. We point out here that this difference has appeared in numerous applications related to convex functions. For example, in [6], the following refinement and reverse of (1.1) were shown

$$f((1-t)a+tb) \leqslant (1-t)f(a)+tf(b)-2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right), \quad (1.9)$$

and

$$(1-t)f(a)+tf(b) \leqslant f((1-t)a+tb)+2R\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right) \quad (1.10)$$

where $f:[a,b]\to\mathbb{R}$ is convex, $0\leqslant t\leqslant 1$, $r=\min\{t,1-t\}$ and $R=\max\{t,1-t\}$.

These inequalities were used and extended in the literature in various ways, as seen in [3, 21, 22, 29].

Noting that

$$\int_0^1 f((1-t)a+tb)dt = \frac{1}{b-a}\int_a^b f(t)dt,$$

then integrating (1.9) and (1.10) imply

$$\frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right),$$

which is a refinement of the second inequality in (1.3), and

$$\frac{f(a)+f(b)}{2} \leqslant \frac{1}{b-a}\int_a^b f(t)dt+\frac{3}{2}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right),$$

which is a trivial reverse of the second inequality in (1.3).

One of the motivations of the current article is to establish that the inequalities (1.5), (1.6), and (1.7) are equivalent. This paper also gives a non-trivial reversed inequality for both inequalities in (1.3), then presents a reverse of (1.8). The obtained results are then used to generate reversed relations among the weighted arithmetic mean, weighted geometric mean, weighted logarithmic mean, and weighted identric mean, which are defined, for $a,b>0$ and $0\leqslant t\leqslant 1$, respectively, by

$$\begin{aligned} &a\nabla_t b=(1-t)a+tb, a\sharp_t b=a^{1-t}b^t,\\ &\mathscr{L}_t(a,b)=\frac{1}{\log b-\log a}\left(\frac{1-t}{t}b^{1-t}\left(b^t-a^t\right)+\frac{t}{1-t}a^t\left(b^{1-t}-a^{1-t}\right)\right),\\ &\mathscr{I}_t(a,b)=\frac{1}{e}((1-t)a+tb)^{\frac{(1-2t)((1-t)a+tb)}{t(1-t)(b-a)}}\left(\frac{b^{\frac{tb}{1-t}}}{a^{\frac{(1-t)a}{t}}}\right)^{\frac{1}{b-a}}. \end{aligned} \quad (1.11)$$

It is well known that if $a,b>0$ and $0\leqslant t\leqslant 1$, then

$$a\sharp_t b\leqslant \mathscr{L}_t(a,b)\leqslant a\nabla_t b \text{ and } a\sharp_t b\leqslant \mathscr{I}_t(a,b)\leqslant a\nabla_t b. \quad (1.12)$$

These relations will be reversed as one of the main targets of our results. Then, we present reversed Hermite-Hadamard inequalities for operators and the Riemann-Liouville fractional integrals.

2. Main results

The following lemmas will be useful to obtain the first main results of this section.

LEMMA 2.1. [25, Theorem 1.1.8] *If a mid convex function* $f:[a,b]\to\mathbb{R}$ *is pointwise continuous in* (a,b)*, then* f *is a convex function.*

LEMMA 2.2. [10, Theorem 125] *A continuous function* $f:(a,b)\to\mathbb{R}$ *is convex on* (a,b) *if and only if the inequality*

$$f(x)\leqslant\frac{1}{2h}\int_{x-h}^{x+h}f(u)du$$

holds for all $x, x\pm h$ *such that* $a\leqslant x-h<x<x+h\leqslant b$.

See also [7, Theorem 2.1] and [25, p. 63].

THEOREM 2.1. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuous function. Then the following are equivalent:*

(i) The inequality

$$f((1-t)a+tb)\leqslant(1-t)f(a)+tf(b)$$

holds for any $0\leqslant t\leqslant 1$.

(ii) The inequality

$$f(a+b-x)\leqslant f(a)+f(b)-f(x)$$

holds for any $x\in[a,b]$.

(iii) The function $f:[a,b]\to\mathbb{R}$ *satisfies*

$$f\left(a+b-\frac{x+y}{2}\right)\leqslant f(a)+f(b)-\int_0^1 f(tx+(1-t)y)\,dt$$

for any $x,y\in[a,b]$ *and* $0\leqslant t\leqslant 1$.

(iv) The function $f:[a,b]\to\mathbb{R}$ *satisfies*

$$f\left(a+b-\frac{x+y}{2}\right)\leqslant\frac{1}{y-x}\int_x^y f(a+b-t)dt$$

for any $x,y\in[a,b]$ *and* $x\leqslant t\leqslant y$.

Proof.

(i) $\Longrightarrow$ (ii) has been shown in [19] as a more general form.

Taking $x := \dfrac{a+b}{2}$ in (ii), we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}.$$

Thus we have (ii) $\Longrightarrow$ (i) by Lemma 2.1.

(i) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iv) have been shown in [13, Theorem 2.1].

If we take $y := x$ in (iii), then we have

$$f(a+b-x) \leqslant f(a)+f(b) - \int_0^1 f(x)dt$$

which is just the same as (ii). Thus, (iii) $\Longrightarrow$ (ii) was shown.

If we take $x := a$ and $y := b$ in (iv), then we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(a+b-t)dt$$

Taking $a+b-t =: u$ in the above, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(u)du.$$

Putting $a := x-h$ and $b := x+h$ for $h > 0$ in the above, we have

$$f(x) \leqslant \frac{1}{2h}\int_{x-h}^{x+h} f(u)du.$$

Thus we have $f((1-t)a+tb) \leqslant (1-t)f(a)+tf(b)$ for $f:(a,b)\to\mathbb{R}$ by Lemma 2.2. Therefore we have $f((1-t)a_k+tb_k) \leqslant (1-t)f(a_k)+tf(b_k)$, where the squences $\{a_k\}$ and $\{b_k\}$ are defined by $a_k := a + \dfrac{(b-a)}{2^k}$ and $b_k := b - \dfrac{(b-a)}{2^k}$ for $k \in \mathbb{N}$. By taking the limit $k \to \infty$ with the continuity of f, we obtain (iv) $\Longrightarrow$ (i).

Thus, the proof is complete. $\square$

Now we proceed to present a reverse of (1.3) and a reverse of (1.8), with applications towards the means mentioned above. In addition, we present some applications in the C^*-algebra of bounded linear operators on a complex Hilbert space. In the end, we present related results for the Riemann-Liouville fractional integrals in a way that reverses well known results in the literature.

2.1. Two results for convex functions

We begin with the following reverse of (1.3).

THEOREM 2.2. *Let $f:[x,y]\to\mathbb{R}$ be a convex function and let $x\leqslant a,b\leqslant y$. Then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right)\\&\leqslant\frac{1}{b-a}\int_a^b f(t)\,dt\\&\leqslant f\left(\frac{a+b}{2}\right)+2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right).\end{aligned}$$

Proof. It follows from (1.4) that

$$f\left(x+y-\frac{a+b}{2}\right)\leqslant f(x)+f(y)-\frac{f(a)+f(b)}{2}. \tag{2.1}$$

If we replace a and b by $(1-t)a+tb$ and $(1-t)b+ta$ with $0\leqslant t\leqslant 1$, in the above inequality, we get

$$\begin{aligned}f\left(x+y-\frac{a+b}{2}\right)&=f\left(x+y-\frac{(1-t)a+tb+(1-t)b+ta}{2}\right)\\&\leqslant f(x)+f(y)-\frac{f((1-t)a+tb)+f((1-t)b+ta)}{2}.\end{aligned} \tag{2.2}$$

Taking integral over $0\leqslant t\leqslant 1$ and using the fact that

$$\int_0^1 f((1-t)a+tb)\,dt=\int_0^1 f((1-t)b+ta)\,dt=\frac{1}{b-a}\int_a^b f(t)\,dt,$$

we get

$$f\left(x+y-\frac{a+b}{2}\right)\leqslant f(x)+f(y)-\frac{1}{b-a}\int_a^b f(t)dt. \tag{2.3}$$

On the other hand, we know that if f is a convex function and $\nu>0$, then

$$f((1+\nu)s-\nu t)\geqslant(1+\nu)f(s)-\nu f(t), \tag{2.4}$$

holds, provided that s, t, and $(1+\nu)s-\nu t$ are contained in the domain of convexity of f; [8]. Notice that if $x\leqslant a,b\leqslant y$, then $x\leqslant\frac{a+b}{2}\leqslant y$. Checking that $x\leqslant x+y-\frac{a+b}{2}\leqslant y$ is not challenging. Therefore, by using (2.4), for $\nu=1$, we get

$$\begin{aligned}f\left(x+y-\frac{a+b}{2}\right)&=f\left(2\left(\frac{x+y}{2}\right)-\frac{a+b}{2}\right)\\&\geqslant 2f\left(\frac{x+y}{2}\right)-f\left(\frac{a+b}{2}\right).\end{aligned} \tag{2.5}$$

Combining the two inequalities (2.3) and (2.5) implies

$$2f\left(\frac{x+y}{2}\right)-f\left(\frac{a+b}{2}\right)\leqslant f\left(x+y-\frac{a+b}{2}\right)$$
$$\leqslant f(x)+f(y)-\frac{1}{b-a}\int_a^b f(t)dt,$$

which implies the second desired inequality. That is,

$$\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\leqslant 2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right).$$

To establish the first inequality, by (2.1), we have

$$f\left(x+y-\frac{a+b}{2}\right)\geqslant 2f\left(\frac{x+y}{2}\right)-f\left(\frac{a+b}{2}\right)$$
$$\geqslant 2f\left(\frac{x+y}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt,$$

where we utilized (1.3) to get the second inequality. Hence

$$2f\left(\frac{x+y}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\leqslant f\left(x+y-\frac{a+b}{2}\right)$$
$$\leqslant f(x)+f(y)-\frac{f(a)+f(b)}{2},$$

where we have used (2.1) to obtain the second inequality. This shows that

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\leqslant 2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right),$$

which completes the proof. □

Having established the reversed version of (1.3), we move to the reverse of (1.8), which can be stated as follows.

THEOREM 2.3. *Let $f:[x,y]\to\mathbb{R}$ be a convex function and let $x\leqslant a,b\leqslant y$. Then for any $0\leqslant t\leqslant 1$,*

$$(1-t)f(a)+tf(b)-2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right)$$
$$\leqslant (1-t)\int_0^1 f(t\alpha(b-a)+a)d\alpha+t\int_0^1 f((1-t)\alpha(b-a)+(1-t)a+tb)d\alpha$$
$$\leqslant f((1-t)a+tb)+2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right).$$

Proof. We have

$$\begin{aligned} &f(x+y-((1-t)a+tb)) \\ &\leqslant f(x)+f(y)-((1-t)f(a)+tf(b)) \\ &\leqslant f(x)+f(y)-\left((1-t)\int_0^1 f(t\alpha(b-a)+a)\,d\alpha \right. \\ &\quad \left. +t\int_0^1 f((1-t)\alpha(b-a)+(1-t)a+tb)\,d\alpha\right), \end{aligned} \tag{2.6}$$

where the first inequality follows from (1.4), while the second inequality is obtained from the second inequality in (1.8). On the other hand, by (2.4), we have

$$f(x+y-((1-t)a+tb)) \geqslant 2f\left(\frac{x+y}{2}\right)-f((1-t)a+tb). \tag{2.7}$$

Now, inequalities (2.6) and (2.7), together, imply

$$\begin{aligned} &(1-t)\int_0^1 f(t\alpha(b-a)+a)\,d\alpha+t\int_0^1 f((1-t)\alpha(b-a)+(1-t)a+tb)\,d\alpha \\ &\quad -f((1-t)a+tb) \\ &\leqslant 2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right). \end{aligned}$$

This shows the second desired inequality. Using (2.7) and (1.8), we can write

$$\begin{aligned} &f(x+y-((1-t)a+tb)) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)-f((1-t)a+tb) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)-\left((1-t)\int_0^1 f(t\alpha(b-a)+a)\,d\alpha \right. \\ &\quad \left. +t\int_0^1 f((1-t)\alpha(b-a)+(1-t)a+tb)\,d\alpha\right). \end{aligned} \tag{2.8}$$

From (1.4), we have

$$f(x+y-((1-t)a+tb)) \leqslant f(x)+f(y)-((1-t)f(a)+tf(b)). \tag{2.9}$$

Both inequalities (2.8) and (2.9) enable us to write

$$\begin{aligned}&(1-t)f(a)+tf(b)-\Bigg((1-t)\int_0^1 f(t\alpha(b-a)+a)\,d\alpha\\&\quad+t\int_0^1 f((1-t)\alpha(b-a)+(1-t)a+tb)\,d\alpha\Bigg)\\&\leqslant 2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right).\end{aligned}$$

This completes the proof. □

2.2. A non-commutative version

Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$, with the identity operator $I_{\mathscr{H}}$. A real-valued continuous function f on an interval J is said to be operator convex if

$$f((1-v)A+vB)\leqslant(1-v)f(A)+vf(B)$$

for all $0\leqslant v\leqslant 1$ and for all self-adjoint operators $A,B\in\mathscr{B}(\mathscr{H})$ whose spectra are contained in J. A map $\Phi:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{K})$ is linear if it is additive and homogeneous and is positive if it preserves the operator order "$\leqslant$", i.e., $A\leqslant B\Rightarrow\Phi(A)\leqslant\Phi(B)$, where, in this context, we say that $A\leqslant B$ if $\langle(B-A)x,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$.

Let $f:J\to\mathbb{R}$ be an operator convex function and let $A,B\in\mathscr{B}(\mathscr{H})$ be self-adjoint operators with spectra in J. It has been shown in [5] that

$$f\left(\frac{A+B}{2}\right)\leqslant\int_0^1 f((1-t)A+tB)\,dt\leqslant\frac{f(A)+f(B)}{2}, \tag{2.10}$$

as an operator Hermite-Hadamard inequality.

Let $X_1,X_2,\ldots,X_n\in\mathscr{B}(\mathscr{H})$ be self-adjoint operators with spectra in $[x,y]$ for some scalars $x<y$, and let $\Phi_1,\Phi_2,\ldots,\Phi_n:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{K})$ be positive linear maps with $\sum_{i=1}^n\Phi_i(I_{\mathscr{H}})=I_{\mathscr{K}}$. If $f:[x,y]\to\mathbb{R}$ is a continuous convex function, then [15]

$$f\left((x+y)I_{\mathscr{K}}-\sum_{i=1}^n\Phi_i(X_i)\right)\leqslant(f(x)+f(y))I_{\mathscr{K}}-\sum_{i=1}^n\Phi_i(f(X_i)), \tag{2.11}$$

which is the operator version of the Jensen-Mercer inequality. To prove (2.11), the authors showed

$$\begin{aligned} & f\left((x+y) I_{\mathscr{K}}-\sum_{i=1}^{n} \Phi_i\left(X_i\right)\right) \\ & \leqslant \frac{y I_{\mathscr{K}}-\sum_{i=1}^n \Phi_i\left(X_i\right)}{y-x} f(y)+\frac{\sum_{i=1}^n \Phi_i\left(X_i\right)-x I_{\mathscr{K}}}{y-x} f(x) \\ & \leqslant(f(x)+f(y)) I_{\mathscr{K}}-\sum_{i=1}^n \Phi_i\left(f\left(X_i\right)\right) . \end{aligned} \tag{2.12}$$

Our main result in this section is the following reverse of (2.10).

THEOREM 2.4. *Let $f:[x,y]\to\mathbb{R}$ be an operator convex function and let $A,B\in\mathscr{B}(\mathscr{H})$ be self-adjoint operators with spectra in $[x,y]$. Then*

$$\begin{aligned} & \frac{f(A)+f(B)}{2}-2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right) I_{\mathscr{H}} \\ & \leqslant \int_0^1 f((1-t) A+t B) d t \\ & \leqslant f\left(\frac{A+B}{2}\right)+2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right) I_{\mathscr{H}} . \end{aligned}$$

Proof. If we take $\Phi_i(X_i)=w_iX_i$ where $w_1,w_2,\ldots,w_n$ are positive scalars such that $\sum_{i=1}^n w_i=1$, in (2.11), then

$$f\left((x+y) I_{\mathscr{H}}-\sum_{i=1}^n w_i X_i\right) \leqslant(f(x)+f(y)) I_{\mathscr{H}}-\sum_{i=1}^n w_i f\left(X_i\right) .$$

In particular,

$$f\left((x+y) I_{\mathscr{H}}-\frac{A+B}{2}\right) \leqslant f(x)+f(y)-\frac{f(A)+f(B)}{2} . \tag{2.13}$$

Utilizing (2.13), we conclude that

$$\begin{aligned} & f\left((x+y) I_{\mathscr{H}}-\frac{A+B}{2}\right) \\ & =f\left((x+y) I_{\mathscr{H}}-\frac{(1-t) A+t B+(1-t) B+t A}{2}\right) \\ & \leqslant(f(x)+f(y)) I_{\mathscr{H}}-\frac{f((1-t) A+t B)+f((1-t) B+t A)}{2}, \end{aligned}$$

where we have used (2.11) to obtain the last inequality.

Taking integral over $0 \leqslant t \leqslant 1$ and using the identity

$$\int_0^1 f((1-t)A+tB)\,dt = \int_0^1 f((1-t)B+tA)\,dt,$$

imply

$$f\left((x+y)I_{\mathscr{H}} - \frac{A+B}{2}\right) \leqslant (f(x)+f(y))I_{\mathscr{H}} - \int_0^1 f((1-t)A+tB)\,dt. \tag{2.14}$$

On the other hand, it has been proved in [8] that if f is an operator convex, $v>0$, and $S,T \in \mathscr{B}(\mathscr{H})$ are self-adjoint operators, then

$$f((1+v)S - vT) \geqslant (1+v)f(S) - vf(T)$$

whenever the spectra of S, T, and $(1+v)S-vT$ are included in the domain of f. This inequality ensures that

$$\begin{aligned} f\left((x+y)I_{\mathscr{H}} - \frac{A+B}{2}\right) &= f\left(2\frac{x+y}{2}I_{\mathscr{H}} - \frac{A+B}{2}\right) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - f\left(\frac{A+B}{2}\right). \end{aligned} \tag{2.15}$$

Combining the two inequalities (2.14) and (2.15) gives

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - f\left(\frac{A+B}{2}\right) &\leqslant f\left((x+y)I_{\mathscr{H}} - \frac{A+B}{2}\right) \\ &\leqslant (f(x)+f(y))I_{\mathscr{H}} - \int_0^1 f((1-t)A+tB)\,dt, \end{aligned}$$

which means

$$\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \leqslant 2\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right)I_{\mathscr{H}}.$$

On the other hand, by (2.15) and (2.10), one can write

$$\begin{aligned} f\left((x+y)I_{\mathscr{H}} - \frac{A+B}{2}\right) &\geqslant 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - f\left(\frac{A+B}{2}\right) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - \int_0^1 f((1-t)A+tB)\,dt. \end{aligned}$$

This, jointly with (2.13), reveals that

$$2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}}-\int_{0}^{1}f\left((1-t)A+tB\right)dt\leqslant f\left((x+y)I_{\mathscr{H}}-\frac{A+B}{2}\right)$$
$$\leqslant (f(x)+f(y))I_{\mathscr{H}}-\frac{f(A)+f(B)}{2},$$

which offers

$$\frac{f(A)+f(B)}{2}-\int_{0}^{1}f\left((1-t)A+tB\right)dt\leqslant 2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right)I_{\mathscr{H}},$$

as expected. □

REMARK 2.1. By (2.12), we conclude that

$$\begin{aligned} f(x+y-t) &\leqslant \frac{y-t}{y-x}f(y)+\frac{t-x}{y-x}f(x)\\ &\leqslant f(y)+f(x)-f(t), \end{aligned} \tag{2.16}$$

for any $x\leqslant t\leqslant y$. On the other hand, if f is a convex function, by (2.4), we have

$$f(x+y-t)\geqslant 2f\left(\frac{x+y}{2}\right)-f(t). \tag{2.17}$$

Combining (2.16) and (2.17), we obtain

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right)-f(t) &\leqslant f(x+y-t)\\ &\leqslant \frac{y-t}{y-x}f(y)+\frac{t-x}{y-x}f(x)\\ &\leqslant f(x)+f(y)-f(t). \end{aligned} \tag{2.18}$$

The following is the operator version of (1.8).

LEMMA 2.3. *Let $f:I\to\mathbb{R}$ be an operator convex function and let $A,B\in\mathscr{B}(\mathscr{H})$ be self-adjoint operators with spectra in an interval I. Then*

$$\begin{aligned} f((1-t)A+tB) &\leqslant (1-t)\int_0^1 f((1-tx)A+txB)\,dx\\ &\quad +t\int_0^1 f((1-t)xA+(1-(1-t)x)B)\,dx\\ &\leqslant (1-t)f(A)+tf(B). \end{aligned}$$

Proof. Note that we have the identity

$$(1-t)((1-tx)a+txb)+t((1-t)xa+(1-(1-t)x)b)=(1-t)a+tb.$$

By the operator convexity, we have

$$\begin{aligned}
f((1-t)A+tB) &= f\left((1-t)((1-tx)A+txB)+t((1-t)xA+(1-(1-t)x)B)\right)\\
&\leqslant (1-t)f\left((1-tx)A+txB\right)+tf\left((1-t)xA+(1-(1-t)x)B\right)\\
&\leqslant \{(1-t)(1-tx)+(1-t)tx\}f(A)\\
&\quad +\{t(1-t)x+t(1-(1-t)x)\}f(B)\\
&= (1-t)f(A)+tf(B)
\end{aligned}$$

for $0\leqslant t\leqslant 1$. We have the desired result by integrating the above inequalities over $x\in[0,1]$. $\square$

Note that the inequalities in Lemma 2.3 recover an operator Hermite–Hadamard inequality 2.10 if we take $t:=1/2$. Applying Lemma 2.3, we give a generalized result for Theorem 2.4 in the following.

THEOREM 2.5. *Let $f:[x,y]\to\mathbb{R}$ be an operator convex function and let $A,B\in\mathscr{B}(\mathscr{H})$ be self-adjoint operators with spectra in $[x,y]$. Then*

$$\begin{aligned}
&(1-t)f(A)+tf(B)-2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right)I_{\mathscr{H}}\\
&\leqslant (1-t)\int_0^1 f\left((1-tx)A+txB\right)dx+t\int_0^1 f\left((1-t)xA+(1-(1-t)x)B\right)dx\\
&\leqslant f\left((1-t)A+tB\right)+2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right)I_{\mathscr{H}}.
\end{aligned}$$

Proof. If we take $\Phi_i(X_i)=w_iX_i$ where $w_1,w_2,\ldots,w_n$ are positive scalars such that $\sum_{i=1}^n w_i=1$, in (2.11), then

$$f\left((x+y)I_{\mathscr{H}}-\sum_{i=1}^n w_iX_i\right)\leqslant (f(x)+f(y))I_{\mathscr{H}}-\sum_{i=1}^n w_if(X_i).$$

In particular,

$$f\left((x+y)I_{\mathscr{H}}-((1-t)A+tB)\right)\leqslant f(x)I_{\mathscr{H}}+f(y)I_{\mathscr{H}}-((1-t)f(A)+tf(B)). \tag{2.19}$$

Using (2.19), we have

$$\begin{aligned}
&f\left((x+y)I_{\mathscr{H}}-((1-t)A+tB)\right)\\
&=f\left((x+y)I_{\mathscr{H}}-((1-t)\left((1-tx)A+txB\right)+t\left((1-t)xA+(1-(1-t)x)B\right))\right)\\
&\leqslant f(x)I_{\mathscr{H}}+f(y)I_{\mathscr{H}}-(1-t)f\left((1-tx)A+txB\right)-tf\left((1-t)xA+(1-(1-t)x)B\right).
\end{aligned}$$

Taking integral over $x \in [0,1]$, we have

$$\begin{aligned} & f((x+y)I_{\mathscr{H}} - ((1-t)A+tB)) \\ & \leqslant (f(x)+f(y))I_{\mathscr{H}} - (1-t)\int_0^1 f((1-tx)A+txB)\,dx \\ & \quad -t\int_0^1 f((1-t)xA+(1-(1-t)x)B)\,dx. \end{aligned}$$

On the other hand, it has been proved in [8] that if f is an operator convex, $\nu > 0$, and $S,T \in \mathscr{B}(\mathscr{H})$ are self-adjoint operators, then

$$f((1+\nu)S-\nu T) \geqslant (1+\nu)f(S) - \nu f(T)$$

whenever the spectra of S, T, and $(1+\nu)S-\nu T$ are included in the domain of f. This inequality ensures that

$$\begin{aligned} f((x+y)I_{\mathscr{H}} - ((1-t)A+tB)) &= f\left(2\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - ((1-t)A+tB)\right) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - f((1-t)A+tB). \end{aligned} \tag{2.20}$$

Thus we have

$$\begin{aligned} & 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - f((1-t)A+tB) \\ & \leqslant (f(x)+f(y))I_{\mathscr{H}} - (1-t)\int_0^1 f((1-tx)A+txB)\,dx \\ & \quad -t\int_0^1 f((1-t)xA+(1-(1-t)x)B)\,dx. \end{aligned}$$

That is, we have the second inequality in the present theorem

$$\begin{aligned} & (1-t)\int_0^1 f((1-tx)A+txB)\,dx + t\int_0^1 f((1-t)xA+(1-(1-t)x)B)\,dx \\ & \leqslant f((1-t)A+tB) + 2\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right)I_{\mathscr{H}}. \end{aligned}$$

On the other hand, from (2.20) and the first inequality in Lemma 2.3 we have

$$\begin{aligned} & f((x+y)I_{\mathscr{H}} - ((1-t)A+tB)) \\ & \geqslant 2f\left(\frac{x+y}{2}\right) - f((1-t)A+tB) \\ & \geqslant 2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}} - (1-t)\int_0^1 f((1-tx)A+txB)\,dx \\ & \quad -t\int_0^1 f((1-t)xA+(1-(1-t)x)B)\,dx. \end{aligned}$$

Thus we have by (2.19),

$$\begin{aligned}
&2f\left(\frac{x+y}{2}\right)I_{\mathscr{H}}-(1-t)\int_0^1 f\left((1-tx)A+txB\right)dx\\
&\quad -t\int_0^1 f\left((1-t)xA+(1-(1-t)x)B\right)dx\\
&\leqslant f\left((x+y)I_{\mathscr{H}}-((1-t)A+tB)\right)\\
&\leqslant \left(f(x)+f(y)\right)I_{\mathscr{H}}-\left((1-t)f(A)+tf(B)\right).
\end{aligned}$$

Therefore, we have the first inequality in the present theorem

$$\begin{aligned}
&(1-t)f(A)+tf(B)-2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right)I_{\mathscr{H}}\\
&\leqslant(1-t)\int_0^1 f\left((1-tx)A+txB\right)dx+t\int_0^1 f\left((1-t)xA+(1-(1-t)x)B\right)dx. \quad\square
\end{aligned}$$

2.3. Means applications

For $a,b>0$, the arithmetic, geometric, logarithmic, and identric means were defined in the introduction as weighted means. For $t=\frac{1}{2}$, we simply write $a\nabla b$, $a\sharp b$, $\mathscr{L}(a,b)$ and $\mathscr{I}(a,b)$ instead of $a\nabla_{\frac{1}{2}}b$, $a\sharp_{\frac{1}{2}}b$, $\mathscr{L}_{\frac{1}{2}}(a,b)$ and $\mathscr{I}_{\frac{1}{2}}(a,b)$, respectively.

Using the given formulas in (1.11), we have $\mathscr{L}(a,b)=\frac{b-a}{\ln b-\ln a}$ and $\mathscr{I}(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$. We know from (1.12) that

$$a\sharp b\leqslant\mathscr{L}(a,b),\ \mathscr{I}(a,b)\leqslant a\nabla b, \tag{2.21}$$

holds. It has been an important topic in the literature to find possible refinements and reverses of means inequalities. In the following, we present reverses of (2.21).

COROLLARY 2.1. *Let* $0<x\leqslant a,b\leqslant y$. *Then*

(i)

$$a\nabla b-\left(\sqrt{x}-\sqrt{y}\right)^2\leqslant\mathscr{L}(a,b)\leqslant a\sharp b+\left(\sqrt{x}-\sqrt{y}\right)^2.$$

(ii)

$$\frac{4xy}{(x+y)^2}\,a\nabla b\leqslant\mathscr{I}(a,b)\leqslant\frac{(x+y)^2}{4xy}\,a\sharp b.$$

Proof. Letting $f(t)=e^t$, and replacing a,b,x,y by $\ln a,\ln b,\ln x,\ln y$, respectively, in Theorem 2.2, we get (i).

Taking $f(t)=-\ln t$, in Theorem 2.2, and then applying $\exp$, we get (ii). $\square$

On the other hand, Theorem 2.3 implies the following weighted versions of Corollary 2.1.

COROLLARY 2.2. *Let* $0 < x \leqslant a, b \leqslant y$ *and* $0 \leqslant t \leqslant 1$. *Then*

$$a\nabla_t b - \left(\sqrt{x} - \sqrt{y}\right)^2 \leqslant \mathscr{L}_t(a,b) \leqslant a\sharp_t b + \left(\sqrt{x} - \sqrt{y}\right)^2,$$

and

$$\frac{4xy}{(x+y)^2}\, a\nabla_t b \leqslant \mathscr{I}_t(a,b) \leqslant \frac{(x+y)^2}{4xy}\, a\sharp_t b.$$

REMARK 2.2. We define the function

$$f(t) = \frac{2(t-1)}{(1+t)\ln t};\ (t \neq 1).$$

One can check that $f > 0$, $f(0) = 0$ (as a limit) and

$$f'(t) = \frac{2(t(2\ln t - t) + 1)}{t(1+t)^2 \ln^2 t}.$$

Consequently,

$$\begin{cases} f'(t) > 0 & \text{if } t < 1 \\ f'(t) < 0 & \text{if } t > 1 \end{cases}.$$

This shows that f increases when $0 < t < 1$ and decreases when $t > 1$.

- If $0 < m' \leqslant b \leqslant m < M \leqslant a \leqslant M'$, then $0 < \frac{m'}{M'} \leqslant \frac{b}{a} \leqslant \frac{m}{M} < 1$. This gives $f\left(\frac{m'}{M'}\right) \leqslant f\left(\frac{b}{a}\right) \leqslant f\left(\frac{m}{M}\right)$. Therefore,

$$\frac{\mathscr{L}(M',m')}{M'\nabla m'}\, a\nabla b \leqslant \mathscr{L}(a,b) \leqslant \frac{\mathscr{L}(M,m)}{M\nabla m}\, a\nabla b.$$

  This indicates that

$$\frac{\mathscr{L}(M',m')}{M'\nabla m'}\, A\nabla B \leqslant \mathscr{L}(A,B) \leqslant \frac{\mathscr{L}(M,m)}{M\nabla m}\, A\nabla B,$$

  provided that $0 < m' I_{\mathscr{H}} \leqslant B \leqslant m I_{\mathscr{H}} < M I_{\mathscr{H}} \leqslant A \leqslant M' I_{\mathscr{H}}$. Here, $A, B \in \mathscr{B}(\mathscr{H})$ are strictly positive operators, $A\nabla B = \frac{A+B}{2}$ is the arithmetic mean of A, B and $\mathscr{L}(A,B) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$, where $f(t) = \frac{1}{e} t^{\frac{t}{t-1}}$ is the representing function of the identric operator mean.

- On the other hand, if $0 < m' \leqslant a \leqslant m < M \leqslant b \leqslant M'$, then $1 < \frac{M}{m} \leqslant \frac{b}{a} \leqslant \frac{M'}{m'}$. This gives $f\left(\frac{M'}{m'}\right) \leqslant f\left(\frac{b}{a}\right) \leqslant f\left(\frac{M}{m}\right)$. Therefore,

$$\frac{\mathscr{L}(M',m')}{M'\nabla m'}\, a\nabla b \leqslant \mathscr{L}(a,b) \leqslant \frac{\mathscr{L}(M,m)}{M\nabla m}\, a\nabla b.$$

  This indicates that

$$\frac{\mathscr{L}(M',m')}{M'\nabla m'}\, A\nabla B \leqslant \mathscr{L}(A,B) \leqslant \frac{\mathscr{L}(M,m)}{M\nabla m}\, A\nabla B,$$

  provided that $0 < m' I_{\mathscr{H}} \leqslant A \leqslant m I_{\mathscr{H}} < M I_{\mathscr{H}} \leqslant B \leqslant M' I_{\mathscr{H}}$.

This gives the multiplicative versions that relate the logarithmic with the arithmetic means.

To study the multiplicative versions that relate the logarithmic with the geometric means, define

$$g(t)=\frac{\sqrt{t}\ln t}{t-1};\ (t\neq 1,0).$$

Then

$$g'(t)=-\frac{(1+t)\ln t+2(1-t)}{2(t-1)^2\sqrt{t}}.$$

Calculations reveal that g increases when $0<t<1$ and decreases when $t>1$.

- Now, if $0<m'\leqslant b\leqslant m<M\leqslant a\leqslant M'$, then $\frac{m'}{M'}\leqslant\frac{b}{a}\leqslant\frac{m}{M}$. This gives $g\left(\frac{m'}{M'}\right)\leqslant g\left(\frac{b}{a}\right)\leqslant g\left(\frac{m}{M}\right)$. Therefore,

  $$\frac{M'\sharp m'}{\mathscr{L}(M',m')}\mathscr{L}(a,b)\leqslant a\sharp b\leqslant\frac{M\sharp m}{\mathscr{L}(M,m)}\mathscr{L}(a,b).$$

  This indicates that

  $$\frac{M'\sharp m'}{\mathscr{L}(M',m')}\mathscr{L}(A,B)\leqslant A\sharp B\leqslant\frac{M\sharp m}{\mathscr{L}(M,m)}\mathscr{L}(A,B),$$

  provided that $0<m'I_{\mathscr{H}}\leqslant B\leqslant mI_{\mathscr{H}}<MI_{\mathscr{H}}\leqslant A\leqslant M'I_{\mathscr{H}}$. Notice that $A\sharp B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$ is the geometric mean of the strictly positive operators $A,B\in\mathscr{B}(\mathscr{H})$.

- Further, if $0<m'\leqslant a\leqslant m<M\leqslant b\leqslant M'$, then $\frac{M}{m}\leqslant\frac{b}{a}\leqslant\frac{M'}{m'}$. This gives $g\left(\frac{M'}{m'}\right)\leqslant g\left(\frac{b}{a}\right)\leqslant g\left(\frac{M}{m}\right)$. Therefore,

  $$\frac{M'\sharp m'}{\mathscr{L}(M',m')}\mathscr{L}(a,b)\leqslant a\sharp b\leqslant\frac{M\sharp m}{\mathscr{L}(M,m)}\mathscr{L}(a,b).$$

  This indicates that

  $$\frac{M'\sharp m'}{\mathscr{L}(M',m')}\mathscr{L}(A,B)\leqslant A\sharp B\leqslant\frac{M\sharp m}{\mathscr{L}(M,m)}\mathscr{L}(A,B),$$

  provided that $0<m'I_{\mathscr{H}}\leqslant A\leqslant mI_{\mathscr{H}}<MI_{\mathscr{H}}\leqslant B\leqslant M'I_{\mathscr{H}}$.

Now, to investigate a possible multiplicative relation relating the identric with the arithmetic means, define

$$h(t)=\frac{2e^{-1}t^{\frac{t}{t-1}}}{1+t};\ (t\neq 0,1).$$

Then

$$h'(t) = -\frac{2e^{-1}t^{\frac{t}{t-1}}\left(1+t\right)\ln t + 2\left(1-t\right)}{\left(1+t\right)^2\left(1-t\right)^2}$$

which shows

$$\begin{cases} h'(t) > 0 & \text{if } t < 1 \\ h'(t) < 0 & \text{if } t > 1 \end{cases}.$$

- Notice that if $0 < m' \leqslant b \leqslant m < M \leqslant a \leqslant M'$, then $\frac{m'}{M'} \leqslant \frac{b}{a} \leqslant \frac{m}{M}$. This gives $h\left(\frac{m'}{M'}\right) \leqslant h\left(\frac{b}{a}\right) \leqslant h\left(\frac{m}{M}\right)$. Therefore,

$$\frac{\mathscr{I}(M',m')}{M'\nabla m'}\, a\nabla b \leqslant \mathscr{I}(a,b) \leqslant \frac{\mathscr{I}(M,m)}{M\nabla m}\, a\nabla b.$$

  This indicates that

$$\frac{\mathscr{I}(M',m')}{M'\nabla m'}\, A\nabla B \leqslant \mathscr{I}(A,B) \leqslant \frac{\mathscr{I}(M,m)}{M\nabla m}\, A\nabla B,$$

  provided that $0 < m'I_{\mathscr{H}} \leqslant B \leqslant mI_{\mathscr{H}} < MI_{\mathscr{H}} \leqslant A \leqslant M'I_{\mathscr{H}}$.

- However, if $0 < m' \leqslant a \leqslant m < M \leqslant b \leqslant M'$, then $\frac{M}{m} \leqslant \frac{b}{a} \leqslant \frac{M'}{m'}$. This gives $h\left(\frac{M'}{m'}\right) \leqslant h\left(\frac{b}{a}\right) \leqslant h\left(\frac{M}{m}\right)$. Therefore,

$$\frac{\mathscr{I}(m',M')}{M'\nabla m'}\, a\nabla b \leqslant \mathscr{I}(a,b) \leqslant \frac{\mathscr{I}(m,M)}{M\nabla m}\, a\nabla b.$$

  This indicates that

$$\frac{\mathscr{I}(m',M')}{M'\nabla m'}\, A\nabla B \leqslant \mathscr{I}(A,B) \leqslant \frac{\mathscr{I}(m,M)}{M\nabla m}\, A\nabla B,$$

  provided that $0 < m'I_{\mathscr{H}} \leqslant A \leqslant mI_{\mathscr{H}} < MI_{\mathscr{H}} \leqslant B \leqslant M'I_{\mathscr{H}}$.

Our last comment in this remark is the possible multiplicative relations that govern the identric and geometric means. For this, define

$$k(t) = e^{-1}t^{\frac{t}{t-1}-\frac{1}{2}};\ (t \neq 1,0).$$

Thus,

$$k'(t) = -\frac{e^{-1}\left(t\left(2\ln t - t\right)+1\right)}{2(1-t)^2 t^{\frac{t-3}{2(t-1)}}}$$

which implies

$$\begin{cases} k'(t) < 0 & \text{if } t < 1 \\ k'(t) > 0 & \text{if } t > 1 \end{cases}.$$

- Clearly, if $0 < m' \leqslant b \leqslant m < M \leqslant a \leqslant M'$, then $\frac{m'}{M'} \leqslant \frac{b}{a} \leqslant \frac{m}{M}$. This gives $k\left(\frac{m}{M}\right) \leqslant k\left(\frac{b}{a}\right) \leqslant k\left(\frac{m'}{M'}\right)$. Therefore,

$$\frac{\mathscr{I}(M,m)}{M\sharp m} a\sharp b \leqslant \mathscr{I}(a,b) \leqslant \frac{\mathscr{I}(M',m')}{M'\sharp m'} a\sharp b.$$

  This indicates that

$$\frac{\mathscr{I}(M,m)}{M\sharp m} A\sharp B \leqslant \mathscr{I}(A,B) \leqslant \frac{\mathscr{I}(M',m')}{M'\sharp m'} A\sharp B,$$

  provided that $0 < m'I_{\mathscr{H}} \leqslant B \leqslant mI_{\mathscr{H}} < MI_{\mathscr{H}} \leqslant A \leqslant M'I_{\mathscr{H}}$.

- Lastly, if $0 < m' \leqslant a \leqslant m < M \leqslant b \leqslant M'$, then $\frac{M}{m} \leqslant \frac{b}{a} \leqslant \frac{M'}{m'}$. This gives $k\left(\frac{M}{m}\right) \leqslant k\left(\frac{b}{a}\right) \leqslant k\left(\frac{M'}{m'}\right)$. Therefore,

$$\frac{\mathscr{I}(m,M)}{M\sharp m} a\sharp b \leqslant \mathscr{I}(a,b) \leqslant \frac{\mathscr{I}(m',M')}{M'\sharp m'} a\sharp b.$$

  This indicates that

$$\frac{\mathscr{I}(m,M)}{M\sharp m} A\sharp B \leqslant \mathscr{I}(A,B) \leqslant \frac{\mathscr{I}(m',M')}{M'\sharp m'} A\sharp B,$$

  provided that $0 < m'I_{\mathscr{H}} \leqslant A \leqslant mI_{\mathscr{H}} < MI_{\mathscr{H}} \leqslant B \leqslant M'I_{\mathscr{H}}$.

2.4. Inequalities for the fractional integral

In this part of the paper, we present some Hermite-Hadamard-type inequalities for the Riemann-Liouville integrals J_{a+}^{ν} and J_{b-}^{ν}, which are defined for $\nu > 0$ as follows

$$J_{a+}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)dt \text{ and } J_{b-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)dt,$$

where $f : [a,b] \to \mathbb{R}$ is continuous, and Γ is the gamma function. The Hermite-Hadamard type's inequalities for fractional integrals have attracted numerous researchers' interest. We refer the reader to [1, 2, 11, 12, 18, 31, 32, 34] as a list of such references.

THEOREM 2.6. *Let $f : [x,y] \to \mathbb{R}$ be convex and continuous. If $x \leqslant a,b \leqslant y$ and $\nu > 0$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - 2\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right) \\ &\leqslant \frac{\Gamma(\nu+1)}{2(b-a)^{\nu}} \left(J_{a+}^{\nu} f(b) + J_{b-}^{\nu} f(a)\right) \\ &\leqslant f\left(\frac{a+b}{2}\right) + 2\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right). \end{aligned}$$

Proof. The two inequalities (2.2) and (2.5) imply, for $0 \leqslant t \leqslant 1$,

$$
\begin{aligned}
&2f\left(\frac{x+y}{2}\right)-f\left(\frac{a+b}{2}\right)\\
&\leqslant f\left(x+y-\frac{a+b}{2}\right)\\
&\leqslant f(x)+f(y)-\frac{f((1-t)a+tb)+f((1-t)b+ta)}{2}.
\end{aligned}
\tag{2.22}
$$

Multiplying both sides of (2.22) by $t^{\nu-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$
\begin{aligned}
&\frac{1}{\nu}\left(2f\left(\frac{x+y}{2}\right)-f\left(\frac{a+b}{2}\right)\right)\\
&\leqslant \frac{1}{\nu}(f(x)+f(y))-\int_0^1 t^{\nu-1}\left(\frac{f((1-t)a+tb)+f((1-t)b+ta)}{2}\right)dt\\
&=\frac{1}{\nu}(f(x)+f(y))-\frac{\Gamma(\nu)}{2(b-a)^{\nu}}\left(J_{a+}^{\nu}f(b)+J_{b-}^{\nu}f(a)\right),
\end{aligned}
\tag{2.23}
$$

where the last equality follows, noting that

$$
\begin{aligned}
\int_0^1 t^{\nu-1}f((1-t)a+tb)dt &= \frac{1}{(b-a)^{\nu}}\int_a^b (y-a)^{\nu-1}f(y)dy\\
&=\frac{\Gamma(\nu)}{(b-a)^{\nu}}J_{b-}^{\nu}(a)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 t^{\nu-1}f(ta+(1-t)b)dt &= \frac{1}{(b-a)^{\nu}}\int_a^b (b-y)^{\nu-1}f(y)dy\\
&=\frac{\Gamma(\nu)}{(b-a)^{\nu}}J_{a+}^{\nu}(b).
\end{aligned}
$$

Rearranging (2.23) and recalling that $\nu\Gamma(\nu)=\Gamma(\nu+1)$, we obtain

$$
\frac{\Gamma(\nu+1)}{2(b-a)^{\nu}}\left(J_{a+}^{\nu}f(b)+J_{b-}^{\nu}f(a)\right)\leqslant f\left(\frac{a+b}{2}\right)+2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right),
$$

which proves the second desired inequality.

To show the first inequality, we notice that

$$\begin{aligned} f(x)+f(y)-\frac{f(a)+f(b)}{2} &\geqslant f\left(x+y-\frac{a+b}{2}\right) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)-f\left(\frac{a+b}{2}\right) \\ &= 2f\left(\frac{x+y}{2}\right)-f\left(\frac{(1-t)a+tb+(1-t)b+ta}{2}\right) \\ &\geqslant 2f\left(\frac{x+y}{2}\right)-\frac{f((1-t)a+tb)+f((1-t)b+ta)}{2}. \end{aligned}$$

That is,

$$2f\left(\frac{x+y}{2}\right)-\frac{f((1-t)a+tb)+f((1-t)b+ta)}{2} \leqslant f(x)+f(y)-\frac{f(a)+f(b)}{2} \tag{2.24}$$

Multiplying both sides of (2.24) by $t^{\nu-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} &\frac{2}{\nu}f\left(\frac{x+y}{2}\right)-\frac{\Gamma(\nu)}{2(b-a)^{\nu}}\left(J_{a^+}^{\nu}f(b)+J_{b^-}^{\nu}f(a)\right) \\ &\leqslant \frac{1}{\nu}\left(f(x)+f(y)-\frac{f(a)+f(b)}{2}\right). \end{aligned}$$

Thus,

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\nu+1)}{2(b-a)^{\nu}}\left(J_{a^+}^{\nu}f(b)+J_{b^-}^{\nu}f(a)\right) \leqslant 2\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right).$$

This completes the proof. □

We point out here that Theorem 2.6 presents a possible reverse of [32, Theorem 2], which states that

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\nu+1)}{2(b-a)^{\nu}}\left(J_{a^+}^{\nu}f(b)+J_{b^-}^{\nu}f(a)\right) \leqslant \frac{f(a)+f(b)}{2}.$$

At this point, we present the fractional integral versions of (1.9) and (1.10).

THEOREM 2.7. *Let $f:[a,b]\to\mathbb{R}$ be convex and continuous. If $\nu>0$, then*

$$\frac{\Gamma(\nu+2)}{(b-a)^{\nu}}J_{b^-}^{\nu}(a)+\frac{2^{1+\nu}-1}{2^{\nu}(2+\nu)}\nu \leqslant f(a)+\nu f(b),$$

and

$$\nu f(a)+f(b) \leqslant \frac{\Gamma(\nu+2)}{(b-a)^{\nu}}J_{a^+}^{\nu}(b)+\frac{1+2^{\nu+1}(\nu+1)}{2^{\nu}(2+\nu)}\nu.$$

In particular,

$$\frac{\Gamma(\nu+2)}{(b-a)^{\nu}}\left(J_{b^-}^{\nu}(a)-J_{a^+}^{\nu}(b)\right)\leqslant(1-\nu)(f(a)-f(b))+\frac{2\nu(\nu-2^{-\nu})}{2+\nu}.$$

Proof. The proof follows by multiplying (1.9) and (1.10) with $t^{\nu-1}$, then integrating on $[0,1]$, with respect to t. □

Following the same logic, the inequalities in (2.18) imply the following version.

THEOREM 2.8. *Let $f:[a,b]\to\mathbb{R}$ be convex and continuous. If $\nu>0$, then*

$$2(\nu+1)f\left(\frac{a+b}{2}\right)-(f(b)+\nu f(a))\leqslant\frac{\Gamma(\nu+2)}{(b-a)^{\nu}}J_{b^-}^{\nu}(a)$$

and

$$\frac{\Gamma(\nu+2)}{(b-a)^{\nu}}J_{b^-}^{\nu}(a)\leqslant f(a)+\nu f(b).$$

Declarations

Availability of data and materials. Not applicable.

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